Mathematical
Inequalities
& Applications
Volume 19, Number 1 (2016), 15–31 doi:10.7153/mia-19-02

FRACTIONAL MAXIMAL OPERATOR AND FRACTIONAL INTEGRAL OPERATOR ON ORLICZ–LORENTZ SPACES

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(*Communicated by B. Opic*)

Abstract. In this paper, we prove the characterization of the weighted modular inequalities for the fractional maximal operator M_α $(0 \leqslant \alpha < n)$ on the Orlicz-Lorentz spaces by atomic decomposition which induces a sufficient condition of the boundedness for this operator on the Orlicz-Lorentz spaces. And we also find the characterization of the weighted modular inequalities for the fractional integral operator I_α $(0 < \alpha < n)$ on the Orlicz-Lorentz spaces in certain case which leads to a sufficient condition of the boundedness for I_α $(0 < \alpha < n)$.

1. Introduction

Let $\mathscr{M}(X,\mu)$ be the class of all measurable and almost everywhere finite functions on X. For $f \in \mathscr{M}(X,\mu)$, a nonincreasing rearrangement of f is a nonincreasing function f^* on $\mathbb{R}_+ \equiv (0,+\infty)$ that is equimeasurable with $|f|$. The rearrangement f^* is defined by the equality (see [1])

$$f^*_\mu(t) = \inf\{s : \lambda^\mu_f(s) \leqslant t\}, \ \ 0 < t < \infty,$$

where

$$\lambda^\mu_f(s) = \mu\{x \in X : |f(x)| > s\}, \ s \geqslant 0.$$

When $(X,\mu) = (\mathbb{R}^n, udx)$ (u is a weight in $\mathbb{R}^n$), we denote $\lambda^\mu_f = \lambda^u_f$, $f^*_\mu = f^*_u$, especially, $f^*_1 = f^*$. We say $\phi : [0,\infty) \to [0,\infty)$ is a Young function if ϕ is nondecreasing and convex with $\phi(0) = 0$, and $\lim_{x\to\infty} \phi(x) = \infty$. An N-function ϕ is a continuous Young function such that $\phi(x) = 0$ if and only if $x = 0$ and $\lim_{x\to 0} \phi(x)/x = 0$, $\lim_{x\to\infty} \phi(x)/x = +\infty$. The Young conjugate $\tilde{\phi}$ of Young function ϕ is defined by

$$\tilde{\phi}(x) = \sup_{y \geqslant 0}\{xy - \phi(y)\}, \ x \geqslant 0.$$

The Orlicz-Lorentz spaces $\Lambda^\phi_X(w)$ (see [18], [19]; if there is no ambiguity, the notation $\Lambda^\phi_\mu(w)$ may be adopted) associated to the Young function ϕ and a weight w on $\mathbb{R}_+$

Mathematics subject classification (2010): 46E30, 46B42.
Keywords and phrases: Orlicz-Lorentz spaces, fractional maximal operator, fractional integral operator, modular inequalities, boundedness of operator.

Paper MIA-19-02

(nonnegative locally integrable function in $\mathbb{R}_+$), is the set of $f \in \mathscr{M}(X,\mu)$ such that for some $\lambda > 0$, we have $I(\lambda f) < \infty$, where

$$I(f) = \int_0^\infty \phi(f_\mu^*(t))w(t)dt,$$

and we let

$$\|f\|_{\Lambda_X^\phi(w)} = \inf\left\{\varepsilon : I\left(\frac{f}{\varepsilon}\right) \leqslant 1\right\}.$$

If $w(t) = 1$, then $\Lambda_X^\phi(w) = L^\phi(X,\mu)$ is an Orlicz space (see [23], [15]); if $\phi(t) = t^p$ $(1 \leqslant p < \infty)$, then $\Lambda_X^\phi(w) = \Lambda_X^p(w)$ is a Lorentz space (see [16], [17] and [4]). If $(X,\mu) = (\mathbb{R}^n, dx)$, we denote $\Lambda_X^\phi(w) = \Lambda^\phi(w)$, and if $(X,\mu) = (\mathbb{R}^+, wdx)$, we write $L^\phi(X,\mu) = L^\phi(w)$.

Given an arbitrary function $G : [0,\infty) \to [0,\infty)$, we say that G satisfies condition Δ_2, in symbol $G \in \Delta_2$ whenever $\sup_{t>0} \frac{G(2t)}{G(t)} < \infty$. A Young function F is said to satisfy Δ' (resp. ∇') condition (e.g., see [10] or [25]), in symbol $F \in \Delta'$ (resp. $F \in \nabla'$) if there exists $C > 0$ such that

$$F(xy) \leqslant CF(x)F(y) \text{ (resp. } F(xy) \geqslant CF(x)F(y)), \ \forall\, x,\ y \geqslant 0.$$

Clearly if $G \in \Delta'$ then $G \in \Delta_2$.

If w is a weight in $\mathbb{R}^+$ and ϕ is a Young function, then we denote $w \in B_\phi$ if there exists a positive constant C such that

$$\int_0^\infty \phi(Sf(x))w(x)dx \leqslant \int_0^\infty \phi(Cf(x))w(x)dx \tag{1.1}$$

for all monotone decreasing non-negative functions, where S is Hardy operator defined as

$$Sf(x) = \frac{1}{x}\int_0^x f(t)dt,\ x \in [0,\infty).$$

The characterization of (1.1) was obtained in [13] and [14]. When $\phi(t) = t^p$ $(p \geqslant 0)$, $B_\phi = B_p$ (e.g., see [4]).

Let us introduce the fractional maximal operator and the fractional integral operator. In this paper all cubes Q considered have their sides parallel to the coordinate axis. The fractional maximal operator M_α $(0 \leqslant \alpha < n)$ is defined as

$$M_\alpha f(x) = \sup_{x\in Q}\left\{|Q|^{\frac{\alpha}{n}-1}\int_Q |f(y)|dy \,\Big|\, Q \text{ is a cube}\right\}.$$

The fractional integral operator I_α $(0 \leqslant \alpha < n)$ is defined as

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}dy,\quad x \in \mathbb{R}^n,\ 0 < \alpha < n.$$

In [3], Bloom and Kerman presented the Hardy-type modular inequality in the Orlicz spaces and got the boundedness of Hardy-Littlewood maximal operator in Orlicz spaces. Heinig and Lai [13] studied the weighted modular inequalities for Hardy-type operators on monotone functions. Li [14] obtained a kind of characterization of strong Hardy-type modular inequality which is different from that of [13]. Sawyer [27] had a dual method to study inequalities on monotone decreasing functions from which the boundedness of Hardy-Littlewood maximal operator on the weighted Lorentz spaces are obtained. In [7] Kamińska and Mastyło found the conditions for boundedness of Hardy-Littlewood maximal operator in some class of Orlicz-Lorentz spaces. Kokilashvili [9] studied weighted inequalities for fractional maximal functions and fractional integrals in the classical Lorentz spaces. Cianchi and Edmunds in [6] got the the boundedness of fractional integration and special singular operators in the classical Lorentz spaces. Rakotondratsimba [24] gave the characterization of the boundedness of fractional maximal operator in the weighted Lorentz spaces. Concerning the convolution operator on Lorentz spaces and Orlicz spaces there are a lot of results and readers may refer to, e.g., [20], [21], [22], [12] and so on.

