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# A GLOBAL VERSION OF GEHRING LEMMA IN ORLICZ SPACES ON SPACES OF HOMOGENEOUS TYPE

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*Abstract.* We extend a global version of Gehring lemma in Orlicz spaces, proved by T. Iwaniec, from the Euclidean case to the setting of spaces of homogeneous type.

## 1. Introduction

The classical Gehring lemma revealed the self-improving property of reverse Hölder inequalities, implying the higher integrability of $p-$integrable functions satisfying such an inequality. Gehring lemma is a powerful tool in proving regularity results for solutions of elliptic systems, nonlinear PDEs and for (quasi)minimizers of variational integrals. The original lemma was proved for Lebesgue functions on $X=\mathbb{R}^n$ with Lebesgue measure [6]. In his seminal paper [6], Gehring used this lemma as a tool to show that the Jacobian, which is in $L^1_{loc}$, of a $K-$quasiconformal mapping on $D\subset\mathbb{R}^n$ actually belongs to $L^{1+\varepsilon}_{loc}$ for some $\varepsilon>0$ depending only on $K$ and $n$.

Let us recall the original Gehring lemma ([6], Lemma 3). Suppose that $Q\subset\mathbb{R}^n$ is a cube, $p>1$ and $g\in L^p(Q)$ is a nonnegative function. Assume that there exists a constant $b>1$ such that the following reverse Hölder inequality holds

$$\left(\frac{1}{m(Q')}\int_{Q'} g^p dm\right)^{1/p} \leqslant b\left(\frac{1}{m(Q')}\int_{Q'} g\,dm\right)$$

for every parallel cube $Q'\subset Q$. Then $g\in L^q(Q)$ whenever $q\in[p,p+c)$, for some $c=c(n,p,b)>0$. Moreover,

$$\left(\frac{1}{m(Q)}\int_{Q} g^q dm\right)^{1/q} \leqslant \left(\frac{c}{p+c-q}\right)^{1/q}\left(\frac{1}{m(Q)}\int_{Q} g^p dm\right)^{1/p}$$

Gehring lemma was extended in several directions, by relaxing the conditions on the underlying space $X$ or on the space of functions where $f$ belongs. Various local

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versions of Gehring lemma have been proved for functions in the local Lebesgue space $L^p_{loc}(X)$, where $X = \mathbb{R}^n$ has a measure determined by a doubling weight (Kinnunen [11]), or more generally, $X$ is a doubling metric measure space (Zatorska-Goldstein [19], Maasalo [14]) or $X$ is a space of homogeneous type with $0 < \mu(X) < \infty$ (Gianazza [7]). An elegant proof of Gehring lemma for Lebesgue functions on a doubling metric measure space has been given by Björn and Björn in the monograph [1, Theorem 3.22].

Local versions of Gehring lemma for functions in Orlicz spaces on $X = \mathbb{R}^n$ have been proved by Cianchi and Fusco [3], by Franchi and Serra Casano [5] and others. Gehring lemma in Orlicz spaces plays a key role in the study of higher integrability properties for the gradient of local minimizers of variational integrals with nonstandard growth [3], [5], [2].

The reformulation of Gehring lemma in the general setting of interpolation theory has led to new methods to obtain Gehring type integrability results that extend the scope of the classical self-improving inequalities. Using interpolation theoretical methods, Mastyło and Milman [16], then Martin and Milman [15], proved a global version, respectively a local version, of Gehring lemma, in weighted $L^p$ spaces on $\mathbb{R}^n$ with a non-doubling weight. Mastyło and Milman [16] also considered a new approach to Gehring type results in Orlicz spaces on $\mathbb{R}^n$, based on an Orlicz space extension of the sharp reverse Hardy inequality for decreasing functions in $L^p$ spaces.

Iwaniec proved in [12] a general version of Gehring lemma in Orlicz spaces and highlighted the significance of Gehring lemma in Lebesgue spaces through its applications to regularity results for $A-$harmonic functions and through its connections to maximal inequalities. Further applications of Iwaniec's version of Gehring lemma are given in the monograph [13].

The purpose of this paper is to extend Iwaniec's version of Gehring lemma from the Euclidean setting, where $X = \mathbb{R}^n$ is endowed with Lebesgue measure, to the setting of spaces of homogeneous type. It is well-known that spaces of homogeneous type represent a natural framework in harmonic analysis. Doubling metric measure spaces are examples of spaces of homogeneous type.

In this paper $(X, \rho, \mu)$ is a space of homogeneous type in the sense of Coifman and Weiss [4]. The function $\rho : X \times X \to \mathbb{R}_+$ is a quasi-metric, i.e. $\rho$ is symmetric, $\rho(x,y) = 0$ if and only if $x = y$, and there is a constant $K \geqslant 1$ such that

$$\rho(x,y) \leqslant K[\rho(x,z) + \rho(z,y)]$$

for all $x, y, z \in X$.

