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HARDY'S TYPE INEQUALITY FOR THE OVER CRITICAL EXPONENT ASSOCIATED WITH THE DUNKL TRANSFORM

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Abstract. For the Hardy space $H^p_{q,s}(\mathbb{R}^d,\mu_k)$, $0<p\leqslant 1$, we shall improve a Hardy's type inequality associated with the Dunkl transform respect to the measures μ_κ homogeneous of degree γ, on the strip $(2\gamma+d)(2-p)\leqslant\sigma<2\gamma+d+p(N+1)$, where $N=[(2\gamma+d)(1/p-1)]$ is the greatest integer not exceeding $(2\gamma+d)(1/p-1)$.

1. Introduction

In recent years the topic of Hardy type inequalities and their applications seem to have grown more and more popular. Although Hardy's original result dates back to the 1920's, some new versions are stated and old ones are still being improved almost a century later. One of the reasons for the popularity of Hardy type inequalities is their usefulness in various applications.

The first definition of Hardy spaces was in terms of analytic functions in the unit disc and their boundary values. In the last two decades, the theory was developed in $\mathbb{R}^d$ by real variable methods like Poisson integrals, Riesz transforms, and maximal functions. The subsequent discovery of the atomic decomposition theory of $H^p(\mathbb{R}^d)$ spaces marks an important step of further developments on its real variable theory. Using the grand maximal function, R. Coifman [4] first shows that an element in $H^p(\mathbb{R}^d)$ can decomposed into a sum of a series of basic elements. Then the study on some analytic problems on $H^p(\mathbb{R}^d)$ is summed up to investigate some properties of these basic elements, and therefore the problems because quite simple. Taibleson and Weiss [21] gave the definition of molecules belonging to H^p, and showed that every molecule is in H^p with continuous embedding map. By the atomic decomposition and the molecule characterization, the proof of H^p boundedness of the operators on Hardy space becomes easier. The theory of H^p have been extensively studied in [9, 10, 11].

In the setting of the Euclidian case, the Fourier transform $\mathscr{F}(f)$ of $f\in H^p(\mathbb{R}^d)$, is a continuous function and satisfies the inequality

$$\int_{\mathbb{R}^d}\frac{|\mathscr{F}(f)(\xi)|^p}{|\xi|^{d(2-p)}}d\xi\leqslant\|f\|^p_{H^p(\mathbb{R}^d)},\qquad 0<p\leqslant 1 \tag{1}$$

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which is well-known as Hardy's inequality for $H^p(\mathbb{R}^d)$ (cf. [11, Corollary 7.23], [19, p. 128]). Recently [20], establish this inequality for the Dunkl transform on the critical exponent case $\sigma = (2\gamma+d)(2-p)$. The aim of this paper is to improve this inequality for the over critical exponent associated with the Dunkl transform.

This kind of inequality has been extended to several different settings. We may cite the following situations: This was done first by Assal [1]; establishes a Hardy's type inequality associated with the Hankel transform for over critical exponent $\sigma > 2-p$. Later, Assal and Rahmouni [2] extended this form of this inequality to other contexts; they are interested in the Laguerre hypergroup with the Euclidean Fourier replaced by the Laguerre Fourier transform, we point out here that an improved of this inequality has been given by [3]. Although, by using the inverse Laguerre Fourier they establish a Hardy's type inequality on the dual of Laguerre hypergroup for the critical and over critical cases [15, 16].

In this paper, we obtain an improved Hardy's type inequality associated with the Dunkl transform. So, for the Hardy space $H^p_{q,s}(\mathbb{R}^d,\mu_\kappa)$, $0 < p \leqslant 1$ we establish a Hardy's type inequality for the strip $(2\gamma+d)(2-p) \leqslant \sigma < 2\gamma+d+p(N+1)$. Throughout this paper, C stands for a positive constant that can be changed from line to line.

2. Preliminaries

In order to confirm the basic and standard notations we briefly overview the theory of Dunkl operators and related harmonic analysis, most of which can be found in [5, 6, 7, 8, 17, 18, 22, 23].

2.1. Reflection groups, Root systems and Multiplicity functions

Let us begin to recall some results concerning the root systems. A useful reference for this topic is the book by Humphreys [12].