The paper is divided into three sections. Section 2 mainly contains characterization of the modular inequalities for the fractional maximal operator on the Orlicz-Lorentz spaces, Theorem 2 and Theorem 3, which generalize the corresponding results of Rakotondratsimba [24]. Proposition 2 gives another characterization under the reverse doubling condition for certain weights. In Section 3 Theorem 4 displays the characterization of the modular inequalities for the fractional integral operator on the Orlicz-Lorentz spaces and Theorem 5 yields a sufficient condition for the fractional integral operator to be bounded on the Orlicz-Lorentz spaces, which fills the gap in this aspect. The methods are also applicable to the singular integral operator.

In the sequel, we will always denote $W(t) = \int_0^x w(t)dt$ if w is a weight on $\mathbb{R}_+$ and $C,\ C_1,\cdots$ denote positive constants but need not be the same at different occurrences.

2. Fractional maximal operator on Orlicz-Lorentz spaces

We shall need an atomic decomposition of dyadic tent spaces associated to the Orlicz-Lorentz spaces whose idea is similar to Soria [29]. Let X be the set $(0,\infty)^n$ minus the dyadic points $z = (z_i)_i \in 2^k\mathbb{Z}^n$, σ be a locally finite positive measure on X, $\tilde{X} = X \times 2^{\mathbb{Z}}$ and a closed dyadic cube be a product of n intervals $[x_i, x_i + 2^k]$ with $x = (x_i)_i \in 2^k\mathbb{Z}^n$ for some $k \in \mathbb{Z}$. For each $x \in X$, we write

$$\tilde{\Gamma}(x) = \bigcup_{x\in Q(y,2^k)} \{(y,2^k)\},$$

where $Q[y,2^k]$ is the unique dyadic cube which contains y and with length 2^k. Also

$$\hat{\Omega} = \tilde{X} \backslash \big(\bigcup \{ \tilde{\Gamma}(x) | x \in \Omega^c \} \big)$$

for each set $\Omega \subset X$. Thus

$$(y,2^k) \in \hat{\Omega} \Leftrightarrow Q[y,2^k] \subset \Omega.$$

The functional $\mathscr{A}_\infty$ is given by

$$(\mathscr{A}_\infty \tilde{f})(x) = \sup\{ |\tilde{f}(y,2^k)| \, \big| (y,2^k) \in \tilde{\Gamma}(x) \}$$

where $\tilde{f}$ is a measurable function of $\tilde{X}$. Now, for each measurable function $\tilde{f}(y,2^k)$ in $\tilde{X}$ and a cube $Q[0,R]=(0,R)^n$, $R>0$, define that

$$\tilde{f}\in T_{d\sigma}^{\phi,dya}(w)[Q[0,R]]$$

if $\tilde{f}(y,2^k)$ is supported by $\widehat{(0,R)^n}$, and the set $\{(\mathscr{A}_\infty\tilde{f})(\cdot)>\lambda\}$, $\lambda>0$, is a union of dyadic cubes and $\int_0^\infty\phi\big((\mathscr{A}_\infty\tilde{f})_\sigma^*(t)\big)w(t)dt<\infty$.

The following lemma is an easy consequence of well-known dyadic cubes properties. It is the key of atomic decomposition of $\mathscr{T}_{d\sigma}^{\phi_1,dya}(w_1)[Q[0,R]]$ $(R>0)$.

LEMMA 1. *For each bounded open set* $\Omega=\bigcup_{Q\in\mathscr{I}}Q\subset X$ *(with* Q *being dyadic cubes) one can find a sequence of (maximal) dyadic cubes* $(Q_i)_i$ $(Q_i\in\mathscr{I})$ *with pairwise disjoint interiors and such that* $\Omega=\bigcup_i Q_i$ *and* $\hat{\Omega}=\bigcup_i\hat{Q}_i$.

In this section we always suppose that the Young functions are N-functions and the measures are locally finite. Furthermore, some restrictions on ϕ_1,ϕ_2,w_1 are done:

(i) $\phi_2\in\Delta'$, $\phi_1\circ\phi_2^{-1}\in\Delta'$;

(ii) there exists a constant $C>0$ such that

$$\phi_1\circ\phi_2^{-1}\Big(\sum_{i=1}^n x_i\Big)\leqslant C\sum_{i=1}^n\phi_1\circ\phi_2^{-1}(x_i);$$

(iii) $w_1\in B_{\phi_1}$, and

$$\sum_{i=1}^n\phi_2\circ\phi_1^{-1}(W_1(a_i))\leqslant C\phi_2\circ\phi_1^{-1}\Big(W_1\Big(\sum_{i=1}^n a_i\Big)\Big),$$

for all $a_i>0$, $n\in\mathbb{Z}^+$.

To prove Proposition 1 which contains all philosophy of weighted inequality for dyadic fractional maximal operator in Theorem 1, the atomic decomposition of $\mathscr{T}_{d\sigma}^{\phi_1,dya}(w_1)[Q[0,\ R]]$ $(R>0)$ is needed.

LEMMA 2. *Let the growth conditions (i)–(iii) be assumed. Then for every constant* $B_1>0$ *there is a constant* $B_2>0$ *such that for all* $\tilde{f}\in\mathscr{T}_{d\sigma}^{\phi_1,dya}(w_1)[Q[0,\ R]]$ $(R>0)$ *one can get* λ_j, *dyadic cubes* Q_j, *and functions* $\tilde{a}_j(y,2^k)$ *with disjoint supports such that*

$$|\tilde{a}_j(y,2^k)|\leqslant\frac{1}{\phi^{-1}(W_1(|Q_j|_\sigma))}\chi_{\hat{Q}_j}(y,2^j), \tag{2.1}$$

$$\tilde{f}(y,2^k)=\sum_j\lambda_j\tilde{a}_j(y,2^k)\ \ a.e., \tag{2.2}$$

and

$$\phi_2^{-1}\big(B_1\sum_j\phi_2(\lambda_j)\big)\leqslant\phi_1^{-1}\left[\int_0^\infty\phi_1(B_2(\mathscr{A}_\infty\tilde{f})_\sigma^*(t))w_1(t)dt\right]. \tag{2.3}$$

Proof. Take $\tilde{f} \in \mathscr{T}_{d\sigma}^{\phi_1,dya}(w_1)[Q[0,R]]$ $(R>0)$. For $j \in \mathbb{N}$, let $\Omega_j = \{x|(\mathscr{A}_\infty \tilde{f})(x) > 2^j\}$. Then

$$\begin{aligned}\infty &> \int_0^\infty \phi_1\left((\mathscr{A}_\infty \tilde{f})^*_\sigma(t)\right) w_1(t)dt = \int_0^\infty W_1(\lambda^\sigma_{\phi_1(\mathscr{A}_\infty \tilde{f})}(t))dt \\ &\geqslant \int_0^{\phi_1(2^j)} W_1(\lambda^\sigma_{\phi_1(\mathscr{A}_\infty \tilde{f})}(t))dt \geqslant \phi_1(2^j)W_1(|\Omega_j|_\sigma),\end{aligned}$$

which implies that

$$W_1(|\Omega_j|_\sigma) < \infty. \tag{2.4}$$

Furthermore,

$$\hat{\Omega}_{j+1} \subset \hat{\Omega}_j \tag{2.5}$$

$$\tilde{f}(y,2^k) \leqslant 2^{j+1} \text{ on } \hat{\Omega}^c_{j+1}. \tag{2.6}$$

