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NEW MULTIPLICATIVE HIGHER ORDER DYNAMIC INEQUALITIES OF OPIAL TYPE

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Abstract. In this paper, we prove some new multiplicative dynamic inequalities of Opial type on a time scale $\mathbb{T}$. The main results will be proved by using Hölder's inequality, the chain rule and some basic dynamic inequalities designed and proved for this purpose. As special cases, we will derive some continuous and discrete inequalities from the main results.

1. Introduction

In [6] Bohner and Kaymakçalan proved some dynamic inequalities of Opial type on time scales. In particular, they proved that if $y:[0,h]\cap\mathbb{T}\longrightarrow\mathbb{R}$ is delta differentiable with $y(0)=0$, then

$$\int_0^h |y(t)+y^\sigma(t)|\left|y^\triangle(t)\right|\triangle t\leqslant h\int_0^h\left|y^\triangle(t)\right|^2\triangle t. \tag{1.1}$$

For extensions and generalizations of (1.1), we refer the reader to the monograph [2] and the recent papers [7, 8] and the references cited therein. There are a few inequalities involving higher order derivatives established in the literature [3, 11, 15, 17]. In the following, we recall some of these results that serve and motivate the contents of this paper. In [17], the authors proved that if $y:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is delta differentiable n times with $y^{\Delta^i}(a)=0$, for $i=0,1,\ldots,n-1$, and f is a positive rd-continuous function on $[a,b]_{\mathbb{T}}$, then

$$\int_a^b f(t)\,|y(t)|^p\left|y^{\Delta^n}(t)\right|^q\Delta t\leqslant\left(\frac{q}{p+q}\right)(b-a)^{np}\int_a^b f(t)\left|y^{\Delta^n}(t)\right|^{p+q}\Delta t. \tag{1.2}$$

In [3], the author proved that if $y:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is delta differentiable n times (n odd) with $y^{\Delta^i}(a)=0$, for $i=0,1,\ldots,n-1$, then

$$\int_a^b|y(t)|\,\Delta t\leqslant\left(\int_a^b\left(\int_a^t h_{n-1}^p(t,\sigma(s))\Delta s\right)^{\frac{q}{p}}\Delta t\right)\int_a^b\left|y^{\Delta^n}(t)\right|^q\Delta t,$$

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where $p,\ q>1$ and satisfy $1/p+1/q=1$ and $h_n(t,s)$ are the Taylor monomials which will be defined later in Section 2. Also in [3] it is proved that if $y:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is delta differentiable n times with $y^{\Delta^i}(a)=0$, for $i=0,1,\ldots,n-1$, and $\left|y^{\Delta^n}(t)\right|$ is increasing, then

$$\begin{aligned}&\int_a^b |y(t)|\left|y^{\Delta^n}(t)\right|\Delta t\\ &\leqslant (b-a)^{\frac{1}{q}}\left(\int_a^b\left(\int_a^t h_{n-1}^p(t,\sigma(s))\Delta s\right)^{\frac{1}{p}}\Delta t\right)\left(\int_a^b\left|y^{\Delta^n}(t)\right|^{2q}\Delta t\right)^{\frac{1}{q}},\end{aligned}\tag{1.3}$$

where $p,\ q>1$ and satisfy $1/p+1/q=1$. As a generalization of (1.3) the authors in [3], proved that if $y:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is delta differentiable n times with $y^{\Delta^{m+i}}(a)=0$, for $i=0,1,\ldots,n-m-1$, and $\left|y^{\Delta^n}(t)\right|$ is increasing, then

$$\begin{aligned}&\int_a^b \left|y^{\Delta^m}(t)\right|\left|y^{\Delta^n}(t)\right|\Delta t\\ &\leqslant (b-a)^{\frac{1}{q}}\left(\int_a^b\left(\int_a^t h_{n-m-1}^p(t,\sigma(s))\Delta s\right)^{\frac{1}{p}}\Delta t\right)\left(\int_a^b\left|y^{\Delta^n}(t)\right|^{2q}\Delta t\right)^{\frac{1}{q}},\end{aligned}\tag{1.4}$$

where $p,\ q>1$ and satisfy $1/p+1/q=1$. In [15], the authors proved that if r and s are positive rd-continuous functions on $[a,b]_{\mathbb{T}}$ such that s is nonincreasing, and $y:[a,b]_{\mathbb{T}}\to\mathbb{R}$ is delta differentiable n times with $y^{\Delta^i}(a)=0$, for $i=0,1,\ldots,n-1$, then

$$\begin{aligned}&\int_a^b s(t)\,|y(t)|^p\left|y^{\Delta^n}(t)\right|\Delta t\\ &\leqslant \frac{1}{p+1}(b-a)^{n-1}\left(\int_a^b r^{1-\gamma}(t)\Delta t\right)^{\frac{1+p}{\gamma}}\left(\int_a^b r(t)\,(s(t))^{\frac{p}{p+1}}\left|y^{\Delta^n}(t)\right|^{\nu}\Delta t\right)^{\frac{1+p}{\nu}},\end{aligned}\tag{1.5}$$

where $p>0$ and $1/\gamma+1/\nu=1$.