For $x \in X$ and $r > 0$, the set $B(x,r) = \{y \in X : \rho(y,x) < r\}$ is called the ball of center $x$ and radius $r$. Every ball comes with a center and a radius, although these are not unique in general. For every ball $B = B(x,r)$ and $\lambda > 0$ we denote $\lambda B = B(x, \lambda r)$.

Here $\mu$ is a Borel regular measure which is positive and finite on balls and is doubling, that is, there is a constant $C_\mu \geqslant 1$ so that

$$\mu(2B) \leqslant C_\mu \mu(B)$$

for all balls $B$ in $X$.

The constants $K$ and $C_\mu$ describing the properties of $\rho$ and $\mu$, respectively, are called the constants of the space of homogeneous type $(X,\rho,\mu)$.

In the following, $\Phi:\mathbb{R}_+\to\mathbb{R}_+$ is a convex function of the form $\Phi(t)=tF(t)$, where $F(t)$ strictly increases from zero to infinity as $t$ runs over the interval $[0,\infty)$ and $F$ satisfies the $\Delta_2-$condition

$$F(2t)\leqslant C_dF(t) \tag{1.1}$$

for some constant $C_d\geqslant 1$ and all $t\geqslant 0$.

We will prove the following extension of Iwaniec's version of Gehring lemma in Orlicz spaces [12, Lemma 3.1].

THEOREM 1. *Let $(X,\rho,\mu)$ be a space of homogeneous type, with constants $K$ and $C_\mu$. Let $g,h\in L^\Phi(X)$ be nonnegative functions satisfying*

$$\Phi^{-1}\left(\frac{1}{\mu(B)}\int_B\Phi(g)\,d\mu\right)\leqslant A\frac{1}{\mu(2B)}\int_{2B}g\,d\mu+\Phi^{-1}\left(\frac{1}{\mu(2B)}\int_{2B}\Phi(h)\,d\mu\right) \tag{1.2}$$

*for every ball $B\subset X$, where the constant $A>1$ is independent of the ball. Assume that $\mu(X)=\infty$. Then there exist positive constants $\varepsilon_0=\varepsilon_0(K,C_\mu,C_d,A)$ and $A'=A'(K,C_\mu,C_d,A,\varepsilon_0)$ such that for each $\varepsilon\in(0,\varepsilon_0]$ we have*

$$\int_X gF^{1+\varepsilon}(g)\,d\mu\leqslant A'\int_X hF^{1+\varepsilon}(h)\,d\mu. \tag{1.3}$$

Actually, it will turn out from the proof of the above theorem that we also can cover the case of spaces with finite measure.

THEOREM 2. *Under the assumptions of Theorem 1, with the exception that $\mu(X)<\infty$ replaces $\mu(X)=\infty$, there exist positive constants $\varepsilon_0=\varepsilon_0(K,C_\mu,C_d,A)$, $A'=A'(K,C_\mu,C_d,A,\varepsilon_0)$ and $A''=A''(K,C_\mu,C_d,A,\varepsilon_0)$ such that for each $\varepsilon\in(0,\varepsilon_0]$ we have*

$$\int_X gF^{1+\varepsilon}(g)\,d\mu\leqslant A'\int_X hF^{1+\varepsilon}(h)\,d\mu+A''F^{\varepsilon}(T+0)\int_X gF(g)\,d\mu. \tag{1.4}$$

*Here $T:=\frac{1}{2A+2}\Phi^{-1}\left(\frac{1}{\mu(X)}\int_X\Phi(g)\,d\mu\right)$.*

Whether or not $X$ has finite measure, the integrability of $hF^{1+\varepsilon}(h)$ over $X$ implies the integrability of $gF^{1+\varepsilon}(g)$ over $X$.

## 2. Preliminary results

The basic covering lemma in spaces of homogeneous types [4, Theorem 1.2] says that, for every family $\mathscr{F}$ of balls that is a covering of a bounded set $E\subset X$ and every

$\lambda > 4K$, there exists a countable family $\mathscr{G} \subset \mathscr{F}$ of pairwise disjoint balls, such that $\{\lambda B : B \in \mathscr{G}\}$ is also a covering of $E$.

For a locally integrable real-valued function $f$ in $X$, define the noncentered maximal function $\mathscr{M}^*(f)$ of $f$ by

$$\mathscr{M}^*(f)(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f| \, d\mu,$$

where the supremum is taken over all balls $B \subset X$ containing $x$.