Since $\Omega_j = \bigcup_{Q\in\mathscr{I}} Q \subset (0,R)^n$, then by Lemma 1

$$\Omega_j = \bigcup_i Q_{ij} \quad \text{and} \quad \sum_i \chi_{Q_{ij}} = \chi_{\Omega_j}, \tag{2.7}$$

$$\hat{\Omega}_j = \bigcup_i \hat{Q}_{ij} \quad \text{and} \quad \sum_i \chi_{\hat{Q}_{ij}} = \chi_{\hat{\Omega}_j}. \tag{2.8}$$

Now define

$$\lambda_{ij} = 2^{j+1}\phi_1^{-1}(W_1(|Q_{ij}|_\sigma)), \tag{2.9}$$

and

$$\tilde{a}_{ij}(y,2^k) = 2^{-(j+1)}\frac{1}{\phi_1^{-1}(W_1(|Q_{ij}|_\sigma))}\tilde{f}(y,2^k) \times \chi_{\hat{Q}_{ij}-\hat{\Omega}_{j+1}}(y,2^k), \tag{2.10}$$

which are well defined by (2.4) and $W_1(|Q_{ij}|_\sigma) > 0$ rooted in the condition $w_1 \in B_{\phi_1}$. Due to the supports of $\tilde{a}_{ij}$, $\tilde{E}_{ij} = \widehat{Q}_{ij} - \widehat{\Omega}_{j+1}$, almost disjoint,

$$|\tilde{a}_{ij}(y,2^k)| \leqslant \frac{1}{\phi_1^{-1}(W_1(|Q_{ij}|_\sigma))}\chi_{\widehat{Q}_{ij}}(y,2^k). \tag{2.11}$$

By the definition of λ_{ij} and $\tilde{a}_{ij}(y,2^k)$ we obtain

$$\tilde{f}(y,2^k) = \sum_{i,j}\lambda_{ij}\tilde{a}_{ij}(y,2^k), \quad a.e..$$

On the other hand, in view of $\phi_2 \in \Delta'$ and

$$\sum_i \phi_2 \circ \phi_1^{-1}(W_1(a_i)) \leqslant C\phi_2 \circ \phi_1^{-1}\Big(W_1\Big(\sum_i a_i\Big)\Big),$$

it follows that

$$\begin{aligned}\phi_2^{-1}\Big(B_1\sum_{i,j}\phi_2(\lambda_{ij}\Big) &\leqslant \phi_2^{-1}\Big[B_1\sum_{i,j}\phi_2(2^{j+1}\phi_1^{-1}(W_1(|Q_{ij}|_\sigma)))\Big]\\ &\leqslant \phi_2^{-1}\Big[C\sum_{i,j}\phi_2(2^{j+1})\phi_2\circ\phi_1^{-1}(W_1(|Q_{ij}|_\sigma))\Big]\\ &\leqslant \phi_2^{-1}\Big[C\sum_{j}\phi_2(2^{j+1})\phi_2\circ\phi_1^{-1}(W_1(\Omega_j|_\sigma))\Big]\\ &\leqslant \phi_2^{-1}\Big[C\sum_{j}\phi_2(2^{j-1})\phi_2\circ\phi_1^{-1}(W_1(|\mathscr{A}_\infty\tilde{f}>2^j|_\sigma))\Big].\end{aligned}$$

Moreover, by the conditions

$$\phi_1\circ\phi_2^{-1}\Big(\sum_{i=1}^n x_i\Big)\leqslant C\sum_{i=1}^n\phi_1\circ\phi_2^{-1}(x_i)$$

and $\phi_1\circ\phi_2^{-1}\in\Delta'$, there holds that

$$\begin{aligned}&\phi_2^{-1}\Big[C\sum_{j}\phi_2(2^{j-1})\phi_2\circ\phi_1^{-1}(W_1(|\mathscr{A}_\infty\tilde{f}>2^j|_\sigma))\Big]\\ &=\phi_1^{-1}\circ\phi_1\circ\phi_2^{-1}\Big[C\sum_{j}\phi_2(2^{j-1})\phi_2\circ\phi_1^{-1}(W_1(|\mathscr{A}_\infty\tilde{f}>2^j|_\sigma))\Big]\\ &\leqslant\phi_1^{-1}\Big(C\sum_{j}\phi_1\circ\phi_2^{-1}\Big[\phi_2(2^{j-1})\phi_2\circ\phi_1^{-1}(W_1(|\mathscr{A}_\infty\tilde{f}>2^j|_\sigma))\Big]\Big)\\ &\leqslant\phi_1^{-1}\Big(C\sum_{j}\phi_1(2^{j-1})W_1(|\mathscr{A}_\infty\tilde{f}>2^j|_\sigma)\Big).\\ &\leqslant\phi_1^{-1}\Big[C\sum_{j}\int_{\phi_1(2^{j-1})}^{\phi_1(2^j)}W_1(\lambda^\sigma_{\phi_1(\mathscr{A}_\infty\tilde{f})}(t))dt\Big]\\ &=\phi_1^{-1}\Big[C\int_0^\infty W_1(\lambda^\sigma_{\phi_1(\mathscr{A}_\infty\tilde{f})}(t))dt\Big]=\phi_1^{-1}\Big[C\int_0^\infty\phi_1((\mathscr{A}_\infty\tilde{f})^*_\sigma(t))w_1(t)dt\Big]\\ &\leqslant\phi_1^{-1}\Big[\int_0^\infty\phi_1(B_2(\mathscr{A}_\infty\tilde{f})^*_\sigma(t))w_1(t)dt\Big],\end{aligned}$$

which completes the proof. □

To get main results of the paper, we bring in the following two operators. The dyadic version of the maximal operator M_α is defined as

$$\mathscr{M}_\alpha f(x)=\sup_{x\in Q}\Big\{|Q|^{\frac{\alpha}{n}-1}\int_Q|f(y)|d\sigma(y)\Big|Q\ \text{ is a closed dyadic cube}\Big\}$$

and the operator $\mathscr{M}_\alpha^{Q[0,R]}$ is defined by

$$\begin{aligned}(\mathscr{M}_\alpha^{Q[0,R]}fd\sigma)(x)=\sup_{x\in Q}\Big\{&|Q|^{\frac{\alpha}{n}-1}\int_Q|f(y)|d\sigma(y)\\ &\Big|Q\text{ is a closed dyadic cube with }Q\subset Q[0,R]\Big\}.\end{aligned}$$

