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CHEBYSHEV AND GRÜSS TYPE INEQUALITIES INVOLVING TWO LINEAR FUNCTIONALS AND APPLICATIONS

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Abstract. In the present paper we prove the Chebyshev inequality involving two isotonic linear functionals. