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ON THE UPPER AND LOWER ESTIMATES OF NORMS IN VARIABLE EXPONENT SPACES

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Abstract. In the present paper we investigate some geometrical properties of the norms in Banach function spaces. Particularly there is shown that if exponent $1/p(\cdot)$ belongs to $BLO^{1/\log}$ then for the norm of corresponding variable exponent Lebesgue space we have the following lower estimate

$$\left\|\sum \chi_Q \|f\chi_Q\|_{p(\cdot)}/\|\chi_Q\|_{p(\cdot)}\right\|_{p(\cdot)} \leqslant C\|f\|_{p(\cdot)}$$

where $\{Q\}$ defines disjoint partition of $[0;1]$. Also we have constructed variable exponent Lebesgue space with above property which does not possess following upper estimation

$$\|f\|_{p(\cdot)} \leqslant C\left\|\sum \chi_Q \|f\chi_Q\|_{p(\cdot)}/\|\chi_Q\|_{p(\cdot)}\right\|_{p(\cdot)}.$$

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ and let $\mathscr{M}$ be the space of all equivalence classes of Lebesgue measurable real-valued functions endowed with the topology of convergence in measure relative to each set of finite measure.

DEFINITION 1. A Banach subspace X of $\mathscr{M}$ is called a Banach function space (BFS) on Ω if

1) the norm $\|f\|_X$ is defined for every measurable function f and $f \in X$ if and only if $\|f\|_X < \infty$. $\|f\|_X = 0$ if and only if $f = 0$ a.e.;

2) $\||f|\|_X = \|f\|_X$ for all $f \in X$;

3) if $0 \leqslant f \leqslant g$ a.e., then $\|f\|_X \leqslant \|g\|_X$;

4) if $0 \leqslant f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$;

5) if E is measurable subset of Ω such that $|E| < \infty$, (below we denote the Lebesgue measure of E by $|E|$) then $\|\chi_E\|_X < \infty$;

6) for every measurable set E, $|E| < \infty$, there is a constant $C_E < \infty$ such that $\int_E f(t)dt \leqslant C_E\|f\|_X$.

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Given a BFS X, its associate space X' is defined by

$$X' = \left\{ g : \int_\Omega |f(x)g(x)|dx < \infty \text{ for all } f \in X \right\}$$

and endowed with the associate norm

$$\|f\|_{X'} = \sup\left\{ \int_\Omega |f(x)g(x)|dx \,:\, \|g\|_X \leqslant 1 \right\}.$$

An immediate consequence of this definition is the generalized Hölder's inequality: for all $f \in X$ and $g \in X'$,

$$\left| \int_\Omega f(x)g(x)dx \right| \leqslant \|f\|_X \|g\|_{X'}.$$

Furthermore, X' is also a BFS on Ω and $(X')' = X$. The associate space of X is closed norming subspace of the dual space X^*, and equality

$$\|f\|_X = \sup\left\{ \int_\Omega |f(x)g(x)|dx \,:\, \|g\|_{X'} \leqslant 1 \right\}$$

holds for all $f \in X$ (see [1]).

Given a Banach function space X, define the scale of spaces X^r, $0 < r < \infty$,

$$X^r = \{ f \in \mathscr{M} \,:\, |f|^r \in X \},$$

with the "norm"

$$\|f\|_{X^r} = \||f|^r\|_X^{1/r}.$$

If $r \geqslant 1$, then $\|\cdot\|_{X^r}$ is again an actual norm and X^r is a Banach function space. However, if $r < 1$, need not be a Banach function space. The simple example is the scale of Lebesgue spaces: if $X = L^p(\Omega)$, $(1 \leqslant p < \infty)$, then $(L^p)^r = L^{pr}$, and so X^r is a Banach space only for $r \geqslant 1/p$.

Let $\mathfrak{I}$ be some fixed family of sequences $\mathscr{Q} = \{Q_i\}$ of disjoint measurable subsets of Ω, $|Q_i| > 0$ such that $\Omega = \cup_{Q_i \in \mathscr{Q}} Q_i$. We ignore the difference in notation caused by a null set.

Everywhere in the sequel $l_{\mathscr{Q}}$ is a Banach sequential space (BSS), meaning that axioms 1)-6) from definition 1 are satisfied with respect to the count measure. Let $e_k = e_{Q_k}$ denote the standard unit vectors in $l_{\mathscr{Q}}$.

Kopaliani in [11] introduced notions of uniformly upper (lower) l-estimates.

DEFINITION 2. 1) Let $l = \{l_{\mathscr{Q}}\}_{\mathscr{Q} \in \mathfrak{I}}$ be a family of BSSs. A BFS X is said to satisfy a uniformly upper l-estimate if there exists a constant $C < \infty$ such that for every $f \in X$ and $\mathscr{Q} \in \mathfrak{I}$ we have

$$\|f\|_X \leqslant C \left\| \sum_{Q_i \in \mathscr{Q}} e_i \|f\chi_{Q_i}\|_X \right\|_{l_{\mathscr{Q}}}.$$

2) BFS X is said to satisfy uniformly lower l-estimate if there exists a constant $C<\infty$ such that for every $f\in X$ and $\mathscr{Q}\in\Im$ we have

$$\|f\|_X \geqslant C\left\|\sum_{Q_i\in\mathscr{Q}} e_i\|f\chi_{Q_i}\|_X\right\|_{l_{\mathscr{Q}}}.$$

Note that if in Definition 2 for all $\mathscr{Q}\in\Im$, we take one discrete Lebesgue space l_p, $(1\leqslant p<\infty)$, we obtain classical definition of upper and lower p-estimates of Banach spaces (see [17], [8]). The existence of upper or lower p-estimates in the Banach spaces is of great interest in study of the structure of the space (see [16]). Berezhnoi [2, 3] investigate uniformly upper (lower) l-estimates of BFS, when discrete $l_{\mathscr{Q}}$ spaces for all partition of Ω coincides to some discrete BSS.