The noncentered maximal operator $\mathscr{M}^*$ maps $L^1(X)$ to weak-$L^1(X)$, as has been proved in [4, Theorem 2.1]. Thus, there exists a constant $C$, depending only on the constants $K$ and $C_\mu$ of the space $X$, such that for all $t > 0$ we have

$$\mu(\{x \in X : \mathscr{M}^*(f)(x)\}) \leqslant \frac{C}{t} \int_X |f| \, d\mu. \tag{2.1}$$

We can take $C = (C_\mu)^{1+\log_2 \lambda}$, where $\lambda > 4K$ is arbitrary.

The following version of Calderón-Zygmund decomposition lemma is from [4, Theorem 2.2], see also [9, Lemma 2.9]. Note that the assumption that $f$ has bounded support can be removed, see [4, page 70].

LEMMA 1 *Let $f \in L^1(X)$ be a nonnegative function and $t > 0$ such that $t > \frac{1}{\mu(X)} \int_X f \, d\mu$ if $\mu(X) < \infty$. Let $C$ be the constant from (2.1). There exists a countable family of balls $\{B_i : i \geqslant 1\}$ satisfying the following conditions:*

*(1) $f(x) \leqslant tC$ for almost every $x \in X \setminus \bigcup_{i \geqslant 1} B_i$;*

*(2) $t \leqslant \frac{1}{\mu(B_i)} \int_{B_i} f \, d\mu \leqslant tC_2$ for each $i \geqslant 1$;*

*(3) There is a positive integer $M$ such that an arbitrary point in $X$ cannot belong to more than $M$ balls from the family $\{B_i : i \geqslant 1\}$;*

*(4) $\sum_{i=1}^{\infty} \mu(B_i) \leqslant \frac{M}{t} \int_X f \, d\mu$.*

*The constants $C_2 > 0$ and $M$ depend only on the constants $K$ and $C_\mu$ of $X$.*

Note that we can take $C_2 = (C_\mu)^{1+\log_2 3K}$, while the positive integer $M$ can be chosen such that $M \leqslant (C_\mu)^{(1+\log_2(4K^3+5K^2))(1+\log_2(2K^2+K))}$. The family $\{B_i : i \geqslant 1\}$ from the previous lemma is called a Calderón-Zygmund decomposition of $X$ for $f$, at the level $t$.

Suppose that the function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$, $\Phi(t) = tF(t)$ satisfies the conditions from the Introduction. The $\Delta_2-$condition implies the estimate

$$F(\lambda t) \leqslant C_d \lambda^{\log_2 C_d} F(t)$$

for every $\lambda \geqslant 1$ and $t \geqslant 0$ [18, Lemma 2.7]. Note that $\Phi(\lambda t) \leqslant \lambda \Phi(t)$ for all $0 \leqslant \lambda \leqslant 1$ and $t \geqslant 0$, since $\Phi$ is convex and $\Phi(0) = 0$. Since $\Phi$ also satisfies a doubling condition, a function $g$ belongs to the Orlicz space $L^{\Phi}(X)$ if and only if $\Phi(|g|)$ is integrable over $X$.

$\Phi$ is a strictly increasing surjective function and its inverse $\Phi^{-1}$ is subadditive [18, Lemma 2.9]. We recall the inequality

$$\Phi^{-1}(v) \leqslant t + \frac{v}{F(t)} \tag{2.2}$$

for $t > 0$ and $v \geqslant 0$, that is equivalent to the obvious inequality $u \cdot F(t) \leqslant t \cdot F(t) + u \cdot F(u)$, where $u = \Phi^{-1}(v)$.

In the following $\{f > t\}$ stands for $\{x \in X : f(x) > t\}$, where $f : X \to \overline{\mathbb{R}}$ and $t \in \mathbb{R}$.

The next lemma extends the result of a basic step from the proof of Lemma 14.3.1 from [13].

LEMMA 2 *Let $g, h \in L^{\Phi}(X)$ be nonnegative functions satisfying (1.2) for every ball $B \subset X$, where the constant $A > 1$ is independent of the ball. Then there exist some positive constants $\alpha$, $\beta$ and $t_0$ such that*

$$\int_{\{g>t\}} \Phi(g)\, d\mu \leqslant \alpha F(t) \int_{\{g>t\}} g\, d\mu + \beta \int_{\{h>t\}} \Phi(h)\, d\mu \tag{2.3}$$

*for all $t \geqslant t_0$. Here $\alpha > 1$ and $\beta > 0$ depend only on the constants $K$, $C_\mu$, $C_d$ and on $A$. We may take an arbitrary $t_0 > 0$ if $\mu(X) = \infty$, or $t_0 > \frac{1}{2A+2}\Phi^{-1}\left(\frac{1}{\mu(X)} \int_X \Phi(g)\, d\mu\right)$ if $\mu(X) < \infty$.*

*Proof.* Denote $G_t = \{g > t\}$ and $H_t = \{h > t\}$ for $t \geqslant 0$.
Let $s > 0$. If $\mu(X) < \infty$ we assume in addition that

$$\Phi(s) > \frac{1}{\mu(X)} \int_X \Phi(g)\, d\mu. \tag{2.4}$$

We consider a Calderón-Zygmund decomposition $\{B_i : i \geqslant 1\}$ of $X$ for the function $f = \Phi(g)$, at the level $\Phi(s)$.