We consider $\mathbb{R}^d$ with the Euclidean inner product $\prec .,. \succ$ and norm $|y| := \sqrt{\prec y,y \succ}$. For $\alpha \in \mathbb{R}^d \setminus \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e

$$\sigma_\alpha y := y - \frac{2 \prec \alpha, y \succ}{|\alpha|^2}\alpha.$$

A finite set $\mathscr{R} \subset \mathbb{R}^d \setminus \{0\}$ is called a root system, if $\mathscr{R} \cap \mathbb{R}.\alpha = \{-\alpha,\alpha\}$ and $\sigma_\alpha \mathscr{R} = \mathscr{R}$ for all $\alpha \in \mathscr{R}$. We assume that it is normalized by $|\alpha|^2 = 2$ for all $\alpha \in \mathscr{R}$. For a root system $\mathscr{R}$, the reflections $\sigma_\alpha, \alpha \in \mathscr{R}$, generate a finite group $G \subset O(d)$, the reflection group associated with $\mathscr{R}$. All reflections in G correspond to suitable pairs of roots.

For a given $\beta \in \mathbb{R}^d \setminus \bigcup_{\alpha\in\mathscr{R}} H_\alpha$, we fix the positive subsystem $\mathscr{R}_+ := \{\alpha \in \mathscr{R} \ : \prec \alpha,\beta \succ > 0\}$. Then, for each $\alpha \in \mathscr{R}$ either $\alpha \in \mathscr{R}_+$ or $\alpha \in \mathscr{R}_-$. Let $k : \mathscr{R} \to \mathbb{C}$ be a multiplicity function on $\mathscr{R}$ (i.e. a function which is constant on the orbits under the action of G).

2.2. Dunkl transform and Dunkl convolution

Let us consider the differential-difference operators $T_j,\ j=1,...,d,$ on $\mathbb{R}^d$ introduced by C. F. Dunkl in [6] and called Dunkl operators in the literature, associated with the finite reflection group G and the multiplicity function κ, are given for a function f of class $\mathscr{C}^1$ on $\mathbb{R}^d$ by

$$T_jf(y) = \frac{\partial}{\partial y_j}f(y) + \sum_{\alpha\in\mathscr{R}_+}\kappa(\alpha)\alpha_j\frac{f(y)-f(\sigma_\alpha y)}{\prec \alpha, y\succ}.$$

For $y\in\mathbb{R}^d$, the initial problem $T_ju(.,y)(x) = y_ju(x,y);\ j=1,...,d$ with $u(0,y)=1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_\kappa(x,y)$ and called Dunkl kernel [7]. This kernel has a unique analytic extension to $\mathbb{C}^d\times\mathbb{C}^d$. The Dunkl kernel has the Laplace-type representation [17]

$$E_\kappa(x,y) = \int_{\mathbb{R}^d} e^{\prec y,z\succ}d\Gamma_x(z);\ \ x\in\mathbb{R}^d,\ y\in\mathbb{C}^d,$$

where $\prec y,z\succ := \sum_{j=1}^d y_jz_j$ and Γ_x is a probability measure on $\mathbb{R}^d$, such that $supp(\Gamma_x)\subset\{z\in\mathbb{R}^d:\ |z|\leqslant|x|\}$. In particular cases, we have

$$|E_\kappa(x,y)|\leqslant 1,\ \ x,y\in\mathbb{R}^d. \tag{2}$$

The Dunkl kernel gives rise to an integral transform, which is called Dunkl transform on $\mathbb{R}^d$, and it was introduced by Dunkl [8], where already many basic properties were established which is taken with respect to a weighted Lebesgue measure invariant under the action of G and which generalizes the Euclidean Fourier transform. Dunkl's results were completed and extended later on by De Jeu [5].

More precisely, let us introduce the measure $d\mu_\kappa(x) := w_\kappa(x)dx$ where the weight given by

$$w_\kappa(y) := \prod_{\alpha\in\mathscr{R}_+}|\prec\alpha,y\succ|^{2k(\alpha)},\ \ y\in\mathbb{R}^d,$$

is homogeneous of degree 2γ with

$$\gamma=\gamma_\kappa := \sum_{\alpha\in\mathscr{R}_+}\kappa(\alpha),$$

and let us denote by $L^p_\kappa(\mathbb{R}^d,\mu_\kappa),\ 0<p\leqslant\infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p_\kappa(\mathbb{R}^d,\mu_\kappa)} = \left(\int_{\mathbb{R}^d}|f(y)|^p d\mu_\kappa(y)\right)^{\frac{1}{p}},\ \text{if } p>0 \quad \text{and} \quad \|f\|_{L^\infty_\kappa(\mathbb{R}^d,\mu_\kappa)} = ess\sup_{y\in\mathbb{R}^d}|f(y)|.$$