In [11], the authors extended the above results and proved some new dynamic multiplicative inequalities by employing a chain rule of the form

$$\left[\prod_{j=1}^n f_j(t)\right]^{\Delta}=\sum_{j=1}^n\left\{\left[\prod_{i=1}^{j-1}f_i^{\sigma}(t)\right]f_j^{\Delta}(t)\left[\prod_{i=j+1}^n f_i(t)\right]\right\},\text{ for } t\in\mathbb{T}.$$

In particular, they proved that if $f_j\in C_{rd}^{(l)}(\mathbb{T})$ are real-valued functions with $f_j^{\Delta^i}(a)=0$ for all $j=1,2,\ldots,n+1$ and $i=0,1,\ldots,l$, then

$$\begin{aligned}&\int_a^b q(t)\sum_{j=1}^{n+1}\left|\left[\prod_{i=1}^{j-1}f_i^{\sigma}(t)\right]f_j^{\Delta^l}(t)\left[\prod_{i=j+1}^{n+1}f_i(t)\right]\right|\Delta t\\ &\leqslant\left(\frac{1}{n+1}\int_a^b\left(q^2(t)\int_a^{\sigma(t)}[h_{l-1}(\sigma(s),\sigma(t))]^2\Delta s\right)^n\Delta t\right)^{1/2}\sum_{j=1}^{n+1}\left(\int_a^b\left[f_j^{\Delta^l}(t)\right]^2\Delta t\right)^{\frac{n+1}{2}},\end{aligned}$$

where $l,\ n\in\mathbb{N},\ a,\ b\in\mathbb{T}$ and $q\in C_{rd}(\mathbb{T})$ is a nonnegative real-valued function.

In this paper, we will prove some new multiplicative dynamic inequalities of higher order of Opial type on time scales. The main results will be proved by using Hölder's inequality, a new chain rule, different from the one used in [1], and some new dynamic inequalities designed and proved for this purpose. As special cases, when $\mathbb{T}=\mathbb{R}$, our results contain some improved inequalities similar to those proved by Cheung in [9]. The paper is divided into two sections. In the next section, we present some concepts related to the notion of time scales and the main results will be proved in Section 3.

2. Preliminaries on time scales

In this section, for completeness, we recall the following concepts related to the notion of time scales. For more details of time scale analysis we refer the reader books by Bohner and Peterson [4], [5] which summarize and organize much of the time scale calculus. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e, when $\mathbb{T}=\mathbb{R},\ \mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t: t\in\mathbb{N}_0\}$ where $q>1$. Without loss of generality, we assume that $\sup\mathbb{T}=\infty$, and define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}}:=[a,b]\cap\mathbb{T}$. The forward jump operator is defined by $\sigma(t):=\inf\{s\in\mathbb{T}:\ s>t\}$. A function $f:\mathbb{T}\to\mathbb{R}$ is said to be right–dense continuous (rd-continuous) provided f is continuous at right–dense points and at left–dense points in $\mathbb{T}$, left hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. For any function $f:\mathbb{T}\to\mathbb{R}$ the notation $f^{\sigma}(t)$ denotes $f(\sigma(t))$. We will refer to the (delta) integral which we can define as follows: If $F^{\Delta}(t)=f(t)$, then the Cauchy (delta) integral of f is defined by $\int_a^t f(s)\Delta s:=F(t)-F(a)$. It can be shown (see [4]) that if $g\in C_{rd}(\mathbb{T})$, then the Cauchy integral $F(t):=\int_{t_0}^t g(s)\Delta s$ exists, $t_0\in\mathbb{T}$, and satisfies $F^{\Delta}(t)=f(t)$, $t\in\mathbb{T}$. An infinite integral is defined as $\int_a^{\infty} f(t)\Delta t=\lim_{b\to\infty}\int_a^b f(t)\Delta t$. The integration by parts formula is given by

$$\int_a^b u(t)v^{\Delta}(t)\Delta t=[u(t)v(t)]_a^b-\int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{2.1}$$

Now, we define the Taylor monomials or generalized polynomials as defined originally by Agarwal and Bohner [1]. These types of monomials are important because they are intimately related to Cauchy functions for certain dynamic equations which are important in variations of constants formulas. The Taylor monomials $h_k:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$, $k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, are defined recursively as follows: The function h_0 is defined by $h_0(t,s)=1$, for all $s,\ t\in\mathbb{T}$, and given h_k for $k\in\mathbb{N}_0$, the function h_{k+1} is defined by

$$h_{k+1}(t,s)=\int_s^t h_k(\tau,s)\Delta\tau,\ \text{ for all } s,\ t\in\mathbb{T}.$$

If we let $h_k^{\Delta}(t,s)$ denotes for each fixed $s\in\mathbb{T}$, the derivative of $h_k(t,s)$ with respect to t, then

$$h_k^{\Delta}(t,s)=h_{k-1}(t,s),\quad k\in\mathbb{N},\ t\in\mathbb{T},$$

for each fixed $s \in \mathbb{T}$. The above definition obviously implies $h_1(t,s) = t - s$, for all s, $t \in \mathbb{T}$. In the following, we give some formulas of $h_k(t,s)$ as determined in [4]. In the case when $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, $y^\Delta(t) = y'(t)$, and

$$h_k(t,s) = \frac{(t-s)^k}{k!}, \text{ for all } s,t \in \mathbb{R}. \tag{2.2}$$

In the case when $\mathbb{T} = \mathbb{N}$, we see that $\sigma(t) = t+1$, $\mu(t) = 1$, $y^\Delta(t) = \Delta y(t) = y(t+1) - y(t)$, and

$$h_k(n,s) := \frac{(n-s)^{(k)}}{k!}, \quad k = 0,1,2,\ldots, \quad t > s, \tag{2.3}$$

where $t^{(k)} = t(t-1)\ldots(t-k+1)$ is the so-called falling function (see [12]). In general for $t \geqslant s$, we have that $h_k(t,s) \geqslant 0$, and

$$h_k(t,s) \leqslant \frac{(t-s)^k}{k!}, \quad \text{for all } t > s, \; k \in \mathbb{N}_0.$$

We also consider the Taylor monomials $g_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, which are defined recursively as follows: The function g_0 is defined by $g_0(t,s) = 1$, for all $s,t \in \mathbb{T}$, and given g_k for $k \in \mathbb{N}_0$, the function g_{k+1} is defined by

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau),s)\Delta\tau, \text{ for all } s,t \in \mathbb{T}.$$