By Lemma 1 (1), there is a set $E \subset X$ with $\mu(E) = 0$ such that $\Phi(g(x)) \leqslant \Phi(s)$ for all $x \in X \setminus \left(\bigcup_{i \geqslant 1} B_i \cup E\right)$. Then $G_s \subset \bigcup_{i \geqslant 1} B_i \cup E$. By Lemma 1 (3), we get $\int_{G_s} \Phi(g)\, d\mu \leqslant M \sum_{i=1}^{\infty} \int_{B_i} \Phi(g)\, d\mu$. But $\int_{B_i} \Phi(g)\, d\mu \leqslant C_2 \Phi(s)\, \mu(B_i)$ for each $i \geqslant 1$, by the second inequality in Lemma 1 (2). It follows that

$$\int_{G_s} \Phi(g)\, d\mu \leqslant C_2 M \Phi(s) \sum_{i=1}^{\infty} \mu(B_i). \tag{2.5}$$

We will use the assumption (1.2) in order to estimate $\sum_{i=1}^{\infty}\mu(B_i)$ by means of $\mu\left(\bigcup_{i\geqslant 1}2B_i\right)$. Since the balls $B_i$ have bounded overlap, by Lemma 1 (3) it follows that

$$\sum_{i=1}^{\infty}\mu(B_i)\leqslant M\mu\left(\bigcup_{i\geqslant 1}B_i\right)\leqslant M\mu\left(\bigcup_{i\geqslant 1}2B_i\right). \tag{2.6}$$

We have $\Phi(s)\leqslant\frac{1}{\mu(B_i)}\int_{B_i}\Phi(g)\,d\mu$ for each $i\geqslant 1$, by the first inequality in Lemma 1 (2). Using (1.2) we get

$$s\leqslant A\frac{1}{\mu(2B_i)}\int_{2B_i}g\,d\mu+\Phi^{-1}\left(\frac{1}{\mu(2B_i)}\int_{2B_i}\Phi(h)\,d\mu\right). \tag{2.7}$$

Let $t>0$. Taking into account that $g\leqslant t$ on $2B_i\setminus G_t$ we get

$$\frac{1}{\mu(2B_i)}\int_{2B_i}g\,d\mu\leqslant t+\frac{1}{\mu(2B_i)}\int_{2B_i\cap G_t}g\,d\mu.$$

Similarly, taking into account the subadditivity of $\Phi^{-1}$ , it follows that

$$\Phi^{-1}\left(\frac{1}{\mu(2B_i)}\int_{2B_i}\Phi(h)\,d\mu\right)\leqslant t+\Phi^{-1}\left(\frac{1}{\mu(2B_i)}\int_{2B_i\cap H_t}\Phi(h)\,d\mu\right). \tag{2.8}$$

From (2.8) and (2.2) we get

$$\Phi^{-1}\left(\frac{1}{\mu(2B_i)}\int_{2B_i}\Phi(h)\,d\mu\right)\leqslant 2t+\frac{1}{F(t)}\frac{1}{\mu(2B_i)}\int_{2B_i\cap H_t}\Phi(h)\,d\mu.$$

Combining (2.7) with the latter inequality we obtain

$$s\leqslant(A+2)t+A\frac{1}{\mu(2B_i)}\int_{2B_i\cap G_t}g\,d\mu+\frac{1}{F(t)}\frac{1}{\mu(2B_i)}\int_{2B_i\cap H_t}\Phi(h)\,d\mu, \tag{2.9}$$

for all $t>0$.