Then for every $f\in L^1_\kappa(\mathbb{R}^d,\mu_\kappa)$, the Dunkl transform of f denoted by $\mathscr{F}_\mathscr{D}(f)$ is defined by

$$\mathscr{F}_\mathscr{D}(f)(x) := c_\kappa\int_{\mathbb{R}^d}E_\kappa(-ix,y)f(y)d\mu_\kappa(y),\ \ x\in\mathbb{R}^d.$$

By using the spherical-polar coordinates $x = ry$, where $y \in S^{d-1}$, we have

$$\int_{\mathbb{R}^d} f(x)d\mu_\kappa(x) = \int_0^\infty \Big(\int_{S^{d-1}} f(ry)w_\kappa(y)d\sigma(y)\Big) r^{d+2\gamma}dr, \tag{3}$$

where $d\sigma$ is the normalized surface measure on the unit sphere S^{d-1} of $\mathbb{R}^d$. From which it follows that the Mehta-type constant c_κ

$$c_\kappa = \Big(\int_{\mathbb{R}^d} e^{-|x|^2/2}d\mu_\kappa(x)\Big)^{-1}.$$

Let us point out that the Dunkl transform coincides with the Euclidean Fourier transform when $\kappa = 0$, whereas if $d = 1$ and $G = \mathbb{Z}_2$ then it is closely related to the Hankel transform on the real line. We list some of the known properties of the Dunkl transform in the following Lemma.

LEMMA 1. [5, 8]

1. *If $f \in L^1_\kappa(\mathbb{R}^d, \mu_\kappa)$, then $\mathscr{F}_\mathscr{D}(f) \in \mathscr{C}(\mathbb{R}^d)$ and $\lim_{\|x\|\to\infty} \mathscr{F}_\mathscr{D}(f)(x) = 0$.*

2. *The Dunkl transform $\mathscr{F}_\mathscr{D}$ is an isomorphism of the Schwartz class $\mathscr{S}(\mathbb{R}^d)$ onto itself, and $\mathscr{F}_\mathscr{D}(f)(x) = f(-x)$.*

3. *The Dunkl transform $\mathscr{F}_\mathscr{D}$ on $\mathscr{S}(\mathbb{R}^d)$ extends uniquely to an isometric isomorphism on $L^2_\kappa(\mathbb{R}^d, \mu_\kappa)$, i.e., $\|f\|_{L^2_\kappa(\mathbb{R}^d,\mu_\kappa)} = \|\mathscr{F}_\mathscr{D}(f)\|_{L^2_\kappa(\mathbb{R}^d,\mu_\kappa)}$.*

4. *Let $f \in L^1_\kappa(\mathbb{R}^d, \mu_\kappa)$. If $\mathscr{F}_\mathscr{D}(f)$ is in $L^1_\kappa(\mathbb{R}^d, \mu_\kappa)$, then we have the following inversion formula*

$$f(x) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(ix, y)\mathscr{F}_\mathscr{D}(f)(y)d\mu_\kappa(y).$$

5. *If $f, g \in L^2_\kappa(\mathbb{R}^d, \mu_\kappa)$, then*

$$\int_{\mathbb{R}^d} \mathscr{F}_\mathscr{D}(f)(y)g(y)d\mu_\kappa(y) = \int_{\mathbb{R}^d} f(y)\mathscr{F}_\mathscr{D}(g)(y)d\mu_\kappa(y).$$

6. *Given $\varepsilon > 0$, let $f_\varepsilon(x) = \varepsilon^{-2-2\gamma}f(\varepsilon^{-1}x)$. Then $\mathscr{F}_\mathscr{D}(f_\varepsilon)(x) = \mathscr{F}_\mathscr{D}(f)(\varepsilon x)$.*

7. *If $f(y) = f_0(\|y\|)$ is radial, then $\mathscr{F}_\mathscr{D}(f)(x) = H_{\frac{d-1}{2}+\gamma-\frac{1}{2}} f_0(\|x\|)$ is again a radial function, where H_α denotes the Hankel transform defined by*

$$H_\alpha g(s) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty g(r)\frac{J_\alpha(rs)}{(rs)^\alpha} r^{2\alpha+1}dr,$$

and J_α denotes the Bessel function of the first kind.