DEFINITION 3. A pair of BFSs (X,Y) is said to have property $G(\Im)$ if there exists a constant $C>0$ such that

$$\sum_{Q_i\in\mathscr{Q}} \|f\chi_{Q_i}\|_X\cdot\|g\chi_{Q_i}\|_{Y'} \leqslant C\cdot\|f\|_X\cdot\|g\|_{Y'}$$

for any $\mathscr{Q}\in\Im$ and every $f\in X$, $g\in Y'$.

Definition 3 was introduced by Berezhnoi [3]. Let us remark that a pair $(L^p(\Omega),L^q(\Omega))$ possesses the property $G(\Im)$ if $p\leqslant q$.

The connections between the property $G(\Im)$ and uniformly upper (lower) l-estimates of BFS-s was investigated in paper [11].

THEOREM 1. ([11]) *Let (X,Y) be a pair of BFSs. Then the following assertions are equivalent:*

1) *The pair (X,Y) of BFSs possesses property $G(\Im)$.*

2) *There is a family $l=\{l_{\mathscr{Q}}\}_{\mathscr{Q}\in\Im}$ of BSSs such that X satisfies uniformly lower l-estimate and Y satisfies uniformly upper l-estimate.*

THEOREM 2. ([11]) *The pair (X,X) of BFSs possesses property $G(\Im)$ if and only if there exist constants $C_1,C_2>0$ such that for every $f\in X$ and $\mathscr{Q}\in\Im$ we have*

$$C_1\|f\|_X \leqslant \left\|\sum_{Q\in\mathscr{Q}} \frac{\|f\chi_Q\|_X}{\|\chi_Q\|_X}\chi_Q\right\|_X \leqslant C_2\|f\|_X. \tag{1.1}$$

Note that the (1.1) type inequalities is very important for studying the boundedness properties of operators of harmonic analysis in variable Lebesgue spaces (see [4], [7]).

DEFINITION 4. We say that BFS X has property $G'(\Im)$ (property $G''(\Im)$) if there exists constant C_1 $(C_2>0)$ such that for every $f\in X$ and $\mathscr{Q}\in\Im$ we have

$$\left\|\sum_{Q\in\mathscr{Q}} \frac{\|f\chi_Q\|_X}{\|\chi_Q\|_X}\chi_Q\right\|_X \leqslant C_1\|f\|_X,\quad \left(\|f\|_X\leqslant C_2\left\|\sum_{Q\in\mathscr{Q}} \frac{\|f\chi_Q\|_X}{\|\chi_Q\|_X}\chi_Q\right\|_X\right). \tag{1.2}$$

The idea of (1.2) type inequalities are to generalize the following property of the Lebesgue norm

$$\|f\|_{L^p} = \left\| \sum_i \frac{\|f\chi_{\Omega_i}\|_{L^p}}{\|\chi_{\Omega_i}\|_{L^p}} \chi_{\Omega_i} \right\|_{L^p},$$

where Ω_i is disjoint measurable partition of Ω.

Below everywhere by $\mathfrak{I}$ we denote the family of all sequences (may be finite) $\{Q_i\}$ disjoint intervals from $[0;1]$. Assume that sets like $[0;a)$ and $(b;1]$ are also intervals. In this paper we will use G, G', G'' notations.

The aim of our paper is to investigate the property G' (property G'') for variable Lebesgue spaces $L^{p(\cdot)}[0;1]$. We have described the class of exponents, for which the correspondent variable exponent Lebesgue spaces has property G' (property G''). Also we have constructed variable exponent Lebesgue space with property G' (G''), which does not possess G'' (G') property. Recently some properties of projection operators in variable Lebesgue spaces were investigated in [9].

Given a function $f \in L^1[0;1]$. Let define its BMO modulus by

$$\gamma(f,r) = \sup_{|Q| \leqslant r} \frac{1}{|Q|} \int_Q |f(x) - f_Q| dx, \ 0 < r \leqslant 1,$$

where the supremum is taken over all intervals of $[0;1]$. We say that $f \in BMO^{1/\log}$ if $\gamma(f,r) \leqslant C/\log(e+1/r)$ and $f \in VMO^{1/\log}$ if $\gamma(f,r)\log(e+1/r) \to 0$ as $r \to 0$.

Given a function $f \in L^1[0;1]$. Let define its BLO modulus by

$$\eta(f,r) = \sup_{|Q| \leqslant r} (f_Q - \operatorname*{essinf}_{x \in Q} f(x)), \ 0 < r \leqslant 1,$$

where the supremum is taken over all intervals of $[0;1]$. We say that $f \in BLO^{1/\log}$ if $\eta(f,r) \leqslant C/\log(e+1/r)$. More about these classes see in [14].

Particularly we will proof following theorems:

THEOREM 3. *Let for exponent $p(\cdot)$ we have $1/p(\cdot) \in BLO^{1/\log}$, $1 \leqslant p_- \leqslant p_+ < \infty$. Then the space $L^{p(\cdot)}[0;1]$ has property G'.*

THEOREM 4. *Let for exponent $p(\cdot), 1 \leqslant p_- \leqslant p_+ < \infty$ we have $1/p(\cdot) \in BLO^{1/\log}$. Then there exists c such that the space $L^{(p(\cdot)+c)'}[0;1]$ has property G''.*

THEOREM 5. 1) *There exists exponent $p(\cdot), 1 \leqslant p_- \leqslant p_+ < \infty$ such that $1/p(\cdot) \in BLO^{1/\log}$ and $L^{p(\cdot)}[0;1]$ has property G' but does not have property G''.*
2) *There exists exponent $p(\cdot), 1 \leqslant p_- \leqslant p_+ < \infty$ such that $1/p(\cdot) \in BLO^{1/\log}$ and $L^{p(\cdot)}[0;1]$ has property G'' but does not have property G'.*

2. Some remarks on properties G' and G''

In this section we will discuss relations between G' and G'' properties for BFS X and its associate space. $\mathfrak{I}$ denotes the family of all sequences of disjoint intervals.