Now fix $t=\frac{s}{2A+2}$. If $\mu(X)<\infty$ the assumption (2.4) becomes

$$t>\frac{1}{2A+2}\Phi^{-1}\left(\frac{1}{\mu(X)}\int_X\Phi(g)\,d\mu\right).$$

For $t=\frac{s}{2A+2}$ inequality (2.9) yields

$$\mu(2B_i)\leqslant\frac{1}{t}\int_{2B_i\cap G_t}g\,d\mu+\frac{1}{A}\cdot\frac{1}{\Phi(t)}\int_{2B_i\cap H_t}\Phi(h)\,d\mu. \tag{2.10}$$

Using the basic covering lemma we extract from $\{2B_i : i \geqslant 1\}$ a subfamily of pairwise disjoint balls $\{2B_{i_k} : k \geqslant 1\}$ such that $\bigcup_{i\geqslant 1} 2B_i \subset \bigcup_{k\geqslant 1} 2\lambda B_{i_k}$. Here $\lambda > 4K$ is a constant. It follows that $\mu\left(\bigcup_{i\geqslant 1} 2B_i\right) \leqslant \mu\left(\bigcup_{k\geqslant 1} 2\lambda B_{i_k}\right) \leqslant \sum_{k=1}^{\infty} \mu\left(2\lambda B_{i_k}\right) \leqslant C_3 \sum_{k=1}^{\infty} \mu\left(2B_{i_k}\right)$, where $C_3 = \left(C_\mu\right)^{1+\log_2 \lambda}$.

Taking account of (2.10) this implies

$$\mu\left(\bigcup_{i\geqslant 1} 2B_i\right) \leqslant C_3 \frac{1}{t} \sum_{k=1}^{\infty} \int_{2B_{i_k}\cap G_t} g d\mu + \frac{C_3}{A} \cdot \frac{1}{\Phi(t)} \sum_{k=1}^{\infty} \int_{2B_{i_k}\cap H_t} \Phi(h)\, d\mu,$$

hence

$$\mu\left(\bigcup_{i\geqslant 1} 2B_i\right) \leqslant C_3 \frac{1}{t} \int_{G_t} g d\mu + \frac{C_3}{A} \cdot \frac{1}{\Phi(t)} \int_{H_t} \Phi(h)\, d\mu. \tag{2.11}$$

Now, from (2.5), (2.6) and (2.11) we obtain

$$\int_{G_s} \Phi(g)\, d\mu \leqslant C_2 M^2 \Phi(s) \left( C_3 \frac{1}{t} \int_{G_t} g d\mu + \frac{C_3}{A} \cdot \frac{1}{\Phi(t)} \int_{H_t} \Phi(h)\, d\mu \right), \tag{2.12}$$

where $t = \frac{s}{2A+2}$.

By the $\Delta_2$-condition we have $F(s) \leqslant (C_d)^{\log_2(2A+2)} F(t)$. Denote $C_4 = (C_d)^{\log_2(2A+2)}$. Then $\frac{\Phi(s)}{t} = \frac{s}{t} F(s) \leqslant (2A+2) C_4 F(t)$ and $\frac{\Phi(s)}{\Phi(t)} \leqslant (2A+2) C_4$.

On the other hand,

$$\int_{G_t\setminus G_s} \Phi(g)\, d\mu \leqslant F(s) \int_{G_t\setminus G_s} g d\mu \leqslant F(s) \int_{G_t} g d\mu.$$

Adding this inequality to (2.12) and using the above estimates, we obtain the claim (2.3), for $t \geqslant t_0$, where $\alpha = C_4 \left[2C_2C_3M^2(A+1)+1\right]$ and $\beta = 2\left(1+\frac{1}{A}\right) C_2C_3C_4M^2$. Note that $\alpha > 1$ and $\beta > 0$.

Here $t_0 > 0$ is arbitrary if $\mu(X) = \infty$, but if $\mu(X) < \infty$ we have to assume that

$$t_0 > \frac{1}{2A+2} \Phi^{-1} \left( \frac{1}{\mu(X)} \int_X \Phi(g)\, d\mu \right).$$

In order to transform integrals of the form $\int_{\{t_0 < f \leqslant t_1\}} \Psi(f)\, d\nu$ into Lebesgue integrals on intervals in $\mathbb{R}$, we use the following consequence of Fubini's theorem, that gives Lemma 3.1 from [11] in the case $\Psi(t) = t^p$, with $p \in [1, \infty)$.

LEMMA 3 *Let* $\Psi:\mathbb{R}_+\to\mathbb{R}_+$ *be a strictly increasing function with* $\lim_{\varepsilon\searrow 0}\Psi(\varepsilon)=0$ *and* $\lim_{t\to\infty}\Psi(t)=\infty$*, which is absolutely continuous on* $(0,\infty)$*. Let* $\nu$ *be a measure on* $X$ *and* $E\subset X$ *be a set with* $\nu(E)<\infty$*. If* $f$ *is a nonnegative* $\nu$*-measurable function on* $E$ *and* $0\leqslant t_0<t_1<\infty$*, then*

$$\int\limits_{\{t_0<f\leqslant t_1\}}\Psi(f)\,d\nu=\int\limits_{t_0}^{t_1}\Psi'(t)\,\nu(\{f>t\})\,dt-\Psi(t)\,\nu(\{f>t\})\big|_{t_0}^{t_1}.$$

The following elementary lemma is proved for the sake of completness.