The Dunkl transform shares many other properties with the Fourier transform. Therefore, it is natural to associate a generalized translation operator with this transform. Let $x \in \mathbb{R}^d$. The Dunkl translation $f \longmapsto \tau_x$ is defined on $L^2_\kappa(\mathbb{R}^d, \mu_\kappa)$ by the equation

$$\mathscr{F}_{\mathscr{D}}(\tau_x f)(y) = E_\kappa(ix, y)\mathscr{F}_{\mathscr{D}}(f)(y), \ \ y \in \mathbb{R}^d.$$

It is known that $\tau_y f(x) = \tau_x f(y)$ for a.e. $x \in \mathbb{R}^d$ and a.e. $y \in \mathbb{R}^d$. In general, the operator τ_y is not positive (see for instance, [22, Proposition 3.10]), and it is still an open problem whether $\tau_y f$ can be extended to a bounded operator on $L^1_\kappa(\mathbb{R}^d, \mu_\kappa)$. On the other hand, however, it was shown in [22, Theorem 3.7] that the generalized translation operator τ_y can be extended to all radial functions in $L^p_\kappa(\mathbb{R}^d, \mu_\kappa)$, $1 \leqslant p \leqslant 2$, and $\tau_y : L^p_{rad}(\mathbb{R}^d, \mu_\kappa) \to L^p_\kappa(\mathbb{R}^d, \mu_\kappa)$ is a bounded operator, where $L^p_{rad}(\mathbb{R}^d, \mu_\kappa)$ denotes the space of all radial functions in $L^p_\kappa(\mathbb{R}^d, \mu_\kappa)$. More properties on the generalized translation operator can be found in [17, 18, 22].

3. Hardy-type inequality

The atom decomposition theory of $H^p(\mathbb{R}^d)$ spaces marks an important step of further developments on its real variable theory. Using the grand maximal function, R. Coifman [4] first shows that an element in $H^p(\mathbb{R})$ can decomposed into a sum of a series of basic elements. Then the study on some analytic problems on $H^p(\mathbb{R}^d)$ is summed up to investigate some properties of these basic elements, and therefore the problems because quite simple. These basic elements are called atoms. Let us now make the definition of an atom.

DEFINITION 1. Let $0 < p \leqslant 1 \leqslant q \leqslant \infty$ with $p \neq q$. A function $a(x) \in L^q_\kappa(\mathbb{R}^d, \mu_\kappa)$ is called a (p,q,s)-atom with the center at x_0, if it satisfies the following conditions

(i) Supp $a \subset B(x_0, r)$;

(ii) $\|a\|_{L^q_\kappa(\mathbb{R}^d,\mu_\kappa)} \leqslant \left[\mu_\kappa B(x_0,r)\right]^{\frac{1}{q}-\frac{1}{p}} = Cr^{(2\gamma+d)(\frac{1}{q}-\frac{1}{p})}$;

(iii) $\int_{\mathbb{R}^d} a(y) y^\ell d\mu_\kappa(y) = 0$, for all monomials y^ℓ with $|\ell| \leqslant s$ with $s \geqslant N = \left[(2\gamma+d)(\frac{1}{p}-1)\right]$, where [.] denotes, as usual, the "greatest integer not exceeding" function.

Here, (i) means that an atom must be a function with compact support, (ii) is the size condition of atoms, and (iii) is called the cancelation moment condition. Moreover, $B(x_0, r)$ is the ball centered at x_0 with radius r. Clearly, $a(p,\infty,s)$ atom must be $a(p,q,s)$ atom, if $p < q < \infty$.

REMARK 1. The volume of the ball at point x_0 it is not translation invariant as the measure is a weight times the Lebesgue measure. Only when the weight is a constant it is translation invariant. So in general cases, $a(p,\ q,\ s)$-atom centered at $x_0 \in \mathbb{R}^d$ is defined to be a $L^q_\kappa(\mathbb{R}^d, \mu_\kappa)$ function a on $\mathbb{R}^d$ such that the translation $\tau_{x_0}(a)(x)$ is not $a(p,\ q,\ s)$-atom centered at the origin.