PROPOSITION 1. *Suppose there exists a constant* $C > 0$ *such that*

$$\phi_1^{-1}\left(\frac{1}{x}\right)\phi_1^{-1}(x) \geqslant C \tag{2.12}$$

for all $0 < x < \infty$ *besides the assumptions of (i)–(iii). Then there exists a constant* $C > 0$ *and for every constant* $B_1 > 0$ *there exists a constant* $B_2 > 0$ *such that for all* $f \in \Lambda_{d\sigma}^{\phi_1}(w_1)$ *and all* $R > 0$ *there exist* $\lambda_j > 0$, *dyadic cubes* Q_j *satisfying*

$$\phi_2(\mathscr{M}_\alpha^{Q[0,R]} f d\sigma)\chi_{Q[0,R]} \leqslant C\sum_j \phi_2(\lambda_j)\phi_2\circ\phi_1^{-1}\left(\frac{1}{W_1(|Q_j|_\sigma)}\right)\phi_2(M_\alpha\chi_{Q_j}d\sigma)\chi_{Q_j} \tag{2.13}$$

and

$$\phi_2^{-1}\left(B_1\sum_j\phi_2(\lambda_j)\right) \leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(B_2 f_\sigma^*(t)\right)w_1(t)dt\right). \tag{2.14}$$

Proof. Let $f \in \Lambda_{d\sigma}^{\phi_1}(w_1)$ and $R > 0$. For $x \in Q[0,R]$,

$$\begin{aligned}(\mathscr{M}_\alpha^{Q[0,R]} f d\sigma)(x) &= \sup\left\{|Q[y,2^k]|^{\frac{\alpha}{n}-1}\int_{Q[y,2^k]}|f(z)|d\sigma(z)\Big| x\in Q[y,2^k]\subset Q[0,R]\right\}\\ &= \sup\left\{\tilde{\Theta}(y,2^k)\tilde{f}(y,2^k)\big| x\in Q[y,2^k]\subset Q[0,R]\right\},\end{aligned}$$

where

$$\tilde{\Theta}(y,2^k) = |Q[y,2^k]|^{\frac{\alpha}{n}-1}\int_{Q[y,2^k]} d\sigma(z)$$

and

$$\tilde{f}(y,2^k) = \begin{cases} |Q[y,2^k]|_\sigma^{-1}\int_{Q[y,2^k]}|f(z)|d\sigma(z), & \text{if } [y,2^k]\in\widehat{Q[0,R]},\\ 0, & \text{else.}\end{cases}$$

Thus $\tilde{f} \in \mathscr{T}_{d\sigma}^{\phi_1,dya}(w_1)[Q[0,R]]$. Indeed,

$$\mathscr{A}_\infty \tilde{f} \leqslant N_\sigma f, \tag{2.15}$$

where $N_\sigma f(x) = \sup_{Q[y,2^k]\ni x}|Q[y,2^k]|_\sigma^{-1}\int_{Q[y,2^k]}|f(z)|d\sigma(z)$. On the other hand, the conditions $\phi_2 \in \Delta'$ and $\phi_1\circ\phi_2^{-1} \in \Delta'$ implies $\phi_1 \in \Delta_2$. So from (5.13) of [24]

$$(N_\sigma f)_\sigma^*(t)) \leqslant \frac{C_2}{t}\int_0^t f_\sigma^*(t),$$

$\phi_1 \in \Delta_2$, $w_1 \in B_{\phi_1}$ and (2.15), we know that

$$\begin{aligned}\int_0^\infty \phi_1((\mathscr{A}_\infty\tilde{f})_\sigma^*(t))w_1(t)dt &\leqslant \int_0^\infty \phi_1((N_\sigma f)_\sigma^*(t))w_1(t)dt\\ &\leqslant \int_0^\infty \phi_1\left(C_2\frac{1}{t}\int_0^t f_\sigma^*(t)dt\right)w_1(t)dt\\ &\leqslant \int_0^\infty \phi_1(C_2C_3 f_\sigma^*(t))w_1(t)dt < \infty.\end{aligned} \tag{2.16}$$

Furthermore, it is obvious that the support of $\mathscr{A}_\infty\tilde{f}$ is contained in $\widehat{Q[0,R]}$ and the set $\{\mathscr{A}_\infty\tilde{f}(\cdot)>\lambda\}$ is a union of dyadic cubes. So the above assertion is true. Consequently by the condition $\phi_2\in\Delta'$, (2.2) and (2.1) of Lemma 2, the definition of $\tilde{\Theta}$ and (2.12), it follows that

$$\begin{aligned}
\phi_2(\mathscr{M}_\alpha^{Q[0,R]}fd\sigma)\chi_{Q[0,R]} &\leqslant \phi_2(\tilde{\Theta}(y,2^k)\tilde{f}(y,2^k)) \leqslant C\phi_2(\tilde{\Theta}(y,2^k))\phi_2(\tilde{f}(y,2^k))\\
&\leqslant C\phi_2(\tilde{\Theta}(y,2^k))\sum_j\phi_2(\lambda_j)\phi_2(\tilde{a}_j(y,2^k))\\
&\leqslant C\sum_j\phi_2(\lambda_j)\phi_2\Big(\frac{1}{\phi_1^{-1}(W_1(|Q_j|_\sigma))}\Big)\phi_2(\tilde{\Theta}(y,2^k))\tilde{\chi}_{\hat{Q}_j}(y,2^j)\\
&\leqslant C\sum_j\phi_2(\lambda_j)\phi_2\Big(\frac{1}{\phi_1^{-1}(W_1(|Q_j|_\sigma))}\Big)\phi_2(\mathscr{M}_\alpha\chi_{Q_j}d\sigma)\tilde{\chi}_{\hat{Q}_j}(y,2^j)\\
&\leqslant C\sum_j\phi_2(\lambda_j)\phi_2\circ\phi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\phi_2(\mathscr{M}_\alpha\chi_{Q_j}d\sigma)\tilde{\chi}_{\hat{Q}_j}(y,2^j).
\end{aligned}$$

On the other hand, by (2.3) and (2.16), we obtain

$$\begin{aligned}
\phi_2^{-1}\Big(B_1\sum_j\phi_2(\lambda_j)\Big) &\leqslant \phi_1^{-1}\Big[\int_0^\infty\phi_1(C(\mathscr{A}_\infty\tilde{f})^*_\sigma(t))w_1(t)dt\Big]\\
&\leqslant \phi_1^{-1}\Big[\int_0^\infty\phi_1(B_2f^*_\sigma(t))w_1(t)dt\Big]. \quad \square
\end{aligned}$$

One of the main results in this section, Theorem 2, is based on the characterization of the weighted modular inequalities for the dyadic version $\mathscr{M}_\alpha$ of the maximal operator M_α, which is the following:

THEOREM 1. *(a) Suppose there is a constant $C>0$ such that*

$$\phi_2^{-1}\left(\int_0^\infty\phi_2\left((\mathscr{M}_\alpha fd\sigma)^*_\omega(t)\right)w_2(t)dt\right)\leqslant\phi_1^{-1}\left(\int_0^\infty\phi_1\left(Cf^*_\sigma(t)\right)w_1(t)dt\right) \tag{2.17}$$

for all $f\in\Lambda^{\phi_1}_{d\sigma}(w_1)$, then there exists a constant $A>0$ such that for all dyadic cubes Q

$$\phi_2^{-1}\left(\int_0^\infty\phi_2\left(((\mathscr{M}_\alpha\chi_Qd\sigma)\chi_Q)^*_\omega(t)\right)w_2(t)dt\right)\leqslant\phi_1^{-1}\left(\int_0^\infty\phi_1\left(A(\chi_Q)^*_\sigma(t)\right)w_1(t)dt\right). \tag{2.18}$$

(b) For the converse, assume $w_2\in B_1$ besides the conditions (i)–(iii) and (2.12). So the test conditions (2.18) implies (2.17).