LEMMA 4 *Let* $\Phi:\mathbb{R}_+\to\mathbb{R}_+$ *be a convex increasing function of the form* $\Phi(t)=tF(t)$ *and* $p>0$*. Then* $F^p$ *is Lipschitz continuous on every interval* $[a,b]\subset(0,\infty)$*.*

*Proof.* Being convex on $\mathbb{R}_+$, the function $\Phi$ is Lipschitz continuous on every compact interval contained in $(0,\infty)$. Let $[a,b]\subset(0,\infty)$. We have

$$L_{\Phi,[a,b]}:=\sup\left\{\frac{|\Phi(t_1)-\Phi(t_2)|}{|t_1-t_2|}:t_1\neq t_2\text{ in }[a,b]\right\}<\infty.$$

First we prove that $F$ is Lipschitz continuous on $[a,b]$. Indeed, given $t_1,t_2\in[a,b]$ we have $|F(t_1)-F(t_2)|=\left|\frac{1}{t_1}(\Phi(t_1)-\Phi(t_2))+\frac{t_2-t_1}{t_1t_2}\Phi(t_2)\right|$, hence

$$|F(t_1)-F(t_2)|\leqslant\frac{1}{a}\left(L_{\Phi,[a,b]}+\frac{1}{a}\Phi(b)\right)|t_1-t_2|.$$

We used the fact that $\Phi$ is increasing and nonnegative. Then $F$ is $L_{F,[a,b]}$-Lipschitz continuous on $[a,b]$, where $L_{F,[a,b]}:=\frac{1}{a}\left(L_{\Phi,[a,b]}+\frac{1}{a}\Phi(b)\right)$.

Let $p>0$ with $p\neq1$. For $a\leqslant x<y\leqslant b$ we have $0<\frac{y^p-x^p}{y-x}<p\max\left\{a^{p-1},b^{p-1}\right\}$. Thus, $F^p$ is $L-$Lipschitz continuous on $[a,b]$, where $L:=p\max\left\{a^{p-1},b^{p-1}\right\}L_{F,[a,b]}$.

## 3. Proof of the main result

Let $t_0>0$. If $\mu(X)<\infty$, we assume that $t_0>\frac{1}{2A+2}\Phi^{-1}\left(\frac{1}{\mu(X)}\int\limits_X\Phi(g)\,d\mu\right)$.

For $\varepsilon>0$ denote $J(\varepsilon)=\int\limits_X gF^{1+\varepsilon}(g)\,d\mu$. Write $J(\varepsilon)=J_1(\varepsilon)+J_2(\varepsilon)$, where $J_1(\varepsilon):=\int\limits_{X\setminus G_{t_0}}gF^{1+\varepsilon}(g)\,d\mu$ and $J_2(\varepsilon):=\int\limits_{G_{t_0}}gF^{1+\varepsilon}(g)\,d\mu$.

We have the trivial estimate

$$J_1(\varepsilon)\leqslant F^{\varepsilon}(t_0)\int\limits_{X\setminus G_{t_0}}gF(g)\,d\mu\leqslant F^{\varepsilon}(t_0)\int\limits_X gF(g)\,d\mu. \tag{3.1}$$

Denote $I_g(\varepsilon,t_1)=\int\limits_{G_{t_0}\setminus G_{t_1}} gF^{1+\varepsilon}(g)\,d\mu$ and $I_h(\varepsilon,t_1)=\int\limits_{H_{t_0}\setminus H_{t_1}} hF^{1+\varepsilon}(h)\,d\mu$. By Lebesgue monotone convergence theorem, we have $J_2(\varepsilon)=\lim\limits_{t_1\to\infty} I_g(\varepsilon,t_1)$.

We will consider the measures $\nu$, $\pi$ and $\omega$ such that $d\nu=\Phi(g)\,d\mu$, $d\pi=g\,d\mu$ and $d\omega=\Phi(h)\,d\mu$. By Lemma 2 we have

$$\nu(G_t)\leqslant \alpha F(t)\,\pi(G_t)+\beta\,\omega(H_t) \tag{3.2}$$

for every $t\geqslant t_0$.