Using the atomic decomposition, we define the Hardy space $H^p_{q,s}(\mathbb{R}^d,\mu_\kappa)$ to be the collection of functions f satisfying $f=\sum_{j=0}^\infty \beta_j a_j$, where a_j are $H^p_{q,s}(\mathbb{R}^d,\mu_\kappa)$-atoms and β_j is a sequence of complex numbers with $\sum_{j=0}^\infty |\beta_j|^p<\infty$. $H^p_{q,s}(\mathbb{R}^d,\mu_\kappa)$ is equipped with a norm as follows

$$\|f\|_{H^p_{q,s}(\mathbb{R}^d,\mu_\kappa)}=\inf\Big\{\sum_{j=0}^\infty |\beta_j|^p\Big\},$$

where the infimum is taken over all atoms decompositions of f. Note that each $H^p_{q,s}(\mathbb{R}^d)$ function has a decomposition into (p,q,s)-atoms, it is natural to compare the spaces of functions admitting decompositions into (p,q,s)-atoms and (p,q',s')-atoms. It was shown in [10] that for each p, these spaces corresponding to different (q,s) all coincide. A well-known that (cf. [20]) for each $p\in(0,1]$ the Dunkl transform $\mathscr{F}_\mathscr{D}(f)$ of $f\in H^p_{q,s}(\mathbb{R}^d,\mu_k)$ is continuous function. Further, the following facts are known. For $f\in\mathscr{S}'(\mathbb{R}^d)$ the Dunkl transform is defined by

$$\langle\mathscr{F}_\mathscr{D}(f),\phi\rangle=\langle f,\mathscr{F}_\mathscr{D}(\phi)\rangle,\quad \phi\in\mathscr{S}(\mathbb{R}^d).$$

Thus the Dunkl transform $\mathscr{F}_\mathscr{D}(f)$, extends to a topological automorphism of $\mathscr{S}'(\mathbb{R}^d)$.

Now we are in a position to give the main result of this paper is the following theorem.

THEOREM 1. *Let $0<p\leqslant 1$, and $N=[(2\gamma+d)(1/p-1)]$, the greatest integer not exceeding $(2\gamma+d)(1/p-1)$. Then for any $f\in H^p_{q,s}(\mathbb{R}^d,\mu_\kappa)$ the Dunkl transform is a continuous function and satisfies the following Hardy's type inequality*

$$\int_{\mathbb{R}^d}\frac{|\mathscr{F}_\mathscr{D}(f)(y)|^p}{|y|^\sigma}d\mu_\kappa(y)\leqslant C\|f\|^p_{H^p_{q,s}(\mathbb{R}^d,\mu_\kappa)}, \tag{4}$$

provide that

$$(2\gamma+d)(2-p)\leqslant\sigma<2\gamma+d+p(N+1) \tag{5}$$

where C is a constant does not depends on f.

REMARKS 1.

1. Note that the collection of all real σ satisfying the condition (5) is a nonempty set since $2\gamma+d+p(N+1)-(2\gamma+d)(2-p)>0$.
2. For the critical case $\sigma_0=(2\gamma+d)(2-p)$ has been extensively studied in [20].
3. It would be interesting to know if this is the best possible improved.

Proof. Let $f=\sum_{j=0}^\infty\beta_j a_j\in H^p_{q,s}(\mathbb{R}^d,\mu_\kappa)$, being element of $H^p_{q,s}(\mathbb{R}^d,\mu_\kappa)$ where a_j are atoms supported by the ball $B(x_0,r)$ centered at point x_0 with radius r. Since $0<p\leqslant 1$, it follows

$$\int_{\mathbb{R}^d}\frac{|\mathscr{F}_\mathscr{D}(f)(y)|^p}{|y|^\sigma}d\mu_\kappa(y)\leqslant C\sum_{j=0}^\infty|\beta_j|^p\int_{\mathbb{R}^d}\frac{|\mathscr{F}_\mathscr{D}(a_j)(y)|^p}{|y|^\sigma}d\mu_\kappa(y).$$

In order to prove the theorem, it is enough to prove,

$$\int_{\mathbb{R}^d} \frac{|\mathscr{F}_{\mathscr{D}}(f)(y)|^p}{|y|^\sigma} d\mu_\kappa(y) \leqslant C. \tag{6}$$

Let us now take ρ an arbitrary nonnegative real number, and decomposing the left hand side of (6) as