If we let $g_k^\Delta(t,s)$ denote for each fixed $s \in \mathbb{T}$, the derivative of $g(t,s)$ with respect to t, then

$$g_k^\Delta(t,s) = g_{k-1}(\sigma(t),s), \quad k \in \mathbb{N}, \; t \in \mathbb{T},$$

for each fixed $s \in \mathbb{T}$. From Theorem 1.112 in [4], we see that

$$h_k(t,s) = (-1)^k g_k(s,t).$$

We denote by $C_{rd}^{(n)}(\mathbb{T})$ the space of all functions $f \in C_{rd}(\mathbb{T})$ such that $f^{\Delta^i} \in C_{rd}(\mathbb{T})$ for $i = 0,1,\ldots,n$ for $n \in \mathbb{N}$. For the function $f : \mathbb{T} \to \mathbb{R}$, we consider the second derivative f^{Δ^2} provided f^Δ is delta differentiable on $\mathbb{T}$ with derivative $f^{\Delta^2} = (f^\Delta)^\Delta$. Similarly, we define the n^{th} order derivative $f^{\Delta^n} = (f^{\Delta^{n-1}})^\Delta$. The Taylor formula that we will need to prove the main results in this paper are adapted from [10] and states that if $f \in C_{rd}^{(n)}(\mathbb{T})$ and $s \in \mathbb{T}$, then

$$f(t) = \sum_{k=0}^{n-1} f^{\Delta^k}(s)h_k(t,s) + \int_s^t h_{n-1}(t,(\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau. \tag{2.4}$$

As a special case if $m < n$, then

$$f^{\Delta^m}(t) = \sum_{k=0}^{n-m-1} f^{\Delta^{k+m}}(s)h_k(t,s) + \int_s^t h_{n-m-1}(t,(\sigma(\tau))f^{\Delta^n}(\tau)\Delta\tau.$$

Also, in [1], the authors proved that if f is n-times differentiable on $\mathbb{T}^{\kappa^n}$, then

$$f(t)=\sum_{k=0}^{n-1}(-1)^k f^{\Delta^k}(s)g_k(s,t)+\int_s^t(-1)^{n-1}g_{n-1}((\sigma(\tau),t)f^{\Delta^n}(\tau)\Delta\tau, \tag{2.5}$$

where $t\in\mathbb{T}$, $s\in\mathbb{T}^{\kappa^{n-1}}$. As a special case if $m<n$, then

$$f^{\Delta^m}(t)=\sum_{k=0}^{n-m-1}(-1)^k f^{\Delta^{k+m}}(s)g_k(t,s)+\int_s^t(-1)^{n-m-1}g_{n-m-1}((\sigma(\tau),t)f^{\Delta^n}(\tau)\Delta\tau.$$

3. Main results

In this section, we will prove the main results and to do this we will need the following chain rule

$$(x^\gamma)^\Delta(t)=\gamma\int_0^1\left[hx^\sigma(t)+(1-h)x(t)\right]^{\gamma-1}dhx^\Delta(t), \tag{3.1}$$

which is a simple consequence of Keller's chain rule [4, Theorem 1.90] and Hölder's inequality [4, Theorem 6.13]

$$\int_a^b|u(t)v(t)|\Delta t\leqslant\left[\int_a^b|u(t)|^\gamma\Delta t\right]^{\frac{1}{\gamma}}\left[\int_a^b|v(t)|^\nu\Delta t\right]^{\frac{1}{\nu}}, \tag{3.2}$$

where $\gamma>1$ and $\frac{1}{\nu}+\frac{1}{\gamma}=1$, $a,\ b\in\mathbb{T}$ and $u,v\in C_{rd}(\mathbb{T},\ \mathbb{R})$.

Throughout the paper, we assume that $n\geqslant 1$ is a fixed number and $f\in C_{rd}^{(n)}([a,b]\cap\mathbb{T})$ is a real-valued rd-continuous function and w is a positive rd-continuous weighted function defined on $[a,b]\cap\mathbb{T}$. We also assume that and $a,\ \tau,\ b\in\mathbb{T}$ such that $[a,\tau]_{\mathbb{T}}=[a,\tau]\cap\mathbb{T}$ and $[\tau,b]_{\mathbb{T}}=[\tau,b]\cap\mathbb{T}$. The following two lemmas will be used in the proof of our main results.

LEMMA 3.1. *For any nonnegative rd-continuous function g and any $\lambda>0$, we have that*

$$\int_a^b g(t)\left(\int_a^t g(s)\Delta s\right)^\lambda\Delta t\leqslant\frac{1}{\lambda+1}\left(\int_a^b g(t)\Delta t\right)^{\lambda+1}<\left(\int_a^b g(t)\Delta t\right)^{\lambda+1}. \tag{3.3}$$

Proof. Let $G(t):=\int_a^t g(s)\Delta s$. Using the chain rule (3.1) and the fact that $G^\Delta(t)>$

0, we see that

$$\left(G^{\lambda+1}\right)^{\triangle}(t) = (\lambda+1)\int_0^1 \left[h\left(G(t)\right)^{\sigma} + (1-h)G(t)\right]^{\lambda} dh G^{\triangle}(t)$$
$$\geqslant (\lambda+1)g(t)\int_0^1 \left[hG(t) + (1-h)G(t)\right]^{\lambda} dh$$
$$= (\lambda+1)g(t)G^{\lambda}(t).$$

That is,

$$\left(G^{\lambda+1}\right)^{\triangle}(t) \geqslant (\lambda+1)g(t)G^{\lambda}(t). \tag{3.4}$$

Integrating both sides of (3.4) from a to b and using the fact that $G(a)=0$, we obtain

$$\int_a^b g(t)G^{\lambda}(t)\Delta t \leqslant \frac{1}{\lambda+1}\int_a^b \left(\left(G(t)\right)^{\lambda+1}\right)^{\triangle}\Delta t = \frac{1}{\lambda+1}G^{\lambda+1}(b)$$
$$= \frac{1}{\lambda+1}\left(\int_a^b g(t)\Delta t\right)^{\lambda+1} < \left(\int_a^b g(t)\Delta t\right)^{\lambda+1},$$

which is the desired inequality (3.3). The proof is complete. □

LEMMA 3.2. *For any nonnegative rd-continuous function g and any $\lambda > 0$, we have that*

$$\int_a^b g(t)\left(\int_t^b g(s)\Delta s\right)^{\lambda}\Delta t \leqslant \left(\int_a^b g(t)\Delta t\right)^{\lambda+1}. \tag{3.5}$$

Proof. Setting $G(t) := \int_t^b g(s)\Delta s$, we see that $G^{\Delta}(t) = -g(t)$ and then

$$G(t) \leqslant G(a), \quad \text{for } a \leqslant t.$$

This implies, for $\lambda > 0$, that

$$\int_a^b g(t)G^{\lambda}(t)\Delta t \leqslant \int_a^b g(t)G^{\lambda}(a)\Delta t = G^{\lambda}(a)\int_a^b g(t)\Delta t = G^{1+\lambda}(a).$$

That is

$$\int_a^b g(t)\left(\int_t^b g(s)\Delta s\right)^{\lambda}\Delta t \leqslant \left(\int_a^b g(t)\Delta t\right)^{\lambda+1},$$

which is the desired inequality (3.5). The proof is complete. □

LEMMA 3.3. *Let $q>0$ and $p>0$ be such that $p+q\geqslant 1$ and let $\Omega(t)$ be a nonnegative rd-continuous function. Then*

$$\int_a^\tau \Omega^{(\frac{p+q-1}{p+q})p}(t)\left[\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{p+q}\Delta s\right]^{\frac{p}{p+q}} w^{\frac{q}{p+q}}(t)\left|f^{\Delta^n}(t)\right|^q\Delta t \leqslant H(\tau)\int_a^\tau w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t, \tag{3.6}$$

where

$$H(\tau):=\left[\int_a^\tau \Omega^{p+q-1}(t)\Delta t\right]^{\frac{p}{p+q}}. \tag{3.7}$$

Proof. Applying Hölder's inequality on the left hand side of the inequality (3.6) with $\gamma=1/\alpha p$ and $\nu=1/\alpha q$ where $\alpha=1/(p+q)$, we have that

$$\begin{aligned}
&\int_a^\tau \Omega^{p(1-\alpha)}(t)\left[\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right]^{\alpha p} w^{\alpha q}(t)\left|f^{\Delta^n}(t)\right|^q\Delta t\\
&\leqslant \left[\int_a^\tau \Omega^{(1-\alpha)/\alpha}(t)\Delta t\right]^{\alpha p}\left[\int_a^\tau\left(\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{p/q} w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t\right]^{\alpha q}\\
&= H(\tau)\left[\int_a^\tau\left(\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{p/q} w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t\right]^{\alpha q}.
\end{aligned}$$

Applying the inequality (3.3) in Lemma 3.1 on the term

$$\int_a^\tau\left(\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{p/q} w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t,$$

with $\lambda=p/q$ and $g(t)=w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}$, we see that

$$\begin{aligned}
&\int_a^\tau \Omega^{p(1-\alpha)}(t)\left[\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right]^{\alpha p} w^{\alpha q}(t)\left|f^{\Delta^n}(t)\right|^q\Delta t\\
&\leqslant H(\tau)\left[\left(\int_a^\tau w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t\right)^{1+\frac{p}{q}}\right]^{\alpha q}\\
&= H(\tau)\left[\left(\int_a^\tau w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t\right)^{1/\alpha q}\right]^{\alpha q}\\
&= H(\tau)\int_a^\tau w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t.
\end{aligned}$$

Using $\alpha=1/(p+q)$, we see that

$$\int_a^\tau \Omega^{(\frac{p+q-1}{p+q})p}(t)\left[\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{p+q}\Delta s\right]^{\frac{p}{p+q}} w^{\frac{q}{p+q}}(t)\left|f^{\Delta^n}(t)\right|^q\Delta t \leqslant H(\tau)\int_a^\tau w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t,$$

which is the desired inequality (3.6). The proof is complete. □

LEMMA 3.4. *Let $q>0$ and $p>0$ be such that $p+q\geqslant 1$, and let $\Lambda(t)$ a non-negative rd-continuous function. Then*

$$\begin{aligned}&\int_\tau^b \Lambda^{(\frac{p+q-1}{p+q})p}(t)\left[\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{p+q}\Delta s\right]^{\frac{p}{p+q}} w^{\frac{q}{p+q}}(t)\left|f^{\Delta^n}(t)\right|^q\Delta t\\ &\leqslant G(\tau)\int_\tau^b w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t, \qquad (3.8)\end{aligned}$$

where

$$G(\tau):=\left[\int_\tau^b \Lambda^{p+q-1}(t)\Delta t\right]^{\frac{p}{p+q}}. \qquad (3.9)$$

Proof. Applying Hölder's inequality on the left hand side of the inequality (3.8) for $\gamma=1/\alpha p$ and $\nu=1/\alpha q$ where $\alpha=1/(p+q)$, we have that

$$\begin{aligned}&\int_\tau^b \Lambda^{p(1-\alpha)}(t)\left[\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right]^{\alpha p} w^{\alpha q}(t)\left|f^{\Delta^n}(t)\right|^q\Delta t\\ &\leqslant \left[\int_\tau^b \Lambda^{(1-\alpha)/\alpha}(t)\Delta t\right]^{\alpha p}\left[\int_\tau^b\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{p/q} w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t\right]^{\alpha q}\\ &= G(\tau)\left[\int_a^b\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{p/q} w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t\right]^{\alpha q}.\end{aligned}$$