DEFINITION 5. ([3]) Let X be a BFS. We say that for BFS X is fulfilled condition A if there exists constant $C > 0$ such that, for all interval $Q \subset [0;1]$

$$\|\chi_Q\|_X \cdot \|\chi_Q\|_{X'} \leqslant C \cdot |Q|$$

THEOREM 6. *Let BFS X has property G' and for X fulfilled condition A. Then associate space of X has property G''.*

Proof. Let $\mathcal{Q} = \{Q_1, Q_2, ...\}$ denotes some partition of $[0;1]$. Let $g \in X'$ and $f \in X$ such that $\|f\|_X \leqslant 1$. Using Hölders inequality and A condition we conclude that $|Q| \asymp \|\chi_Q\|_X \cdot \|\chi_Q\|_{X'}$. Using this fact and property G' we obtain

$$\begin{aligned}
\int_{[0;1]} |f(x)g(x)|dx &= \sum_k \int_{Q_k} |f(x)g(x)|dx \leqslant \sum_k \|f\chi_{Q_k}\|_X \cdot \|g\chi_{Q_k}\|_{X'} \\
&\leqslant C_1 \int\limits_{[0;1]} \sum_k \frac{\|f\chi_{Q_k}\|_X}{\|\chi_{Q_k}\|_X} \frac{\|g\chi_{Q_k}\|_{X'}}{\|\chi_{Q_k}\|_{X'}} \chi_{Q_k} dx \\
&\leqslant C_1 \left\| \sum_k \frac{\|f\chi_{Q_k}\|_X}{\|\chi_{Q_k}\|_X} \chi_{Q_k} \right\|_X \left\| \sum_k \frac{\|g\chi_{Q_k}\|_{X'}}{\|\chi_{Q_k}\|_{X'}} \chi_{Q_k} \right\|_{X'} \\
&\leqslant C_2 \left\| \sum_k \frac{\|g\chi_{Q_k}\|_{X'}}{\|\chi_{Q_k}\|_{X'}} \chi_{Q_k} \right\|_{X'} .
\end{aligned}$$

Consequently X' possess G'' property. □

Note that if BFS X has property G (in the meaning of (X,X) has the property $G(\mathfrak{I})$) then for X' we have property G without condition A (see [11]).

DEFINITION 6. Let $\mathcal{Q} \in \mathfrak{I}$. We define the averaging operator with respect to $\mathcal{Q}$ by

$$T_{\mathcal{Q}} f(x) = \sum_i |f|_{Q_i} \chi_{Q_i}(x)$$

where $|f|_Q$ denotes the average of $|f|$ on Q.

THEOREM 7. *Let BFS X has property G'' and the averaging operators $T_{\mathcal{Q}} : X \to X$, $\mathcal{Q} \in \mathfrak{I}$ are uniformly bounded. Then associate space of X has property G'.*

Proof. Let $g \in X$ is nonnegative function such that $||g||_X \leqslant 1$. For any $\varepsilon > 0$ and i we choose nonnegative function $h_i \in X$ such that $||h_i||_X \leqslant 1$ and $||f\chi_{Q_i}||_{X'} \leqslant$

$(1+\varepsilon)\int_{Q_i} fh_i$. Note that uniformly boundedness of the averaging operator implies condition A for space X (see [3]). So by property G'' and Hölder inequality we get

$$\begin{aligned}
\int\limits_{[0;1]} g(x)\sum_i \frac{||f\chi_{Q_i}||_{X'}}{||\chi_{Q_i}||_{X'}}\chi_{Q_i}(x)dx &\leqslant \int\limits_{[0;1]} g(x)\sum_i \frac{(1+\varepsilon)\int_{Q_i} f(t)h_i(t)dt}{||\chi_{Q_i}||_{X'}}\chi_{Q_i}(x)dx \\
&= (1+\varepsilon)\int_{[0;1]} f(t)\sum_i \frac{h_i(t)\int_{Q_i} g(x)dx}{||\chi_{Q_i}||_{X'}}dt \\
&\leqslant (1+\varepsilon)||f||_{X'}\left\|\sum_i \frac{h_i(\cdot)\int_{Q_i} g(x)dx}{||\chi_{Q_i}||_{X'}}\chi_{Q_i}\right\|_X \\
&\leqslant (1+\varepsilon)||f||_{X'}\left\|\sum_i \frac{||h_i||_X\int_{Q_i} g(x)dx}{||\chi_{Q_i}||_X||\chi_{Q_i}||_{X'}}\chi_{Q_i}\right\|_X \\
&\leqslant C_1(1+\varepsilon)||f||_{X'}\left\|\sum_i \frac{\chi_{Q_i}}{|Q_i|}\int_{Q_i} g(x)dx\right\|_X \\
&\leqslant C_2(1+\varepsilon)||f||_{X'}||g||_X \\
&\leqslant C_2(1+\varepsilon)||f||_{X'}.
\end{aligned}$$

By the fact that ε is arbitrary small we conclude that X' has property G'. □

Note that if $0<r<\infty$ then for any $f\in X$ we have $\|f\|_X = \|f^{1/r}\|^r_{X^r}$ and the inequalities in definition 4 can be written in following form

$$\left\|\sum_{Q\in\mathscr{Q}} \frac{\|f^{1/r}\chi_Q\|_{X^r}}{\|\chi_Q\|_{X^r}}\chi_Q\right\|^r_{X^r} \leqslant C_1\|f^{1/r}\|^r_{X^r},$$

$$\|f^{1/r}\|^r_{X^r} \leqslant C_2\left\|\sum_{Q\in\mathscr{Q}} \frac{\|f^{1/r}\chi_Q\|_{X^r}}{\|\chi_Q\|_{X^r}}\chi_Q\right\|^r_{X^r}.$$

Consequently if BFS has property G' (G''), then the “norms” $\|\cdot\|_{X^r}$ $(0<r<\infty)$ have also property G' (G'').