Proof. The part (a) is evident. To prove Part (b) of Theorem 1, by translation and reflection, it is sufficient to verify that there exists a constant $C>0$ such that

$$\phi_2^{-1}\left(\int_0^\infty\phi_2\left((\mathscr{M}_\alpha^{Q[0,R]}fd\sigma\chi_{Q[0,R]})^*_\omega(t)\right)w_2(t)dt\right)\leqslant\phi_1^{-1}\left(\int_0^\infty\phi_1\left(Cf^*_\sigma(t)\right)w_1(t)dt\right). \tag{2.19}$$

Let $f_\mu^{**}(t) = \frac{1}{t}\int_0^t f_\mu^*(s)ds,\ t>0$. By (2.13), $w_2 \in B_1$, $\phi_1 \circ \phi_2^{-1} \in \Delta'$ which implies $\phi_2 \circ \phi_1^{-1} \in \nabla'$, (2.18) and (2.14), (2.19) appears since

$$\begin{aligned}
&\phi_2^{-1}\left(\int_0^\infty \phi_2\left((\mathscr{M}_\alpha^{Q[0,R]} f d\sigma \chi_{Q[0,R]})_\omega^*(t)\right) w_2(t)dt\right)\\
&= \phi_2^{-1}\left(\int_0^\infty \left(\phi_2(\mathscr{M}_\alpha^{Q[0,R]} f d\sigma \chi_{Q[0,R]})\right)_\omega^*(t) w_2(t)dt\right)\\
&\leqslant \phi_2^{-1}\left(\int_0^\infty \left(C\sum_j \phi_2(\lambda_j)\phi_2\circ\phi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\phi_2(M_\alpha \chi_Q d\sigma)\chi_{Q_j}\right)_\omega^*(t) w_2(t)dt\right)\\
&\leqslant \phi_2^{-1}\left(\int_0^\infty \left(C\sum_j \phi_2(\lambda_j)\phi_2\circ\phi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\phi_2(M_\alpha \chi_Q d\sigma)\chi_{Q_j}\right)_\omega^{**}(t) w_2(t)dt\right)\\
&\leqslant \phi_2^{-1}\left(C\sum_j \phi_2(\lambda_j)\phi_2\circ\phi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\int_0^\infty (\phi_2(M_\alpha \chi_Q d\sigma)\chi_{Q_j})_\omega^*(t) w_2(t)dt\right)\\
&\leqslant \phi_2^{-1}\left(C\sum_j \phi_2(\lambda_j)\phi_2\circ\phi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\phi_2\circ\phi_1^{-1}\left(\int_0^\infty \phi_1\left(A\left(\chi_{Q_j}\right)_\sigma^*\right)(t) w_1(t)dt\right)\right)\\
&= \phi_2^{-1}\left(C\phi_2(A)\sum_j \phi_2(\lambda_j)\right) \leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(Cf_\sigma^*(t)\right) w_1(t)dt\right). \quad \square
\end{aligned}$$

For the fractional maximal operator on Orlicz-Lorentz spaces, we have the following results.

THEOREM 2. *(a) If there is a constant $C_1>0$ such that*

$$\phi_2^{-1}\left(\int_0^\infty \phi_2\left((M_\alpha f d\sigma)_u^*(t)\right) w_2(t)dt\right) \leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(C_1 f_\sigma^*(t)\right) w_1(t)dt\right) \tag{2.20}$$

for all $f \in \Lambda_{d\sigma}^{\phi_1}(w_1)$, then there exists a constant $C_2>0$ such that for all cubes Q

$$\phi_2^{-1}\left(\int_0^\infty \phi_2\left(((M_\alpha \chi_Q d\sigma)\chi_Q)_u^*(t)\right) w_2(t)dt\right) \leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(C_2(\chi_Q)_\sigma^*(t)\right) w_1(t)dt\right). \tag{2.21}$$

(b) On the contrary, assume $w_2 \in B_1 \cap B_{\phi_2}$ besides the conditions (i)–(iii) and (2.12). Then the test condition (2.21) implies (2.20).

Proof. We only need to prove the part (b). Due to the monotone convergence theorem, it suffices that we can find a constant $C>0$ such that

$$\phi_2^{-1}\left(\int_0^\infty \phi_2\left((M_\alpha^{2^N} f d\sigma)_u^*(t)\right) w_2(t)dt\right) \leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(Cf_\sigma^*(t)\right) w_1(t)dt\right), \tag{2.22}$$

where truncated maximal function M^R is defined by

$$M^R_\alpha f(x) = \sup_{x\in Q}\left\{|Q|^{\frac{\alpha}{n}-1}\int_Q |f(y)|dy\,\middle|\,|Q|^{\frac{1}{n}}\leqslant R\right\}.$$

To get (2.22), the first point is Lemma 2 of [26]

$$(M^{2^N}_\alpha g)(x)\leqslant C_1\int_{[-2^{N+2},2^{N+2}]^n}({}^zM_\alpha g)(x)\frac{dz}{2^{n(N+3)}}, \tag{2.23}$$

where $C_1>0$ does not depend on x,z and $N\in\mathbb{N}$, and ${}^zM_\alpha$ is defined as

$$({}^zM_\alpha f)(x) = \sup_{x\in Q}\left\{|Q|^{\frac{\alpha}{n}-1}\int_Q |f(y)|dy\,\middle|\,Q-z \text{ a closed dyadic cube}\right\}.$$

The second point is

$$\phi_2^{-1}\left(\int_0^\infty \phi_2\left(({}^zM_\alpha f d\sigma)^*_u(t)\right)w_2(t)dt\right)\leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(Cf^*_\sigma(t)\right)w_1(t)dt\right) \tag{2.24}$$

for all $f\geqslant 0$, and all $z\in\mathbb{R}^n$.

For the time being we assume (2.24). Then by (2.23)

$$\begin{aligned}
&\phi_2^{-1}\left(\int_0^\infty \phi_2\left((M^{2^N}_\alpha f d\sigma)^*_u(t)\right)w_2(t)dt\right)\\
&\leqslant \phi_2^{-1}\left(\int_0^\infty \phi_2\left(\left(C_1\int_{[-2^{N+2},2^{N+2}]^n}({}^zM_\alpha f)(\cdot)\frac{dz}{2^{n(N+3)}}\right)^*_u(t)\right)w_2(t)dt\right)\\
&\leqslant \phi_2^{-1}\left(\int_0^\infty \phi_2\left(\left(C_1\int_{[-2^{N+2},2^{N+2}]^n}({}^zM_\alpha f)(\cdot)\frac{dz}{2^{n(N+3)}}\right)^{**}_u(t)\right)w_2(t)dt\right)\\
&\leqslant \phi_2^{-1}\left(\int_0^\infty \phi_2\left(C_1\int_{[-2^{N+2},2^{N+2}]^n}({}^zM_\alpha f)^{**}_u(t)\frac{dz}{2^{n(N+3)}}\right)w_2(t)dt\right).
\end{aligned}$$

By Jensen's inequality, the assumption $w_2\in B_{\phi_2}$ and (2.24), it follows that the right hand side of the last inequality is

$$\begin{aligned}
&\leqslant \phi_2^{-1}\left(\int_0^\infty C_1\int_{[-2^{N+2},2^{N+2}]^n}\phi_2\left(({}^zM_\alpha f)^{**}_u(t)\right)\frac{dz}{2^{n(N+3)}}w_2(t)dt\right)\\
&\leqslant \phi_2^{-1}\left(\int_{[-2^{N+2},2^{N+2}]^n}\frac{dz}{2^{n(N+3)}}\int_0^\infty \phi_2\left(C({}^zM_\alpha f)^*_u(t)\right)w_2(t)dt\right)\\
&\leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(Cf^*_\sigma(t)\right)w_1(t)dt\right).
\end{aligned}$$

This completes the proof of (2.22).