Applying the inequality (3.5) in Lemma 3.2 on the term

$$\int_\tau^b\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{p/q} w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t,$$

with $\lambda=p/q$ and $g(t)=w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}$, we see that

$$\begin{aligned}&\int_\tau^b \Lambda^{p(1-\alpha)}(t)\left[\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right]^{\alpha p} w^{\alpha q}(t)\left|f^{\Delta^n}(t)\right|^q\Delta t\\ &\leqslant G(\tau)\left[\left(\int_\tau^b w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t\right)^{1+\frac{p}{q}}\right]^{\alpha q}\\ &= G(\tau)\left[\left(\int_\tau^b w(t)\left|f^{\Delta^n}(t\right|^{1/\alpha}\Delta t\right)^{1/\alpha q}\right]^{\alpha q}\\ &= G(\tau)\int_\tau^b w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t.\end{aligned}$$

Using $\alpha = 1/(p+q)$, we see that

$$\int_a^b \Lambda^{(\frac{p+q-1}{p+q})p}(t)\left[\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{p+q}\Delta s\right]^{\frac{p}{p+q}} w^{\frac{q}{p+q}}(t)\left|f^{\Delta^n}(t)\right|^q \Delta t$$
$$\leqslant G(\tau)\int_a^b w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t,$$

which is the desired inequality (3.8). The proof is complete. □

For the remaining results we assume that $p \geqslant 1$ and $q > 0$ are any real numbers and $r_k \geqslant 0$ for $k = 0, 1, \ldots, n-1$ are real numbers with $\Sigma_{k=0}^{n-1} r_k = 1$.

THEOREM 3.1. *Assume that* $f^{\Delta^{k+i}}(a) = 0$, *for all* $i = 0, 1, \ldots, n-k-1$. *If* w *is a positive, rd-continuous, and nonincreasing on* $[a,\tau]_{\mathbb{T}}$, *then*

$$\int_a^\tau w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \left|f^{\Delta^n}(t)\right|^q \Delta t \leqslant \sum_{k=0}^{n-1} r_k H_k(a,\tau)\int_a^\tau w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t, \tag{3.10}$$

where

$$H_k(a,\tau) := \left[\int_a^\tau \Omega_k^{p+q-1}(a;t)\Delta t\right]^{\frac{p}{p+q}}, \tag{3.11}$$

and

$$\Omega_k(a;t) := \int_a^t \left|h_{n-k-1}(t,\sigma(s))\right|^{\frac{p+q}{p+q-1}}\Delta s.$$

Proof. Applying the inequality (see [9, Lemma 1]),

$$\prod_{k=0}^{n-1} a_k^{r_k} \leqslant \sum_{k=0}^{n-1} r_k a_k \leqslant \left(\sum_{k=0}^{n-1} r_k a_k^p\right)^{1/p}, \tag{3.12}$$

for any real numbers $a_k \geqslant 0,$ for $k = 0, 1, \ldots, n-1$, and any $p \geqslant 1$, we see that

$$\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \leqslant \sum_{k=0}^{n-1} r_k \left|f^{\Delta^k}(t)\right|^p. \tag{3.13}$$

From Taylor's formula (2.4) and $f^{\Delta^{k+i}}(a) = 0$ for $i = 0, 1, \ldots, n-k-1,$ we have

$$f^{\Delta^k}(t) = \int_a^t h_{n-k-1}(t,\sigma(s))\, f^{\Delta^n}(s)\Delta s.$$

That is

$$\left|f^{\Delta^k}(t)\right| = \left|\int_a^t h_{n-k-1}(t,\sigma(s))\, f^{\Delta^n}(s)\Delta s\right|$$
$$\leqslant \int_a^t \left|h_{n-k-1}(t,\sigma(s))\right|\left|f^{\Delta^n}(s)\right|\Delta s. \tag{3.14}$$

From (3.13) and (3.14) and using the fact that w is nonincreasing, we get

$$\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \leqslant \sum_{k=0}^{n-1} r_k\left(\int_a^t |h_{n-k-1}(t,\sigma(s))|\left|f^{\Delta^n}(s)\right|\Delta s\right)^p$$
$$\leqslant \sum_{k=0}^{n-1} r_k w^{-\alpha p}(t)\left(\int_a^t |h_{n-k-1}(t,\sigma(s))|\, w^{\alpha}(s)\left|f^{\Delta^n}(s)\right|\Delta s\right)^p. \tag{3.15}$$

Applying Hölder's inequality with $\gamma = 1/(1-\alpha)$, and $\nu = 1/\alpha$ on the right hand side where $\alpha = 1/(p+q)$, we obtain

$$\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \leqslant \sum_{k=0}^{n-1} r_k w^{-\alpha p}(t)\Omega_k^{(1-\alpha)p}(a;t)\left(\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}. \tag{3.16}$$

Multiplying (3.16) by $w(t)\left|f^{\Delta^n}(t)\right|^q$ and integrating from a to τ and then applying the inequality in Lemma 3.3 where $\Omega(t)$ is replaced by $\Omega_k(a;t)$ and $H(t)$ is replaced by $H_k(a,\tau)$, we have

$$\int_a^\tau w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p\left|f^{\Delta^n}(t)\right|^q\Delta t$$
$$\leqslant \sum_{k=0}^{n-1} r_k\int_a^\tau w^{(1-\alpha p)}(t)\left|f^{\Delta^n}(t)\right|^q\Omega_k^{(1-\alpha)p}(a;t)\left(\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}\Delta t$$
$$= \sum_{k=0}^{n-1} r_k\int_a^\tau w^{\alpha q}(t)\left|f^{\Delta^n}(t)\right|^q\Omega_k^{(1-\alpha)p}(a;t)\left(\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}\Delta t$$
$$= \left[\sum_{k=0}^{n-1} r_k H_k(a,\tau)\right]\int_a^\tau w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t.$$

Using $\alpha = 1/(p+q)$, we have

$$\int_a^\tau w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p\left|f^{\Delta^n}(t)\right|^q\Delta t \leqslant \sum_{k=0}^{n-1} r_k H_k(a,\tau)\int_a^\tau w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t,$$

which is the desired inequality (3.10). The proof is complete. □

THEOREM 3.2. *Assume that* $f^{\Delta^{k+i}}(b) = 0$, *for all* $i = 0,1,\ldots,n-k-1$. *If* w *is a positive, rd-continuous and nondecreasing on* $[\tau,b]_{\mathbb{T}}$, *then*

$$\int_\tau^b w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p\left|f^{\Delta^n}(t)\right|^q\Delta t$$
$$\leqslant \sum_{k=0}^{n-1} r_k G_k(\tau,b)\int_\tau^b w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t, \tag{3.17}$$

where

$$G_k(\tau,b)=\left[\int_\tau^b \Lambda_k^{p+q-1}(t;b)\Delta t\right]^{\frac{p}{p+q}}, \tag{3.18}$$

and

$$\Lambda_k(t;b)=\int_t^b |g_{n-k-1}(\sigma(s),t)|^{\frac{p+q}{p+q-1}}\Delta s.$$

Proof. As in the proof of Theorem 3.1, by applying Taylor's formula (2.5), we see that

$$\left|f^{\Delta^k}(t)\right|\leqslant \int_t^b |g_{n-k-1}(\sigma(s),t)|\left|f^{\Delta^n}(s)\right|\Delta s.$$

By applying the inequality (3.12), we have

$$\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p\leqslant \sum_{k=0}^{n-1} r_k\left(\int_t^b |g_{n-k-1}(\sigma(s),t)|\left|f^{\Delta^n}(s)\right|\Delta s\right)^p. \tag{3.19}$$

Applying Hölder's inequality for $\gamma=1/(1-\alpha)$ and $\nu=1/\alpha$ where $\alpha=1/(p+q)$ and since $w(t)$ is nondecreasing, we have

$$\begin{aligned}
&\left(\int_t^b |g_{n-k-1}(\sigma(s),t)|\left|f^{\Delta^n}(s)\right|\Delta s\right)^p\\
&\leqslant \left(w^{-\alpha}(t)\int_t^b w^{\alpha}(s)\,|g_{n-k-1}(\sigma(s),t)|\left|f^{\Delta^n}(s)\right|\Delta s\right)^p\\
&\leqslant w^{-\alpha p}(t)\left(\int_t^b |g_{n-k-1}(\sigma(s),t)|^{1/(1-\alpha)}\Delta s\right)^{(1-\alpha)p}\\
&\quad\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}\\
&= w^{-\alpha p}(t)\Lambda(t;b)^{(1-\alpha)p}\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}.
\end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.19), we have

$$\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p\leqslant \sum_{k=0}^{n-1} r_k w^{-\alpha p}(t)\Lambda(t;b)^{(1-\alpha)p}\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}. \tag{3.21}$$

Multiplying (3.21) by $w(t)\left|f^{\Delta_n}(t)\right|^q$ and integrating from τ to b and then applying the inequality in Lemma 3.4, where $\Lambda(t)$ is replaced by $\Lambda_k(t;b)$ and $G(\tau)$ is replaced by

$G_k(\tau,b)$, we have

$$\begin{aligned}
&\int_\tau^b w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \left|f^{\Delta^n}(t)\right|^q \Delta t \\
&\leqslant \sum_{k=0}^{n-1} r_k \int_\tau^b w(t)\left|f^{\Delta^n}(t)\right|^q w^{-\alpha p}(t)\Lambda(t;b)^{(1-\alpha)p}\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}\Delta t \\
&= \sum_{k=0}^{n-1} r_k \int_\tau^b w^{\alpha q}(t)\left|f^{\Delta^n}(t)\right|^q \Lambda(t;b)^{(1-\alpha)p}\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}\Delta t \\
&\leqslant \sum_{k=0}^{n-1} r_k G_k(\tau,b)\int_\tau^b w(t)\left|f^{\Delta^n}(t)\right|^{1/\alpha}\Delta t.
\end{aligned}$$

Using $\alpha = 1/(p+q)$, we have

$$\begin{aligned}
&\int_\tau^b w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \left|f^{\Delta^n}(t)\right|^q \Delta t \\
&\leqslant \sum_{k=0}^{n-1} r_k G_k(\tau,b)\int_\tau^b w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t,
\end{aligned}$$

which is the desired inequality (3.17). The proof is complete. □

THEOREM 3.3. *Assume that* $f^{\Delta^{k+i}}(a) = 0$, *for all* $i = 0,1,\dots,n-k-1$. *If* $0 < A \leqslant w(t) \leqslant B$ *for all* $t \in [a,\tau]_{\mathbb{T}}$, *then*

$$\begin{aligned}
&\int_a^\tau w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \left|f^{\Delta^n}(t)\right|^q \Delta t \\
&\leqslant \sum_{k=0}^{n-1} r_k H_k(a,\tau)\left(\frac{B}{A}\right)^{\frac{p}{p+q}}\int_a^\tau w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t, \qquad (3.22)
\end{aligned}$$

where $H_k(a,\tau)$ *is defined as in (3.11).*

Proof. We proceed as in the proof of Theorem 3.1 to get

$$\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \leqslant \sum_{k=0}^{n-1} r_k\left(\int_a^t |h_{n-k-1}(t,\sigma(s))|\left|f^{\Delta^n}(s)\right|\Delta s\right)^p.$$