3. Variable Lebesgue spaces

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ and the corresponding variable exponent Sobolev spaces $W^{k,p(\cdot)}$ are of interest for their applications to the problems in fluid dynamics, partial differential equations with non-standard growth conditions, calculus of variations, image processing and etc (see [7]).

Given a measurable function $p:[0;1]\to[1;+\infty)$, $L^{p(\cdot)}[0;1]$ denotes the set of measurable functions f on $[0;1]$ such that for some $\lambda>0$

$$\int_{[0;1]}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx<\infty.$$

This set becomes a Banach function spaces when equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_{[0;1]} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leqslant 1\right\}.$$

For the given $p(\cdot)$, the conjugate exponent $p'(\cdot)$ is defined pointwise $p'(x) = p(x)/(p(x)-1)$, $x \in [0;1]$. Given a set $Q \subset [0;1]$ we define some standard notations:

$$p_-(Q) := \operatorname*{essinf}_{x\in Q} p(x), \quad p_+(Q) := \operatorname*{esssup}_{x\in Q} p(x), \quad p_- := p_-([0;1]), \quad p_+ := p_+([0;1]).$$

In the notation introduced above, an exponent $p(\cdot)$, $1 \leqslant p_- \leqslant p_+ < \infty$, the associate space of $L^{p(\cdot)}[0;1]$ contains all measurable functions f such that

$$\|f\|'_{(L^{p(\cdot)})'} = \sup\left\{\int_{[0;1]} |f(x)g(x)|dx : g \in L^{p(\cdot)}[0;1],\, \|g\|_{p(\cdot)} \leqslant 1\right\} < \infty.$$

Note that in this case the associate space of $L^{p(\cdot)}[0;1]$ is equal to $L^{p'(\cdot)}[0;1]$, and $\|\cdot\|'_{(L^{p(\cdot)})'}$ and $\|\cdot\|_{p'(\cdot)}$ are equivalent norms (see [4], [7]). We have also

$$\int_{[0;1]} |f(x)g(x)|dx \leqslant C\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}, \; f \in L^{p(\cdot)}[0;1],\; g \in L^{p'(\cdot)}[0;1].$$

Conversely for all $f \in L^{p(\cdot)}[0;1]$

$$\|f\|_{p(\cdot)} \leqslant C \sup \int_{[0;1]} |f(x)g(x)|dx,$$

where the supremum is taken over all $g \in L^{p'(\cdot)}[0;1]$ such that $\|g\|_{p'(\cdot)} \leqslant 1$.

Given exponent $p(\cdot)$, $1 \leqslant p_- \leqslant p_+ < \infty$ and a Lebesgue measurable function f define the modular associated with $p(\cdot)$ on the set $E \subset [0;1]$ by

$$\rho_{p(\cdot),E} f = \int_E |f(x)|^{p(x)} dx.$$

In case of constant exponents, the L^p norm and the modular differ only by an exponent. In the variable Lebesgue spaces their relationship is more subtle as the next result shows (see [4], [7]).

PROPOSITION 8. *Given exponent $p(\cdot)$, suppose $1 \leqslant p_- \leqslant p_+ < \infty$. Let E measurable subset of $[0;1]$. Then:*

(1) $\|f\chi_E\|_{p(\cdot)} = 1$ *if and only if* $\rho_{p(\cdot),E} f = 1$;
(2) *if* $\rho_{p(\cdot),E} f \leqslant C$, *then* $\|f\chi_E\|_{p(\cdot)} \leqslant \max(C^{1/p_-(E)}, C^{1/p_+(E)})$;
(3) *if* $\|f\|_{p(\cdot)} \leqslant C$, *then* $\rho_{p(\cdot),E} f \leqslant \max(C^{p_+(E)}, C^{p_-(E)})$.

The next result is a necessary and sufficient condition for the embedding $L^{q(\cdot)}[0;1] \subset L^{p(\cdot)}[0;1]$ (see [4], [7]).

PROPOSITION 9. *Given the exponents* $p(\cdot), q(\cdot)$, *suppose* $1 \leqslant p_- \leqslant p_+ < \infty, 1 \leqslant q_- \leqslant q_+ < \infty$. *Then* $L^{q(\cdot)}[0;1] \subset L^{p(\cdot)}[0;1]$ *if and only if* $p(\cdot) \leqslant q(\cdot)$ *almost everywhere. Furthermore, in this case we have*

$$\|f\|_{p(\cdot)} \leqslant 2\|f\|_{q(\cdot)}.$$

For our results we need to impose some regularity on the exponent function $p(\cdot)$. The most important condition, one widely used in the study of variable Lebesgue spaces, is log-Hölder continuity. Let $C^{1/\log}$ denotes the set of exponents $p:[0;1] \to [1,+\infty)$ with log-Hölder condition

$$|(p(x)-p(y))\ln|x-y|| \leqslant C,\ x,y \in [0;1], x \neq y. \tag{3.1}$$

For Lebesgue integrable function f define Hardy-Littlewood maximal function

$$Mf(x) = \sup_{x \in Q} |f|_Q,$$

where supremum is taken over all $Q \subset [0;1]$ intervals containing point x and f_Q denotes the average of function f on Q. Let by $\mathscr{B}$ denote set of all exponents $p(\cdot)$ for which Hardy-Littlewood maximal operator is bounded on the space $L^{p(\cdot)}[0;1]$. Diening [5] proved a key consequence of log-Hölder continuity of $p(\cdot)$. If $1 < p_-$ and $p(\cdot) \in C^{1/\log}$, then $p(\cdot) \in \mathscr{B}$. For an overview about subject we refer to the monographs [4], [7].