Now we begin to prove (2.24). Due to the following equality:

$$\int_{\{{}^zM_\alpha f d\sigma>\lambda\}}u(y)dy = \int_{\{\mathscr{M}_\alpha f(\cdot+z)d\sigma(\cdot+z)>\lambda\}}u(y+z)dy$$

we know that

$$\phi_2^{-1}\left(\int_0^\infty \phi_2\left(({}^zM_\alpha f d\sigma)_u^*(t)\right) w_2(t)dt\right)$$
$$= \phi_2^{-1}\left(\int_0^\infty \phi_2\left((\mathscr{M}_\alpha f(\cdot+z)d\sigma_z)_{u(\cdot+z)}^*(t)\right) w_2(t)dt\right). \tag{2.25}$$

On the other hand, by (2.21) we get that

$$\phi_2^{-1}\left(\int_0^\infty \phi_2\left(((\mathscr{M}_\alpha d\sigma_z\chi_Q)\chi_Q)_{u(\cdot+z)}^*(t)\right) w_2(t)dt\right)$$
$$\leqslant \phi_2^{-1}\left(\int_0^\infty \phi_2\left(((M_\alpha d\sigma_z\chi_Q)\chi_Q)_{u(\cdot+z)}^*(t)\right) w_2(t)dt\right)$$
$$= \phi_2^{-1}\left(\int_0^\infty \phi_2\left(((M_\alpha d\sigma\chi_{Q+z})\chi_{Q+z})_u^*(t)\right) w_2(t)dt\right)$$
$$\leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(A(\chi_{Q+z})_\sigma^*(t)\right) w_1(t)dt\right)$$
$$\leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(A(\chi_Q)_{d\sigma_z}^*(t)\right) w_1(t)dt\right) \text{ with } d_{\sigma_z}(\cdot)=\sigma(\cdot+z). \tag{2.26}$$

So by (2.25), (2.26) and Theorem 2.4 we obtain that

$$\phi_2^{-1}\left(\int_0^\infty \phi_2\left(({}^zM_\alpha f d\sigma)_u^*(t)\right) w_2(t)dt\right)$$
$$\leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(A(f(\cdot+z))_{\sigma_z}^*(t)\right) w_1(t)dt\right) = \phi_1^{-1}\left(\int_0^\infty \phi_1\left(Af_\sigma^*(t)\right) w_1(t)dt\right). \quad \square$$

REMARK 1. Let $0<p_2\leqslant p_1<\infty$, $\phi_1(t)=t^{p_1}$ and $\phi_2(t)=t^{p_2}$. Then $B_1\cap B_{\phi_2}=B_1$ and (i), (ii), (2.12) are naturally true. Furthermore, Section 4 of [5] proves that (iii) can be established for some kinds of weights w_1. So Theorem 2 is generalization of Theorem 2.1 of [24].

Let $\sigma\in RD_\rho$ $(\rho>0)$ indicate that there is $c>0$ such that

$$\frac{\int_{Q_1} w(y)dy}{\int_Q w(y)dy}\leqslant c\left(\frac{|Q_1|}{|Q|}\right)^\rho$$

for all cubes Q_1 and Q with $Q_1\subset Q$. Observing that

$$(M_\alpha\chi_Q d\sigma)\chi_Q\leqslant C|Q|^{\frac{\alpha}{n}-1}\int_Q \sigma(y)dy$$

for all cubes (see (3.1) of [24]), we get

PROPOSITION 2. *Suppose* $\sigma\in RD_\rho$ *with* $1-\frac{\alpha}{n}\leqslant\rho$. *Then condition (2.21) in Theorem 2.5 can be substituted by*

$$\phi_2^{-1}\left(\phi_2\left(|Q|^{\frac{\alpha}{n}-1}\int_Q\sigma(y)dy\right)W_2(|Q|_u)\right)\leqslant\phi_1^{-1}(CW_1(|Q|_\sigma))$$

for all cubes Q.

The next theorem generalizes Corollary 2.7 of [24] by using Orlicz-Lorentz spaces replacing classical Lorentz spaces.

THEOREM 3. *(a) Suppose there is a constant $C>0$ such that*

$$\phi_2^{-1}\left(\int_0^\infty \phi_2\left((M_\alpha f)_u^*(t)\right)w_2(t)dt\right)\leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(Cf_v^*(t)\right)w_1(t)dt\right) \tag{2.27}$$

for all $f\in \Lambda_{d\sigma}^{\phi_1}(w_1)$. Then there exists a constant $A>0$ such that for all cubes

$$\phi_2^{-1}\left(\phi_2(|Q|^{\frac{\alpha}{n}})W_2(|Q|_u)\right)\leqslant A\phi_1^{-1}(W_1(|Q|_v)). \tag{2.28}$$

(b) For the converse, suppose $v\in A_t$ $(t\geqslant 1)$, and ψ_1,ψ_2 are N-functions with $\psi_1=\phi_1\circ G$, $\psi_2=\phi_2\circ G$ where $G(s)=s^{1/t}$ besides the assumptions (i)–(iii), (2.12) and $w_2\in B_1\cap B_{\phi_2}$. Then (2.28) implies (2.27).