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{|\mathscr{F}_{\mathscr{D}}(a_j)(y)|^p}{|y|^\sigma} d\mu_\kappa(y) &= \int_{|y|<\rho} \frac{|\mathscr{F}_{\mathscr{D}}(a_j)(y)|^p}{|y|^\sigma} d\mu_\kappa(y) + \int_{|y|\geqslant\rho} \frac{|\mathscr{F}_{\mathscr{D}}(a_j)(y)|^p}{|y|^\sigma} d\mu_\kappa(y)\\ &:= S_1 + S_2.\end{aligned}$$

To estimate S_1; we may use Taylor's theorem in several variables with integral's remainder for the function $y \longmapsto E_\kappa(ix,y)$, we obtain

$$E_\kappa(ix,y) = \sum_{n=0}^{N} \frac{V_\kappa(\prec iy, . \succ)^n(x)}{n!} + R_{N+1}(x,y),$$

where

$$R_{N+1}(x,y) = \frac{1}{(N+1)!} \int_0^1 (1-t)^{N+1} \Big[\int_{\mathbb{R}^d} \prec iy, z \succ^{N+1} e^{t\prec iy,z\succ} d\Gamma_x(z) \Big] dt,$$

and V_κ is the intertwining operator (see, [7, 18]), defined on $\mathbb{C}[\mathbb{R}^d]$ (the algebra of polynomial functions on $\mathbb{R}^d$) by

$$V_\kappa(p) = \int_{\mathbb{R}^d} f(z) d\Gamma_x(z), \quad x \in \mathbb{R}^d.$$

Since $\int_{\mathbb{R}^d} a_j(y) y^\ell d\mu_\kappa(y) = 0$, for every $|\ell| \leqslant N$ where $N = \Big[(2\gamma+d)(\frac{1}{p}-1)\Big]$, we can write

$$\mathscr{F}_{\mathscr{D}}(a_j)(x) = \int_{B(x_0,r)} \Big[E_\kappa(-ix,y) - \sum_{n=0}^{N} \frac{V_\kappa(\prec iy, . \succ)^n(x)}{n!} \Big] a_j(y) d\mu_\kappa(y), \quad x \in \mathbb{R}^d.$$

Hence, from (2) it follows that

$$|\mathscr{F}_{\mathscr{D}}(a_j)(x)| \leqslant c_\kappa \int_{B(x_0,r)} |R_{N+1}(x,y)||a_j(y)| d\mu_\kappa(y).$$

But it is clear that

$$|R_{N+1}(x,y)| \leqslant \frac{1}{(N+1)!} [|x|.|y|]^{N+1}.$$

Now with the help of properties $(i), (ii)$ and (iii) for $a(p,\infty,s)$-atoms of $H^p_{q,s}(\mathbb{R}^d, \mu_\kappa)$, we get the following results

$$\begin{aligned}|\mathscr{F}_{\mathscr{D}}(a_j)(x)| &\leqslant C \int_{B(x_0,r)} |x|^{N+1} |y|^{N+1} [\mu_\kappa B(x_0,r)]^{-\frac{1}{p}} d\mu_\kappa(y)\\ &\leqslant C r^{N+1+(2\gamma+d)(1-\frac{1}{p})} |x|^{N+1},\end{aligned}$$

where we have used (see, [22, 13]),

$$\mu_\kappa B(x_0,r) = Cr^{2\gamma+d}.$$

Integrating with respect to the measure $d\mu_\kappa$ over the domain $0<|y|<\rho$, we obtain

$$S_1 := \int_{|y|<\rho} \frac{|\mathscr{F}_{\mathscr{D}}(a_j)(y)|^p}{|y|^\sigma} d\mu_\kappa(y) \leqslant Cr^{p(N+2\gamma+d+1)-(2\gamma+d)} \int_{|y|<\rho} |y|^{p(N+1)-\sigma}\, d\mu_\kappa(y)$$
$$\leqslant Cr^{-(2\gamma+d)+p(N+2\gamma+d+1)}\rho^{2\gamma+d+p(N+1)-\sigma}$$

that is

$$S_1 \leqslant Cr^{-(2\gamma+d)+p(N+2\gamma+d+1)}\rho^{2\gamma+d+p(N+1)-\sigma} \tag{7}$$

provide that $\sigma < 2\gamma+d+p(N+1)$ which follows from the inequality (5).