From the boundedness of $w(t)$ and since $A \leqslant w(t)$, we have

$$\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \leqslant \sum_{k=0}^{n-1} r_k A^{-\alpha p}\left(\int_a^t w^\alpha(s)\,|h_{n-k-1}(t,\sigma(s))|\left|f^{\Delta^n}(s)\right|\Delta s\right)^p.$$

Applying Hölder's inequality with $\gamma = 1/(1-\alpha)$ and $\nu = 1/\alpha$, we have

$$\begin{aligned}
&\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \\
&\leqslant \sum_{k=0}^{n-1} r_k A^{-\alpha p}\left(\int_a^t |h_{n-k-1}(t,\sigma(s))|^{1/(1-\alpha)}\Delta s\right)^{(1-\alpha)p}\left(\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p} \\
&= \sum_{k=0}^{n-1} r_k A^{-\alpha p}\Omega^{(1-\alpha)p}(a;t)\left(\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}.
\end{aligned} \tag{3.23}$$

Multiplying (3.23) by $w(t)\left|f^{\Delta^n}(t)\right|^q$ and integrating from a to τ, we obtain

$$\begin{aligned}
&\int_a^\tau w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}\right|^{r_k}\right)^p\left|f^{\Delta^n}(t)\right|^q\Delta t \\
&\leqslant \sum_{k=0}^{n-1}\frac{r_k}{A^{\alpha p}}\int_a^\tau w(t)\Omega^{(1-\alpha)p}(a;t)\left(\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}\left|f^{\Delta^n}(t)\right|^q\Delta t.
\end{aligned} \tag{3.24}$$

Since $w(t) \leqslant B$ and $\alpha(p+q) = 1$, then we have

$$w^{\alpha p}(t) \leqslant B^{\alpha p}, \text{ and } w(t) \leqslant B^{\alpha p} w^{\alpha q}(t). \tag{3.25}$$

This and (3.24) imply that

$$\begin{aligned}
&\int_a^\tau w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p\left|f^{\Delta^n}\right|^q\Delta t \\
&\leqslant \sum_{k=0}^{n-1} r_k\frac{B^{\alpha p}}{A^{\alpha p}}\int_a^\tau w(t)^{\alpha q}\Omega^{(1-\alpha)p}(a;t)\left(\int_a^t w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}\left|f^{\Delta^n}(t)\right|^q\Delta t.
\end{aligned}$$

Applying the inequality in Lemma 3.3 and using $\alpha = 1/(p+q)$, we have

$$\begin{aligned}
&\int_a^\tau w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p\left|f^{\Delta^n}(t)\right|^q\Delta t \\
&\leqslant \sum_{k=0}^{n-1} r_k H_k(a,\tau)\left(\frac{B}{A}\right)^{\frac{p}{p+q}}\int_a^\tau w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t,
\end{aligned}$$

which is the desired inequality (3.22). This completes our proof. □

THEOREM 3.4. *Assume that* $f^{\Delta^{k+i}}(b) = 0$, *for all* $i = 0,1,\ldots,n-k-1$. *If* $0 < A \leqslant w(t) \leqslant B$ *for all* $t \in [\tau,b]_{\mathbb{T}}$, *then*

$$\begin{aligned}
&\int_\tau^b w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p\left|f^{\Delta^n}(t)\right|^q\Delta t \\
&\leqslant \sum_{k=0}^{n-1} r_k\left(\frac{B}{A}\right)^{\frac{p}{p+q}} G_k(\tau,b)\int_\tau^b w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t,
\end{aligned} \tag{3.26}$$

where $G_k(\tau,b)$ is defined as in (3.18).

Proof. We proceed as in the proof of Theorem 3.2 to get

$$\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \leqslant \sum_{k=0}^{n-1} r_k\left(\int_t^b |g_{n-k-1}(\sigma(s),t)|\left|f^{\Delta^n}(s)\right|\Delta s\right)^p.$$

From the boundedness of $w(t)$, since $A \leqslant w(t)$, we have

$$\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \leqslant \sum_{k=0}^{n-1} r_k A^{-\alpha p}\left(\int_t^b w^{\alpha}(s)|g_{n-k-1}(\sigma(s),t)|\left|f^{\Delta^n}(s)\right|\Delta s\right)^p.$$

Applying Hölder's inequality with $\gamma = 1/(1-\alpha)$ and $\nu = 1/\alpha$ where $\alpha = 1/(p+q)$, we have

$$\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p \leqslant \sum_{k=0}^{n-1} r_k A^{-\alpha p}\Lambda(t;b)^{(1-\alpha)p}\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}. \quad (3.27)$$

Multiplying (3.27) by $w(t)\left|f^{\Delta^n}(t)\right|^q$ and integrating from τ to b, we obtain

$$\int_\tau^b w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}\right|^{r_k}\right)^p\left|f^{\Delta_n}(t)\right|^q\Delta t$$
$$\leqslant \sum_{k=0}^{n-1}\frac{r_k}{A^{\alpha p}}\int_\tau^b w(t)\Lambda^{(1-\alpha)p}(t;b)\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}\left|f^{\Delta^n}(t)\right|^q\Delta t. \quad (3.28)$$

Since $w(t) \leqslant B$ and $\alpha(p+q) = 1$, then by using (3.25) in (3.28), we have that

$$\int_\tau^b w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p\left|f^{\Delta^n}(t)\right|^q\Delta t$$
$$\leqslant \sum_{k=0}^{n-1} r_k\frac{B^{\alpha p}}{A^{\alpha p}}\int_\tau^b w^{\alpha q}(t)\Lambda^{(1-\alpha)p}(t;b)\left(\int_t^b w(s)\left|f^{\Delta^n}(s)\right|^{1/\alpha}\Delta s\right)^{\alpha p}\left|f^{\Delta^n}(t)\right|^q\Delta t.$$