Proof. The part (a) is obvious. We only prove the part (b). In view of $v\in A_t$, we get by using the characterization of A_t

$$M_\alpha f\leqslant C(N_{\alpha t,v}f^t)^{\frac{1}{t}},$$

where

$$N_{\lambda,v}f(x)=\sup_{x\in Q}\left\{|Q|^{\frac{\alpha}{n}}\left(\int_Q v(z)dz\right)^{-1}\int_Q|f(y)|v(y)dy\,\middle|\,Q \text{ is a cube}\right\}.$$

Now in the light of $G(s)=s^{1/t}$, $\psi_1=\phi_1\circ G$, $\psi_2=\phi_2\circ G$, the proof of

$$\phi_2^{-1}\left(\int_0^\infty \phi_2\left((M_\alpha f)_u^*(t)\right)w_2(t)dt\right)\leqslant \phi_1^{-1}\left(\int_0^\infty \phi_1\left(Cf_v^*(t)\right)w_1(t)dt\right)$$

is reduced to

$$\psi_2^{-1}\left(\int_0^\infty \psi_2\left((N_{\alpha t,v}g)_u^*(t)\right)w_2(t)dt\right)\leqslant \psi_1^{-1}\left(\int_0^\infty \psi_1\left(Cg_v^*(t)\right)w_1(t)dt\right)\ \text{ for all } g\geqslant 0.$$

As in Theorem 2, to obtain the last inequality, the idea is also to prove the corresponding dyadic version

$$\psi_2^{-1}\left(\int_0^\infty \psi_2\left((\mathcal{N}_{\alpha t,v}g)_u^*(t)\right)w_2(t)dt\right)\leqslant \psi_1^{-1}\left(\int_0^\infty \psi_1\left(Cg_v^*(t)\right)w_1(t)dt\right)\ \text{ for all } g\geqslant 0.$$

The above modular inequality is based on

$$\psi_2(\mathcal{N}_{\alpha t,v}^{Q[0,R]}g)(\cdot)\chi_{Q[0,R]}(\cdot)\leqslant C\sum_j\psi_2(\lambda_j)\psi_2\circ\psi_1^{-1}\left(\frac{1}{W_1(|Q_j|_\sigma)}\right)\psi_2(|Q_j|^{\frac{\alpha t}{n}})\chi_{Q_j}, \tag{2.29}$$

and

$$\psi_2^{-1}\big(\sum_j \psi_2(\lambda_j)\big) \leqslant \psi_1^{-1}\left(\int_0^\infty \psi_1\left(Cg_v^*(t)\right) w_1(t)dt\right). \tag{2.30}$$

Here $\lambda_j > 0$ and the Q_j's are dyadic cubes and $\mathscr{N}_{\lambda,v}^{Q[0,R]}$ is the maximal operator defined as $\mathscr{N}_{\lambda,v}$ by means of dyadic cubes $Q \subset Q[0,R] = (0,R)^n$. (2.29) and (2.30) can be obtained by atomic decomposition of a suitable tent space as in the proof of Proposition 2.3. Now, we have by (2.29) and the condition $w_2 \in B_1$

$$\begin{aligned}
&\psi_2^{-1}\left(\int_0^\infty \psi_2\left(((N_{\alpha t,v}g)\chi_{Q_j})_u^*(t)\right) w_2(t)dt\right) \\
&= \psi_2^{-1}\left(\int_0^\infty \left(\psi_2((N_{\alpha t,v}g)\chi_{Q_j})\right)_u^*(t) w_2(t)dt\right) \\
&\leqslant \psi_2^{-1}\left(\int_0^\infty C\left(\sum_j \psi_2(\lambda_j)\psi_2\circ\psi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\psi_2(|Q_j|^{\frac{\alpha t}{n}})\chi_{Q_j}\right)_u^*(t)w_2(t)dt\right) \\
&\leqslant \psi_2^{-1}\left(\int_0^\infty C\left(\sum_j \psi_2(\lambda_j)\psi_2\circ\psi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\psi_2(|Q_j|^{\frac{\alpha t}{n}})\chi_{Q_j}\right)_u^{**}(t)w_2(t)dt\right) \\
&\leqslant \psi_2^{-1}\left(C\sum_j \psi_2(\lambda_j)\psi_2\circ\psi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\psi_2(|Q_j|^{\frac{\alpha t}{n}})\int_0^\infty \left(\chi_{Q_j}\right)_u^*(t)w_2(t)dt\right) \\
&= \psi_2^{-1}\left(C\sum_j \psi_2(\lambda_j)\psi_2\circ\psi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\psi_2(|Q_j|^{\frac{\alpha t}{n}})W_2(|Q_j|_u)\right).
\end{aligned} \tag{2.31}$$

Furthermore, (2.28) implies

$$\phi_2(|Q|^{\frac{\alpha}{n}})W_2(|Q|_u) \leqslant A\phi_2\circ\phi_1^{-1}(W_1(|Q|_v))$$

for all cubes. Thus by the condition $\phi_2\circ\phi_1^{-1} \in \nabla'$ we get

$$\begin{aligned}
&\psi_2\circ\psi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\psi_2(|Q_j|^{\frac{\alpha t}{n}})W_2(|Q_j|_u) \\
&= \phi_2\circ\phi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\phi_2(|Q_j|^{\frac{\alpha}{n}})W_2(|Q_j|_u) \\
&\leqslant A'\phi_2\circ\phi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\phi_2\circ\phi_1^{-1}(W_1(|Q|_v)) \leqslant A''\phi_2\circ\phi_1^{-1}(1).
\end{aligned} \tag{2.32}$$

(2.32), together with (2.30), leads to

$$\begin{aligned}
&\psi_2^{-1}\left(C\sum_j \psi_2(\lambda_j)\psi_2\circ\psi_1^{-1}\Big(\frac{1}{W_1(|Q_j|_\sigma)}\Big)\psi_2(|Q_j|^{\frac{\alpha t}{n}})W_2(|Q_j|_u)\right) \\
&\leqslant \psi_1^{-1}\left(\int_0^\infty \psi_1\left(Cg_v^*(t)\right) w_1(t)dt\right),
\end{aligned}$$

which combined with (2.31) completes the proof. □

Some papers, e.g. [3], [8] and [11], gave the comparison between norm and modular inequalities: norm inequalities are weaker than modular inequalities. So we get a sufficient condition for M_α to be bounded on Orlicz-Lorentz spaces.

COROLLARY 1. *Suppose* $v \in A_t$ $(t \geqslant 1)$, *and* ψ_1, ψ_2 *are* N*-functions with* $\psi_1 = \phi_1 \circ G, \psi_2 = \phi_2 \circ G$ *where* $G(s) = s^{1/t}$ *besides the assumptions (i)–(iii), (2.12) and* $w_2 \in B_1 \cap B_{\phi_2}$. *Then (2.28) implies that* M_α *is bounded from* $\Lambda_v^{\phi_1}(w_1)$ *into* $\Lambda_u^{\phi_2}(w_2)$.

3. Fractional integral operator on Orlicz-Lorentz spaces

Let ϕ be an N-function, w_1, w_2 two weights on $(0,\infty)$. Consider the characterization of the following modular inequality

$$\int_0^\infty \phi\left((I_\alpha f)^*(t)\right) w_2(t)dt \leqslant \int_0^\infty \phi\left(Cf^*(t)\right) w_1(t)dt. \tag{3.1}$$

In the light of the evaluation (see [N1] or [Sa2])

$$(I_\alpha f)^*(t) \leqslant C\left(t^{-1+\alpha/n}\int_0^t f^*(\tau)dt + \int_t^\infty f^*(\tau)\tau^{-1+\alpha/n}d\tau\right) \leqslant (I_\alpha \tilde{f})^*(t) \tag{3.2}$$

for all $0 < t < \infty$, where $\tilde{f}(x) = f(A|x|^n)$ with A the volume of the n-dimensional unit ball.