We have $I_g(\varepsilon,t_1)=\int\limits_{G_{t_0}\setminus G_{t_1}} F^{\varepsilon}(g)\,d\nu$. By Lemma 4, the function $F^{\varepsilon}$ is Lipschitz continuous on every compact interval contained in $(0,\infty)$, hence $F^{\varepsilon}$ is absolutely continuous on $(0,\infty)$. According to Lemma 3, we can write $I_g(\varepsilon,t_1)=\int\limits_{t_0}^{t_1}(F^{\varepsilon})'(t)\,\nu(G_t)\,dt - F^{\varepsilon}(t)\,\nu(G_t)\big|_{t_0}^{t_1}$. But $(F^{\varepsilon})'(t)=\varepsilon F^{\varepsilon-1}(t)F'(t)$ for a.e. $t\in[t_0,t_1]$. Using (3.2) we get for all $t\geqslant t_0$

$$(F^{\varepsilon})'(t)\,\nu(G_t)\leqslant \alpha\frac{\varepsilon}{\varepsilon+1}\left(F^{1+\varepsilon}\right)'(t)\,\pi(G_t)+\beta\,(F^{\varepsilon})'(t)\,\omega(H_t).$$

The occurence of $F^{1+\varepsilon}$ in the right hand side of the above inequality is the key to the self-improving phenomenon occuring here .

Using the previous inequality and applying Lemma 3 twice more we get

$$\begin{aligned} I_g(\varepsilon,t_1)\leqslant{}& -F^{\varepsilon}(t)\,\nu(G_t)\big|_{t_0}^{t_1}+\frac{\alpha\varepsilon}{\varepsilon+1}\int\limits_{G_{t_0}\setminus G_{t_1}} F^{1+\varepsilon}(g)\,d\pi+\frac{\alpha\varepsilon}{\varepsilon+1}\,F^{1+\varepsilon}(t)\,\pi(G_t)\big|_{t_0}^{t_1}+\\ &\beta\int\limits_{H_{t_0}\setminus H_{t_1}} F^{\varepsilon}(h)\,d\omega+\beta\,F^{\varepsilon}(t)\,\omega(H_t)\big|_{t_0}^{t_1}. \end{aligned}$$

Note that $\int\limits_{G_{t_0}\setminus G_{t_1}} F^{1+\varepsilon}(g)\,d\pi=I_g(\varepsilon,t_1)$ and $\int\limits_{H_{t_0}\setminus H_{t_1}} F^{\varepsilon}(h)\,d\omega=I_h(\varepsilon,t_1)$. The previous inequality writes as

$$\begin{aligned} \left(1-\frac{\alpha\varepsilon}{\varepsilon+1}\right)I_g(\varepsilon,t_1)\leqslant{}& \beta I_h(\varepsilon,t_1)+F^{\varepsilon}(t_1)\left[\frac{\alpha\varepsilon}{\varepsilon+1}F(t_1)\,\pi(G_{t_1})-\nu(G_{t_1})\right]+\\ & F^{\varepsilon}(t_0)\left[\nu\left(G_{t_0}\right)-\frac{\alpha\varepsilon}{\varepsilon+1}F(t_0)\,\pi\left(G_{t_0}\right)\right]+\beta\,F^{\varepsilon}(t)\,\omega(H_t)\big|_{t_0}^{t_1}. \end{aligned} \tag{3.3}$$

Let $\varepsilon>0$ such that $\frac{\alpha\varepsilon}{\varepsilon+1}<1$, i.e. $\varepsilon<1/(\alpha-1)$.

Then

$$\frac{\alpha\varepsilon}{\varepsilon+1}F(t_1)\,\pi(G_{t_1})-\nu(G_{t_1})\leqslant F(t_1)\,\pi(G_{t_1})-\nu(G_{t_1})=\int\limits_{G_{t_1}} g\,[F(t_1)-F(g)]\,d\mu,$$

hence

$$\frac{\alpha\varepsilon}{\varepsilon+1}F(t_1)\,\pi(G_{t_1})-\nu(G_{t_1})\leqslant 0. \tag{3.4}$$

By (3.2) for $t = t_0$ we get $\nu\left(G_{t_0}\right) - \frac{\alpha\varepsilon}{\varepsilon+1} F(t_0)\,\pi\left(G_{t_0}\right) \leqslant \frac{1}{\varepsilon+1}\nu\left(G_{t_0}\right) + \frac{\beta\varepsilon}{\varepsilon+1}\omega\left(H_{t_0}\right)$, therefore

$$\begin{aligned} &F^{\varepsilon}(t_0)\left[\nu\left(G_{t_0}\right) - \frac{\alpha\varepsilon}{\varepsilon+1}F(t_0)\,\pi\left(G_{t_0}\right)\right] + \beta\, F^{\varepsilon}(t)\,\omega(H_t)\Big|_{t_0}^{t_1} \\ &\leqslant \frac{1}{\varepsilon+1}F^{\varepsilon}(t_0)\left[\nu\left(G_{t_0}\right) - \beta\omega\left(H_{t_0}\right)\right] + \beta F^{\varepsilon}(t_1)\,\omega\left(H_{t_1}\right) \\ &\leqslant \frac{\alpha}{\varepsilon+1}F^{1+\varepsilon}(t_0)\,\pi\left(G_{t_0}\right) + \beta F^{\varepsilon}(t_1)\,\omega\left(H_{t_1}\right). \end{aligned} \tag{3.5}$$