Applying the inequality in Lemma 3.4 and using $\alpha = 1/(p+q)$, we have

$$\int_\tau^b w(t)\left(\prod_{k=0}^{n-1}\left|f^{\Delta^k}(t)\right|^{r_k}\right)^p\left|f^{\Delta^n}(t)\right|^q\Delta t$$
$$\leqslant \sum_{k=0}^{n-1}\left(\frac{B}{A}\right)^{\frac{p}{p+q}} r_k G_k(\tau,b)\int_\tau^b w(t)\left|f^{\Delta^n}(t)\right|^{p+q}\Delta t,$$

which is the desired inequality (3.26). The proof is complete. □

As a special case of the above Theorems 3.1, 3.2, 3.3 and 3.4 respectively, when $\mathbb{T} = \mathbb{R}$, by setting

$$h_k(t,s) = \frac{(t-s)^k}{k!}, \text{ for all } s \leqslant t, \quad (3.29)$$

we have the the following modified versions of the results obtained in [9].

THEOREM 3.5. *Assume that* $f^{(k+i)}(a)=0$ *for all* $i=0,1,\ldots,n-k-1$. *If* w *is nonincreasing, then*

$$\int_a^\tau w(t)\left(\prod_{k=0}^{n-1}\left|f^{(k)}(t)\right|^{r_k}\right)^p\left|f^{(n)}(t)\right|^q dt \leqslant \sum_{k=0}^{n-1} m_k(\tau-a)^{(n-k)p}\int_a^\tau w(t)\left|f^{(n)}(t)\right|^{p+q} dt,$$

where

$$m_k:=\left(\frac{1}{p+q}\right)^{\frac{p}{p+q}}\frac{r_k}{[(n-k)!]^p}\left[\frac{(n-k)(1-\alpha)}{n-k-\alpha}\right]^{(\frac{p+q-1}{p+q})p}. \tag{3.30}$$

THEOREM 3.6. *Assume that* $f^{(k+i)}(b)=0$ *for all* $i=0,1,\ldots,n-k-1$. *If* w *is nondecreasing for all* $t\in[\tau,b]\cap\mathbb{R}$, *then*

$$\int_\tau^b w(t)\left(\prod_{k=0}^{n-1}\left|f^{(k)}(t)\right|^{r_k}\right)^p\left|f^{(n)}(t)\right|^q dt \leqslant \sum_{k=0}^{n-1} m_k(b-\tau)^{(n-k)p}\int_\tau^b w(t)\left|f^{(n)}(t)\right|^{p+q} dt,$$

where m_k *is defined in (3.30).*

THEOREM 3.7. *Assume that* $f^{(k+i)}(a)=0$ *for all* $i=0,1,\ldots,n-k-1$. *If* $0<A\leqslant w(t)\leqslant B$ *for all* $t\in[a,\tau]\cap\mathbb{R}$, *then*

$$\int_a^\tau w(t)\left(\prod_{k=0}^{n-1}\left|f^{(k)}(t)\right|^{r_k}\right)^p\left|f^{(n)}(t)\right|^q dt \leqslant \sum_{k=0}^{n-1} m_k^*(\tau-a)^{(n-k)p}\int_a^\tau w(t)\left|f^{(n)}(t)\right|^{p+q} dt,$$

where

$$m_k^*:=\left(\frac{B}{A}\right)^{\frac{p}{p+q}}\left(\frac{1}{p+q}\right)^{\frac{p}{p+q}}\frac{r_k}{[(n-k)!]^p}\left[\frac{(n-k)(1-\alpha)}{n-k-\alpha}\right]^{(\frac{p+q-1}{p+q})p}. \tag{3.31}$$

THEOREM 3.8. *Assume that* $f^{(k+i)}(b)=0$ *for all* $i=0,1,\ldots,n-k-1$. *If* $0\leqslant A\leqslant w(t)\leqslant B$ *for all* $t\in[\tau,b]\cap\mathbb{R}$, *then*

$$\int_\tau^b w(t)\left(\prod_{k=0}^{n-1}\left|f^{(k)}(t)\right|^{r_k}\right)^p\left|f^{(n)}(t)\right|^q dt \leqslant \sum_{k=0}^{n-1} m_k^*(b-\tau)^{(n-k)p}\int_\tau^b w(t)\left|f^{(n)}(t)\right|^{p+q} dt,$$

where m_k^* *is defined as in (3.31).*

As a special case, when $\mathbb{T} = \mathbb{N}$, by setting

$$h_k(m,n) = \frac{(m-n)^{(k)}}{k!}, \text{ for all } m \leqslant n, \tag{3.32}$$

in Theorem 3.1, we have the following result.

THEOREM 3.9. *Assume that* $\Delta^{k+i}u(a) = 0$ *for all* $i = 0,1,\ldots,n-k-1$*. If* w *is nonincreasing for* $t \in [a,b] \cap \mathbb{N}$*, then*

$$\sum_{t=a}^{b-1} w(t) \left(\prod_{k=0}^{n-1} |\Delta_k u(t)|^{r_k} \right)^p |\Delta_n u(t)|^q \leqslant \sum_{k=0}^{n-1} r_k H_k \sum_{t=a}^{b-1} w(t) \, |\Delta_n u(t)|^{p+q},$$

where

$$H_k = \left[\sum_{t=a}^{b-1} \left(\sum_{s=a}^{t-1} \left(\frac{(t-s-1)^{(n-k-1)}}{(n-k-1)!} \right)^{\frac{p+q}{p+q-1}} \right)^{p+q-1} \right]^{p/(p+q)}. \tag{3.33}$$

REMARK 3.1. When $\mathbb{T} = \mathbb{N}$, and using (3.32) in Theorems 3.2, 3.3 and 3.4, we can obtain new results in discrete time scales. The details are left to the interested reader.

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