Due to the evaluation (3.2), (3.1) is equivalent to

$$\int_0^\infty \phi\left(t^{-1+\alpha/n}\int_0^t f^*(\tau)dt + \int_t^\infty f^*(\tau)\tau^{-1+\alpha/n}d\tau\right) w_2(t)dt \leqslant \int_0^\infty \phi\left(Cf^*(t)\right) w_1(t)dt. \tag{3.3}$$

If $\phi \in \Delta_2$ which implies that $\phi(x+y) \leqslant C(\phi(x)+\phi(y))$, $0 < x,\ y < \infty$, then it is easy to find that (3.3) is equivalent to

$$\int_0^\infty \phi\left(t^{-1+\alpha/n}\int_0^t f^*(\tau)dt\right) w_2(t)dt \leqslant \int_0^\infty \phi\left(C_1 f^*(t)\right) w_1(t)dt \tag{3.4}$$

and

$$\int_0^\infty \phi\left(\int_t^\infty f^*(\tau)\tau^{-1+\alpha/n}d\tau\right) w_2(t)dt \leqslant \int_0^\infty \phi\left(C_2 f^*(t)\right) w_1(t)dt. \tag{3.5}$$

To obtain the characterization of (3.4) and (3.5), we introduce the following concept. A strictly increasing positive sequence $\{x_j\}_{j\in\mathbb{Z}}$ is called a covering sequence if the sequence is of the form $\{x_j\}_{j=-\infty}^{j=\infty}$ or of the form $\{x_j\}_{j=N}^{j=M}$, where M and/ or N is finite. In the latter case we define $x_{M+1} = \infty$ and/ or $x_{N-1} = 0$. Thanks to Corollaries 3.5–3.6 of [13], we obtain

LEMMA 3. *Let ϕ be an N-function, $\phi,\ \tilde{\phi}\in\Delta_2$. Then (3.4) holds if and only if there exists a constant $B>0$ such that for all decreasing sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and the covering sequence $\{x_j\}$ satisfying $W_1(x_j)=2^j$,*

$$\sum_j\int_{x_j}^{x_{j+1}}\phi\left[\frac{\varepsilon_j}{B}x^{\alpha/n}\right]w_2(x)dx\leqslant\sum_j\phi(\varepsilon_j)\int_{x_j}^{x_{j+1}}w_1(y)dy \tag{3.6}$$

holds, and

$$\sum_j\int_{x_j}^{x_{j+1}}\phi\left[\frac{x^{-1+\alpha/n}}{B}\left\|\frac{i\chi_{(x_{j-1},x_j)}}{\varepsilon_jW_1}\right\|_{L^{\tilde{\phi}}(\varepsilon_jw_1)}\right]w_2(x)dx\leqslant\sum_j\frac{1}{\varepsilon_j} \tag{3.7}$$

is satisfied for all positive sequences $\{\varepsilon_j\}$ and all covering sequences $\{x_j\}$ where $i(x)=x$.

LEMMA 4. *Let ϕ be an N-function, $\phi,\ \tilde{\phi}\in\Delta_2$. Then (3.5) holds if and only if there exists a constant $B>0$ such that*

$$\sum_j\int_{x_{j-1}}^{x_j}\phi\left[\frac{1}{B}\left\|\frac{k(\cdot,x_j)\chi_{(x_j,x_{j+1})}}{\varepsilon_jW_1}\right\|_{L^{\tilde{\phi}}(\varepsilon_jw_1)}\right]w_2(x)dx\leqslant\sum_j\frac{1}{\varepsilon_j} \tag{3.8}$$

and

$$\sum_j\int_{x_{j-1}}^{x_j}\phi\left[\frac{k(x_j,x)}{B}\left\|\frac{\chi_{(x_j,x_{j+1})}}{\varepsilon_jW_1}\right\|_{L^{\tilde{\phi}}(\varepsilon_jw_1)}\right]w_2(x)dx\leqslant\sum_j\frac{1}{\varepsilon_j} \tag{3.9}$$

holds for all positive sequences $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ and all covering sequences $\{x_j\}_{j\in\mathbb{Z}}$. Here $k(y,x)=y^{\alpha/n}-x^{\alpha/n}$.

Combining (3.3) with Lemmas 3-4 induces the next result.

THEOREM 4. *Let ϕ be an N-function, $\phi,\ \tilde{\phi}\in\Delta_2$. Then (3.1) holds if and only if (3.6)–(3.9) hold.*

As we pointed out that norm inequalities are weaker than modular inequalities, the following result holds.

THEOREM 5. *Let ϕ be an N-function, $\phi,\ \tilde{\phi}\in\Delta_2$. If (3.6)–(3.9) hold, then I_α is bounded from $\Lambda^\phi(w_1)$ into $\Lambda^\phi(w_2)$.*

Proof. By Lemmas 3–4, we obtain that the modular inequalities (3.4) and (3.5) hold which imply that

$$\left\|t^{-1+\alpha/n}\int_0^t f^*(\tau)d\tau\right\|_{L^\phi(w_2)}\leqslant C\|f^*\|_{L^\phi(w_1)}, \tag{3.10}$$

and

$$\left\|\int_t^\infty f^*(\tau)\tau^{-1+\alpha/n}d\tau\right\|_{L^\phi(w_2)}\leqslant C\|f^*\|_{L^\phi(w_1)}. \tag{3.11}$$

Thus by (3.10), (3.11)

$$\begin{aligned}\|I_\alpha f\|_{\Lambda^\phi(w_2)} &\leqslant C\Big\|t^{-1+\alpha/n}\int_0^t f^*(\tau)dt+\int_t^\infty f^*(\tau)\tau^{-1+\alpha/n}d\tau\Big\|_{L^\phi(w_2)}\\ &\leqslant C\Big\|t^{-1+\alpha/n}\int_0^t f^*(\tau)dt\Big\|_{L^\phi(w_2)}+C\Big\|\int_t^\infty f^*(\tau)\tau^{-1+\alpha/n}d\tau\Big\|_{L^\phi(w_2)}\\ &\leqslant C\|f^*\|_{L^\phi(w_1)}=C\|f\|_{\Lambda^\phi(w_1)}.\quad\square\end{aligned}$$

The methods in this section also apply to the singular integral operator

$$Sf(x)=\lim_{\varepsilon\to 0^+}\int_{|y|\geqslant\varepsilon}\frac{K(y)}{|y|^n}f(x-y)dy,\ x\in\mathbb{R}^n,$$

where K is an odd function on $\mathbb{R}^n$ which is homogeneous of degree 0 and satisfies Dini-type condition on the unit sphere $\mathbf{S}^{n-1}$ of $\mathbb{R}^n$:

$$\int_0^1\frac{\omega(\delta)}{\delta}d\delta<\infty,\quad\text{with } \omega(\delta)=\sup_{x,y\in\mathbf{S}^{n-1},|x-y|\leqslant\delta}|K(x)-K(y)|,$$

since

$$(Sf)^*(t)\leqslant C\left(\frac{1}{t}\int_0^t f^*(\tau)dt+\int_t^\infty\frac{f^*(\tau)}{\tau}d\tau\right)$$

for all $0<t<\infty$ (see Theorem 16.12 of [2]).

Acknowledgements. This work is supported by National Natural Science Foundation of China (11401530, 11461033), Natural Science Foundation of Zhejiang Province of China (LQ13A010018). The author would like to express thanks to the referees for valuable suggestions which helped to improve the presentation of the paper.

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(Received June 17, 2014)

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