From (3.3), (3.4) and (3.5) it follows that

$$I_g(\varepsilon, t_1) \leqslant \frac{\alpha}{1+\varepsilon-\alpha\varepsilon}F^{1+\varepsilon}(t_0)\,\pi\left(G_{t_0}\right) + \frac{\beta(1+\varepsilon)}{1+\varepsilon-\alpha\varepsilon}\left[I_h(\varepsilon, t_1) - F^{\varepsilon}(t_1)\,\omega\left(H_{t_1}\right)\right]. \tag{3.6}$$

It remains to let $t_1$ tend to infinity in (3.6). If $\int_X hF^{1+\varepsilon}(h)\,d\mu = \infty$, then the claim of the theorem is obvious. Assume that $\int_X hF^{1+\varepsilon}(h)\,d\mu < \infty$. We prove that

$$\lim_{t_1\to\infty} F^{\varepsilon}(t_1)\,\omega\left(H_{t_1}\right) = 0. \tag{3.7}$$

Since every locally integrable function $f$ is dominated a.e. by its noncentered maximal function $\mathscr{M}^* f$, we have the inclusion $H_{t_1} \subset \left\{\mathscr{M}^*\left(hF^{1+\varepsilon}(h)\right) > t_1 F^{1+\varepsilon}(t_1)\right\}$. By (2.1), this implies

$$\mu\left(H_{t_1}\right) \leqslant \frac{C}{t_1 F^{1+\varepsilon}(t_1)} \int_X hF^{1+\varepsilon}(h)\,d\mu.$$

Since $F^{1+\varepsilon}(t_1)\,\omega\left(H_{t_1}\right) \leqslant \int_{H_{t_1}} hF^{1+\varepsilon}(h)\,d\mu$, using the absolute continuity of the integral, the integrability of $hF^{1+\varepsilon}(h)$ and the fact that $\lim_{t_1\to\infty}\mu\left(H_{t_1}\right) = 0$, we obtain $\lim_{t_1\to\infty} F^{1+\varepsilon}(t_1)\,\omega\left(H_{t_1}\right) = 0$. Then we also have (3.7), since there is $\tau_1 > 0$ such that $F(t) \geqslant 1$ for every $t \geqslant \tau_1$.

Letting $t_1$ tend to infinity in (3.6) and using (3.1) we obtain

$$\int_X gF^{1+\varepsilon}(g)\,d\mu \leqslant \frac{1+\varepsilon+\alpha-\alpha\varepsilon}{1+\varepsilon-\alpha\varepsilon}F^{\varepsilon}(t_0)\int_X gF(g)\,d\mu + \frac{\beta(1+\varepsilon)}{1+\varepsilon-\alpha\varepsilon}\int_X hF^{1+\varepsilon}(h)\,d\mu. \tag{3.8}$$

If $\mu(X) = \infty$, then we let $t_0$ tend to zero in (3.8), as we may. Taking into account that $F(0) = 0$ we get (1.3) with $A' = \frac{\beta(1+\varepsilon)}{1+\varepsilon-\alpha\varepsilon}$.

Fix any $\varepsilon_0 < 1/(\alpha-1)$. Since the constants $\alpha$ and $\beta$ depend only on $K$, $C_\mu$, $C_d$ and $A$, the same holds true for $\varepsilon_0$. For every $\varepsilon \in (0, \varepsilon_0]$ we have $\frac{\beta(1+\varepsilon)}{1+\varepsilon-\alpha\varepsilon} \leqslant \frac{\beta(1+\varepsilon_0)}{1+\varepsilon_0-\alpha\varepsilon_0}$, consequently (1.3) holds for $A' = \frac{\beta(1+\varepsilon_0)}{1+\varepsilon_0-\alpha\varepsilon_0}$.

If $\mu(X) < \infty$, we let $t_0$ tend to $T := \frac{1}{2A+2}\Phi^{-1}\left(\frac{1}{\mu(X)}\int_X \Phi(g)\,d\mu\right)$ from the right in (3.8), hence we get (1.4) with $A'' = \frac{1+\varepsilon_0+\alpha-\alpha\varepsilon_0}{1+\varepsilon_0-\alpha\varepsilon_0}$.

REMARK 1 *If $\mu(X) = \infty$, the only function $g \in L^{\Phi}(X)$ satisfying the reverse Jensen inequality $\Phi^{-1}\left(\frac{1}{\mu(B)}\int_B \Phi(g)\,d\mu\right) \leqslant A\frac{1}{\mu(2B)}\int_{2B} g d\mu$ for all balls $B \subset X$ is the zero function.*

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