Now to estimate S_2, we may apply Hölder's inequality for $q=\frac{2}{p}$ and Plancherel formula to get

$$\begin{aligned} S_2 &\leqslant \left(\int_{\mathbb{R}^d} \left(|a_j(y)|^p\right)^{\frac{2}{p}} d\mu_\kappa(y)\right)^{\frac{p}{2}} \left(\int_{|y|\geqslant\rho} |y|^{\frac{2\sigma}{p-2}} d\mu_\kappa(y)\right)^{\frac{2-p}{2}} \\ &\leqslant C\|a_j\|^p_{L^2_\kappa(\mathbb{R}^d,\mu_\kappa)} \left(\int_{y\geqslant\rho} |y|^{\frac{2\sigma}{p-2}} d\mu_\kappa(y)\right)^{\frac{2-p}{2}} \\ &\leqslant C\|a_j\|^p_{L^2_\kappa(\mathbb{R}^d,\mu_\kappa)}\, \rho^{\frac{(2\gamma+d)(2-p)}{2}-\sigma}, \end{aligned}$$

provide that $\frac{(2\gamma+d)(2-p)}{2} < \sigma$, which is a consequence of the left hand side of (5). Taking into account that

$$\begin{aligned} \|a_j\|^2_{L^2_\kappa(\mathbb{R}^d,\mu_\kappa)} &= \int_{\mathbb{R}^d} |a_j(y)|^2\, d\mu_\kappa(y) \\ &\leqslant \int_{B(x_0,r)} \left[\mu_\kappa B(x_0,r)\right]^{-\frac{2}{p}} d\mu_\kappa(y) \\ &\leqslant C\, r^{-\frac{(2\gamma+d)(2-p)}{p}}. \end{aligned}$$

We obtain $\|a_j\|^p_{L^2_\kappa(\mathbb{R}^d,\mu_\kappa)} \leqslant C\, r^{-\frac{(2\gamma+d)(2-p)}{2}}$ and hence,

$$S_2 \leqslant C\, r^{-\frac{(2\gamma+d)(2-p)}{2}}\rho^{\frac{(2\gamma+d)(2-p)}{2}-\sigma}. \tag{8}$$

Case 1. If $\sigma_0 = (2\gamma+d)(2-p)$. We put $\rho=\frac{1}{r}$, $\forall\, r>0$, then we have $S_1 \leqslant C$ and $S_2 \leqslant C$.

Case 2. If $(2\gamma+d)(2-p) < \sigma < (2\gamma+d)+p(N+1)$. We shall discuss the cases $0<r<1$ and $r\geqslant 1$.

• Hence, in order to deal with the case $0<r<1,$ we need more precise estimates, so we consider the set Υ_ρ; the collection of all numbers ρ satisfying

$$\Upsilon_\rho=\left\{\rho>0:\frac{(2\gamma+d)(2-p)}{(2\gamma+d)(2-p)-2\sigma}\log(r)\leqslant\log(\rho)\leqslant\frac{(2\gamma+d)-p(N+2\gamma+d+1)}{(2\gamma+d)+p(N+1)-\sigma}\log(r)\right\}. \quad (9)$$

To prove that the collection Υ_ρ above is nonempty set it is enough to prove that

$$\frac{(2\gamma+d)(2-p)}{(2\gamma+d)(2-p)-2\sigma}\times\frac{(2\gamma+d)+p(N+1)-\sigma}{(2\gamma+d)-p(N+1+(2\gamma+d))}\leqslant 1 \quad (10)$$

which is a different formulation of the hand side of (5), that is $(2\gamma+d)(2-p)\leqslant\sigma.$

Using the fact that $(2\gamma+d)+p(N+1)-\sigma>0$ and the right hand side of (9) it follows that

$$S_1\leqslant Cr^{-(2\gamma+d)+p(N+2\gamma+d+1)}\rho^{2\gamma+d+p(N+1)-\sigma}\leqslant C. \quad (11)$$

Using the left hand side of (9) and the fact that $\frac{(2\gamma+d)(2-p)}{2}-\sigma<0,$ we obtain

$$S_2\leqslant C. \quad (12)$$

Combining (11) and (12) we obtain

$$S_1\leqslant C \text{ and } S_2\leqslant C, \text{ for } \quad 0<r<1. \quad (13)$$