Kopaliani [11] proved that if exponent $p(\cdot)$ satisfies log-Hölder conditions then the pair of BFSs $(L^{p(\cdot)}[0;1], L^{p(\cdot)}[0;1])$ has property G. Note that there are another classes of exponents $p(\cdot)$ such that pair of BFSs $(L^{p(\cdot)}[0;1], L^{p(\cdot)}[0;1])$ has property G. For instance, if exponent $p(\cdot)$ is absolutely continuous on $[0;1]$, then the pair of BFSs $(L^{p(\cdot)}[0;1], L^{p(\cdot)}[0;1])$ has property G (see [13]). Note also that there exists continuous exponent on $[0;1]$ such that the pair of BFSs $(L^{p(\cdot)}[0;1], L^{p(\cdot)}[0;1])$ does not have property G (see [11]).

The class $BMO^{1/\log}$ is very important for investigation of exponents from $\mathscr{B}$.

THEOREM 10. ([15], [10]) *Let* $p:[0;1] \to [1;+\infty)$, *then*
1) *if* $p(\cdot) \in \mathscr{B}$, *then* $1/p(\cdot) \in BMO^{1/\log}$;
2) *if* $p(\cdot) \in VMO^{1/\log}$, *then* $p(\cdot) \in \mathscr{B}$;
3) *if* $p(\cdot) \in BMO^{1/\log}$, *then there exists* c *such that* $p(\cdot)+c \in \mathscr{B}$.

4. Proof of results

Proof of theorem 3. We begin with some auxiliary estimations.

LEMMA 1. *Let* $p(\cdot)$ *be a exponent on* $[0;1]$ *with* $1 \leqslant p_- \leqslant p_+ < \infty$. *Then for all* $t \geqslant 0$ *and* $Q \subset [0;1]$ *interval*

$$\frac{1}{|Q|}\int_Q t^{p(x)}dx \geqslant e^{2(p_-(Q)-p_+(Q))} t^{\overline{p}_Q}, \tag{4.1}$$

where $\overline{p}_Q$ *is defined as* $\frac{1}{\overline{p}_Q} = \frac{1}{|Q|}\int_Q \frac{1}{p(x)}dx$.

This lemma is proved in [6] (see Lemma 4.1) in case of $1 < p_- \leqslant p_+ < \infty$, but analogously may be proved in presented case. If in (4.1) we take $t = \frac{1}{\|\chi_Q\|_{p(\cdot)}}$, we obtain

$$\|\chi_Q\|_{p(\cdot)} \geqslant C_1 |Q|^{(1/p(\cdot))_Q}, \tag{4.2}$$

for some constant $C_1 > 0$.

Now assume that $1/p(\cdot) \in BLO^{1/\log}$, then there exists C_2 such that

$$\begin{aligned} |Q|^{(1/p)_Q} &= |Q|^{\frac{1}{|Q|}\int\limits_Q \frac{1}{p(x)}dx - \frac{1}{p_+(Q)} + \frac{1}{p_+(Q)}} \\ &\geqslant |Q|^{\frac{C}{\ln(e+1/|Q|)} + \frac{1}{p_+(Q)}} \geqslant C_2 \cdot |Q|^{\frac{1}{p_+(Q)}}. \end{aligned} \tag{4.3}$$

From (4.2) and (4.3) we obtain

$$C_3 \cdot |Q|^{1/p_+(Q)} \leqslant ||\chi_Q||_{p(\cdot)} \leqslant C_4 \cdot |Q|^{1/p_+(Q)}. \tag{4.4}$$

Let $\mathscr{Q} = \{Q_1, Q_2, \dots\}$ denotes some partition of $[0;1]$. Define on $[0;1]$ function $\tilde{p}(\cdot)$ in following way: $\tilde{p}(x) = p_+(Q_i)$ when $x \in Q_i$.

Without restriction of generality let consider the case when $||f||_{p(\cdot)} = 1$. By Proposition 8 $\int_0^1 |f(x)|^{p(x)}dx = 1$. Then we only need to prove that

$$\left\| \sum_i \frac{||f\chi_{Q_i}||_{p(\cdot)}}{||\chi_{Q_i}||_{p(\cdot)}} \chi_{Q_i}(x) \right\|_{p(\cdot)} \leqslant C.$$

By Proposition 8 we have

$$||f\chi_{Q_i}||_{p(\cdot)} \leqslant \left(\int\limits_{Q_i} |f(x)|^{p(x)}dx \right)^{1/p_+(Q_i)}. \tag{4.5}$$

Then by (4.4) and (4.5)

$$\begin{aligned} \int_{[0;1]} \left(\sum_i \frac{||f\chi_{Q_i}||_{p(\cdot)}}{||\chi_{Q_i}||_{p(\cdot)}} \chi_{Q_i}(x) \right)^{\tilde{p}(x)} dx &= \sum_i \int_{Q_i} \left(\frac{||f\chi_{Q_i}||_{p(\cdot)}}{||\chi_{Q_i}||_{p(\cdot)}} \right)^{p_+(Q_i)} \chi_{Q_i}(x)dx \\ &= \sum_i |Q_i| \left(\frac{||f\chi_{Q_i}||_{p(\cdot)}}{||\chi_{Q_i}||_{p(\cdot)}} \right)^{p_+(Q_i)} \\ &\leqslant \sum_i |Q_i| \frac{\int\limits_{Q_i} |f(x)|^{p(x)}dx}{C_1 |Q_i|} \\ &= \frac{1}{C_1} \int_{[0;1]} |f(x)|^{p(x)}dx = \frac{1}{C_1}. \end{aligned}$$

Consequently we obtain

$$\left\|\sum_i \frac{||f\chi_{Q_i}||_{p(\cdot)}}{||\chi_{Q_i}||_{p(\cdot)}}\chi_{Q_i}(x)\right\|_{\tilde{p}(\cdot)} \leqslant C.$$

Using the fact that $p(x) \leqslant \tilde{p}(x), x \in [0;1]$ and proposition 9 we obtain desired result. □

Proof of theorem 4. The proof of theorem can be obtained from analogous arguments as in proof of theorem 3. But we will obtain this proof from more general proposition.

Consider exponent $p(\cdot)$ such that $1/p(\cdot) \in BLO^{1/\log}$, then by theorem 10 there exists a constant c such $p(\cdot)+c \in \mathscr{B}$. Using theorem 6 and theorem 3 we obtain desired result. □

Proof of theorem 5. Let us show that the function

$$f(x) = \begin{cases} \ln\ln(1/x) \text{ if } x \in (0, e^{-1}]; \\ 0 \text{ if } x \in (e^{-1}, 1], \end{cases}$$

belongs to $BLO^{1/\log}$.

Let $(a;b) \subset [0;1]$. Without loss of generality assume that $0 \leqslant a < b \leqslant e^{-1}$. On $(a;b]$ define the function

$$h(x) = \int_a^x \ln\ln(1/t)dt - (x-a)\ln\ln(1/x) - \frac{2(x-a)}{\ln(1/(x-a))}.$$

We have

$$h'(x) = \frac{x-a}{x\ln(1/x)} - 2\cdot\frac{\ln(1/(x-a))+1}{(\ln(1/(x-a)))^2},\ a < x \leqslant b.$$

Since the function $x\ln(1/x)$ on $(0;1)$ is increasing

$$(\ln(1/(x-a)))^2(x-a) - 2x\ln(1/x)(\ln(1/(x-a))+1)$$
$$\leqslant \ln\frac{1}{x-a}\left((x-a)\ln\frac{1}{x-a} - 2x\ln\frac{1}{x}\right) \leqslant -x\cdot\ln\frac{1}{x}\cdot\ln\frac{1}{x-a} < 0.$$

This means that function h is decreasing. From monotonicity of h and $h(a+) = 0$ follows thats

$$\int_a^b \ln\ln(1/x)dx - (b-a)\ln\ln(1/b) - \frac{2(b-a)}{\ln(1/(b-a))} \leqslant 0.$$

By the last inequality we get

$$\frac{1}{b-a}\int_a^b \ln\ln(1/x)dx - \ln\ln(1/b) \leqslant \frac{4}{\ln(e+1/(b-a))}, \tag{4.6}$$

and consequently $f \in BLO^{1/\log}$.

Note that the function f is a classical example discontinuous functions from $BMO^{1/\log}$ (see [18]). From the well-known observation that a Lipschitz function preserves mean oscillations it follows that the function $\sin(f(x))$ provides an example of a discontinuous bounded function from $BMO^{1/\log}$. Lerner [15] proved that if $p(x) = p_0 + \mu \sin(f(x))$, $x \in [0;1]$ where $p_0 > 0$ and μ sufficiently close to 0, then Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}[0;1]$. It is unknown whether $p(\cdot) \in BLO^{1/\log}$. Bellow we will construct a bounded function (some sense analogous of $\sin(f(x))$) which belongs to $BLO^{1/\log}$.

Let $d_n = e^{-e^n}$, $n \in \{0\} \cup \mathbb{N}$ and $c_0 = 2/e$, $c_{2n+1} = c_{2n} - (d_n - d_{n+1})$, $c_{2n+2} = c_{2n+1} - (d_n - d_{n+1})$, $n \in \{0\} \cup \mathbb{N}$. Let us show that the non-negative bounded function

$$g(x) = \begin{cases} \ln\ln \frac{1}{c_{2n}+c_{2n+2}-x-d_n} - n \text{ if } x \in (c_{2n+2}; c_{2n+1}], n \in \{0\} \cup \mathbb{N}; \\ \ln\ln \frac{1}{x-d_n} - n \text{ if } x \in (c_{2n+1}; c_{2n}], n \in \{0\} \cup \mathbb{N}; \\ 0 \text{ if } x \in (2/e, 1]. \end{cases}$$

belongs to $BLO^{1/\log}$ i.e. for all $(a;b) \subset [0;1]$ we have

$$\frac{1}{b-a} \int_{(a;b)} g(x)dx - \inf_{x \in (a;b)} g(x) \leqslant \frac{C}{\ln(e+1/(b-a))}. \tag{4.7}$$

Note that $g(c_{2n}) = 0$, $g(c_{2n+1}) = 1$, $n \in \{0\} \cup \mathbb{N}$ and on each set $[c_{2n+1}; c_{2n}]$ function g is strictly monotonic and continuous.

Let $(a;b) \subset [0;1]$, without lose of generality suppose that $b \leqslant 2/e$. Consider three cases:

Case 1. At least one point c_{2n} belongs to interval $(a;b)$, where $n \in \{0\} \cup \mathbb{N}$;

Case 2. Interval $(a;b)$ contains only one point like c_{2n+1}, where $n \in \{0\} \cup \mathbb{N}$;

Case 3. Interval $(a;b)$ does not contain point c_n for any $n \in \{0\} \cup \mathbb{N}$.

Define $m_a = \sup\{k : a \leqslant c_k\}$, $m_b = \min\{k : c_k \leqslant b\}$. Note that if $a > 0$ then $m_a = \max\{k : a \leqslant c_k\}$ and $m_a = \infty$ if $a = 0$.