• Now, to deal with the case $r\geqslant 1,$ we may take

$$\rho=r^{\frac{2\gamma+d-p(N+1+2\gamma+d)}{2\gamma+d+p(N+1)-\sigma}} \quad (14)$$

so, using the fact that $r\geqslant 1,$ we obtain

$$\rho\leqslant r^{\frac{(2\gamma+d)(2-p)}{(2\gamma+d)(2-p)-2\sigma}}, \quad (15)$$

which leads to

$$S_1\leqslant C \text{ and } S_2\leqslant C, \text{ for } r\geqslant 1. \quad (16)$$

Hence to prove (6), it is enough to combine (13) and (16). The proof of the main theorem is completed.

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REFERENCES

[1] M. ASSAL, *Hardy's type inequality associated with the Hankel transform for over critical exponent*, Integr. Transf. Spec. F., **22** (2011), 45–50.

[2] M. ASSAL AND A. RAHMOUNI, *Hardy's type inequality associated with the Laguerre Fourier transform*, Integr. Transf. Spec. F., **24** (2013), 156–163.

[3] M. ASSAL AND A. RAHMOUNI, *An improved Hardy's inequality associated with the Laguerre Fourier transform*, Collect. Math., (2013), 1–11.

[4] R. R. COIFMAN, *A real-variable characterization of H^p*, Studia Math., **51** (1974), 269–274.
[5] M. F. E. DE JEU, *The Dunkl transform*, Invent. Math., **113** (1993), 147–162.
[6] C. F. DUNKL, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc., **311** (1989), 167–183.
[7] C. F. DUNKL, *Integral kernels with reflection group invariance*, Canad. J. Math., **43** (1991), 1213–1227.
[8] C. F. DUNKL, *Hankel transforms associated to finite reflection groups*, Contemp. Math., **138** (1992), 123–138.
[9] C. FEFFERMAN AND E. M. STEIN, *H^p spaces of several variables*, Acta Math., **129**, (1972), 137–193.
[10] G. B. FOLLAND AND E. M. STEIN, *Hardy spaces on homogeneous groups*, Princeton University Press, Princeton, NJ, 1982.
[11] J. GARCIA-CUERVA AND J. RUBIO DE FRANCIA, *Weighted norm inequalities and related topics*, North Holland, 1985.
[12] J. E. HUMPHREYS, *Reflection groups and coxter groups*, Cambridge Univ. Press, 1990, 1–204.
[13] H. MEJJAOLI, *Littlewood–Paley decomposition associated with the Dunkl operators and paraproduct operators*, J. Inequal. Pure and Appl. Math., **9** (4) (2008), 1–25.
[14] A. RAHMOUNI AND M. ASSAL, *An improved Hardy's inequality associated with the Euclidean Fourier transform*, Acta Mathematica Scientia, **33** B (5) (2013), 1382–1386.
[15] A. RAHMOUNI AND M. ASSAL, *Hardy's type inequality for the critical exponent associated with the inverse Laguerre Fourier transform*, Integr. Transf. Spec. F., (2013) 1–9.
[16] A. RAHMOUNI, M. ASSAL AND M. N. LAZHARI, *An improved Hardy's inequality associated with the dual of Laguerre hypergroup*, (preprint).
[17] M. RÖSLER, *Positivity of Dunkl's intertwining operator*, Duke Math. J., **98** (1999), 445–463.
[18] M. RÖSLER, *A positive radial product formula for the Dunkl kernel*, Trans. Amer. Math. Soc., **355** (2003), 2413–2438.
[19] E. M. STEIN, *Harmonic Analysis, real variable methods, orthogonality and oscillatory integrals*, Princeton Univ. Press, Princeton, NJ, 1993.
[20] F. SOLTANI, *Paley type inequality on the Hardy type space in the Dunkl setting*, Arch. Math., **95** (2010), 35–44.
[21] M. H. TAIBLESON AND G. WEISS, *The molecular characterization of certain Hardy spaces*, Astérisque **77** (1980), Société Math. de France, Paris, 67–149.
[22] S. THANGAVELU AND Y. XU, *Convolution operator and maximal function for the Dunkl transform*, J. Anal. Math., **97** (2005), 25–56.
[23] K. TRIMÈCHE, *The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual*, Integr. Transf. and Special Funct., **12** (2001), 349–374.

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