Consider case 1. Suppose that $m_a < \infty$, define $m'_a = \max\{k : a \leqslant c_k \wedge g(c_k) = 0\}$ and $m'_b = \min\{k : c_k \leqslant b \wedge g(c_k) = 0\}$. It is clear that $c_{m_a} \leqslant c_{m'_a} \leqslant c_{m'_b} \leqslant c_{m_b}$. We have

$$\begin{aligned} \frac{1}{b-a} \int_{(a;b)} g(x)dx - \inf_{x \in (a;b)} g(x) &= \frac{1}{b-a} \int_{(a;b)} g(x)dx \\ &= \frac{1}{b-a} \left(\int_a^{c_{m'_a}} + \int_{c_{m'_a}}^{c_{m'_b}} + \int_{c_{m'_b}}^{b} \right) g(x)dx \\ &= A_1 + A_2 + A_3. \end{aligned} \tag{4.8}$$

Let $c_{m'_a} < c_{m'_b}$. Using the fact that $g(2c_{2k+1} - x) = g(x)$ when $x \in [c_{2k+2}; c_{2k+1}]$

we get

$$(b-a)A_2=\int_{c_{m'_a}}^{c_{m'_b}} g(x)dx=\sum_{k=m'_b/2}^{(m'_a-2)/2}\int_{c_{2k+2}}^{c_{2k}} g(x)dx=\sum_{k=m'_b/2}^{(m'_a-2)/2}\left(\int_{c_{2k+2}}^{c_{2k+1}}+\int_{c_{2k+1}}^{c_{2k}}\right)g(x)dx$$

$$=\sum_{k=m'_b/2}^{(m'_a-2)/2} 2\int_{c_{2k+1}}^{c_{2k}} g(x)dx=2\sum_{k=m'_b/2}^{(m'_a-2)/2}\int_{c_{2k+1}}^{c_{2k}}\left(\ln\ln\frac{1}{x-d_k}-k\right)dx.$$

Note that by $c_{2k}-d_k=d_k$ and $c_{2k+1}-d_k=d_{k+1}$ we have

$$(b-a)A_2=2\sum_{k=m'_b/2}^{(m'_a-2)/2}\int_{d_{k+1}}^{d_k}\left(\ln\ln\frac{1}{t}-k\right)dt\leqslant 2\sum_{k=m'_b/2}^{(m'_a-2)/2}\int_{d_{k+1}}^{d_k}\left(\ln\ln\frac{1}{t}-\frac{m'_b}{2}\right)dt$$

$$=2\sum_{k=m'_b/2}^{(m'_a-2)/2}\int_{d_{k+1}}^{d_k}\left(\ln\ln\frac{1}{t}-\ln\ln\frac{1}{d_{m'_b/2}}\right)dt=2\int_{d_{m'_a/2}}^{d_{m'_b/2}}\left(\ln\ln\frac{1}{t}-\ln\ln\frac{1}{d_{m'_b/2}}\right)dt.$$

Now by the following estimation $b-a>(b-a)/2\geqslant d_{m'_b/2}-d_{m'_a/2}$ and by (4.6) we have

$$A_2\leqslant\frac{1}{d_{m'_b/2}-d_{m'_a/2}}\int_{d_{m'_a/2}}^{d_{m'_b/2}}\left(\ln\ln\frac{1}{t}-\ln\ln\frac{1}{d_{m'_b/2}}\right)dt \tag{4.9}$$

$$\leqslant\frac{4}{\ln(e+1/(d_{m'_b/2}-d_{m'_a/2}))}\leqslant\frac{4}{\ln(e+1/(b-a))}.$$

If $m_a=\infty$ then

$$A_2\leqslant\frac{1}{d_{m'_b/2}-0}\int_0^{d_{m'_b/2}}\left(\ln\ln\frac{1}{t}-\ln\ln\frac{1}{d_{m'_b/2}}\right)dt\leqslant\frac{4}{\ln(e+1/(b-a))}. \tag{4.10}$$

Consider A_1. Let $c_{m_a}=c_{m'_a}$. Since $c_{m'_a}-d_{m'_a/2}=d_{m'_a/2}$ and using (4.6) we get

$$A_1=\frac{1}{b-a}\int_a^{c_{m'_a}}\left(\ln\ln\frac{1}{x-d_{m'_a/2}}-\frac{m'_a}{2}\right)dx \tag{4.11}$$

$$=\frac{1}{b-a}\int_{a-d_{m'_a/2}}^{c_{m'_a}-d_{m'_a/2}}\left(\ln\ln\frac{1}{t}-\frac{m'_a}{2}\right)dx$$

$$\leqslant\frac{4}{\ln(e+1/(c_{m'_a}-a))}\leqslant\frac{4}{\ln(e+1/(b-a))}.$$

Let $c_{m_a} \neq c_{m'_a}$ then $m_a = m'_a + 1$ and $g(c_{m_a}) = 1$. Since $c_{m'_a} - d_{m'_a/2} = d_{m'_a/2}$ we get

$$A_1 \leqslant \frac{2}{b-a} \int_{c_{m_a}}^{c_{m'_a}} \left(\ln\ln \frac{1}{x - d_{m'_a/2}} - \frac{m'_a}{2} \right) dx \tag{4.12}$$
$$= \frac{2}{b-a} \int_{c_{m_a} - d_{m'_a/2}}^{c_{m'_a} - d_{m'_a/2}} \left(\ln\ln \frac{1}{t} - \ln\ln \frac{1}{d_{m'_a/2}} \right) dx$$
$$\leqslant \frac{8}{\ln(e + 1/(b-a))}.$$

Consider A_3. Let $c_{m_b} = c_{m'_b}$. Since $c_{m_b-2} - d_{(m_b-2)/2} = d_{(m_b-2)/2}$ we get

$$A_3 = \frac{1}{b-a} \int_{c_{m_b}}^{b} \left(\ln\ln \frac{1}{c_{m_b} + c_{m_b-2} - x - d_{(m_b-2)/2}} - \frac{m_b - 2}{2} \right) dx \tag{4.13}$$
$$= \frac{1}{b-a} \int_{c_{m_b} + c_{m_b-2} - b - d_{(m_b-2)/2}}^{c_{m_b-2} - d_{(m_b-2)/2}} \left(\ln\ln \frac{1}{t} - \frac{m_b - 2}{2} \right) dt$$
$$\leqslant \frac{4}{\ln(e + 1/(b-a))}.$$

If $c_{m_b} \neq c_{m'_b}$ then $m_b = m'_b - 1$ and $g(m_b) = 1$ we have

$$A_3 \leqslant \frac{2}{b-a} \int_{c_{m'_b}}^{c_{m_b}} \left(\ln\ln \frac{1}{c_{m'_b} + c_{m'_b-2} - x - d_{(m'_b-2)/2}} - \frac{m'_b - 2}{2} \right) dx \tag{4.14}$$
$$= \frac{2}{b-a} \int_{c_{m'_b} + c_{m'_b-2} - c_{m_b} - d_{(m'_b-2)/2}}^{c_{m'_b-2} - d_{(m'_b-2)/2}} \left(\ln\ln \frac{1}{t} - \frac{m'_b - 2}{2} \right) dt$$
$$\leqslant \frac{8}{\ln(e + 1/(b-a))}.$$

In case of $m'_a = m'_b$ desired result can be obtained from estimations of A_1 and A_3.

Case 2. It is clear that in this case $c_{m_a} = c_{m_b} = c_n$ where n is odd. Note that restriction of function g on the interval $(c_{n+1}; c_{n-1})$ has symmetry about $x = c_n$ line, therefore without loss of generality we can assume that $g(a) \geqslant g(b)$ then

$$\begin{aligned} \frac{1}{b-a}\int_a^b g(x)dx - g(b) &= \frac{1}{b-a}\int_a^b (g(x)-g(b))dx \qquad (4.15)\\ &\leqslant \frac{2}{b-a}\int_{c_n}^b \left(\left(\ln\ln\frac{1}{x-d_{(n-1)/2}} - \frac{n-1}{2}\right) - g(b)\right)dx \\ &\leqslant \frac{2}{b-c_n}\int_{c_n}^b \left(\ln\ln\frac{1}{x-d_{(n-1)/2}} - \ln\ln\frac{1}{b-d_{(n-1)/2}}\right) \\ &\leqslant \frac{4}{\ln(e+1/(b-a))}. \end{aligned}$$

Case 3. In this case by (4.6) we get desired estimation.

Finally by the estimates (4.8)–(4.15) and (4.6) we get (4.7).

Now let construct exponent $p(\cdot)$ such that $1/p(\cdot) \in BLO^{1/\log}$ but G'' property fails.

We choose real numbers a and b such that $0 < a < b < 1$, $a+b < 1$. Consider sets A and B

$$A = \{x : g(x) \leqslant a\}, \qquad B = \{x : g(x) \geqslant b\}.$$

It is clear that these sets are union of intervals and let denote they by Δ_n^a and Δ_n^b i.e.

$$A = \bigcup_{n\geqslant 1}\Delta_n^a, \qquad B = \bigcup_{n\geqslant 1}\Delta_n^b.$$

Let now construct exponent p in following way

$$p(x) = \begin{cases} 1/a \text{ if } x \in A; \\ 1/b \text{ if } x \in B; \\ 1/g(x) \text{ if } x \in [0;1]\setminus(A\cup B). \end{cases}$$

It is clear that $p(\cdot)$ is continuous except point 0, where it has discontinuity and $1/p(\cdot) \in BLO^{1/\log}$.

Let consider the set of right side endpoints of intervals from A. Let make partition of $[0;1]$ by these points. So we will get sequence of disjoint intervals Δ_n such that $\Delta_n^a \cup \Delta_n^b \subset \Delta_n$.

Let $\delta_k = \min\{|\Delta_k^a|, |\Delta_k^b|\}$. Since $\delta_k \leqslant \min\{|\Delta_n^a|, |\Delta_n^b|\}$ for all $n \leqslant k$ then for each n, $n \leqslant k$ we can choose intervals $\Delta_n^{a'} \subset \Delta_n^a$ and $\Delta_n^{b'} \subset \Delta_n^b$ such that $\delta_k = |\Delta_n^{a'}| = |\Delta_n^{b'}|$.

Now for each k we construct functions f_k and g_k in following way $f_k(x) = \chi_{\cup_{n\leqslant k}\Delta_n^{a'}}(x)$ and $g_k(x) = \chi_{\cup_{n\leqslant k}\Delta_n^{b'}}(x)$.

Let now check property G of $L^{p(\cdot)}[0;1]$

$$\sum_{n=1}^{k}||f_k\chi_{\Delta_n}||_{L^{1/a}}\cdot||g_k\chi_{\Delta_n}||_{L^{1/b}} = \sum_{n=1}^{k}||\chi_{\Delta_n^{a'}}||_{L^{1/a}}\cdot||\chi_{\Delta_n^{b'}}||_{L^{1/b}}$$
$$= \sum_{n=1}^{k}|\Delta_n^{a'}|^{a}\cdot|\Delta_n^{b'}|^{b} = k\cdot\delta_k^{a+b}.$$

On the other hand

$$||f_k||_{L^{1/a}}\cdot||g_k||_{L^{1/b}} = \left(\sum_{n=1}^{k}|\Delta_n^{a'}|\right)^{a}\cdot\left(\sum_{n=1}^{k}|\Delta_n^{b'}|\right)^{b} = (k\cdot\delta_k)^{a+b}.$$

Property G states that, there exits absolute constant C such that

$$k\cdot\delta_k^{a+b}\leqslant C\cdot(k\cdot\delta_k)^{a+b},$$

we have

$$k^{1-a-b}\leqslant C.$$

The last estimation is impossible since $a+b<1$ and $k^{1-a-b}\to+\infty$, $k\to+\infty$.

Using Theorem 1 we conclude that $L^{p(\cdot)}[0;1]$ does not have property G''.

Note that $1/(p(\cdot)+c)\in BLO^{1/\log}$ for all $c>0$. Consequently exponents $p(\cdot)+c$ give us the spaces with same property.

Proof of the second part of Theorem 5. Note that by Theorem 10 and Theorem 6 we conclude that space $L^{(p(\cdot)+c)'}[0;1]$ possesses property G'' for some constant $c>0$. It is clear that space $L^{(p(\cdot)+c)'}[0;1]$ does not have property G' (because $L^{(p(\cdot)+c)}[0;1]$ does not have property G''). $\square$

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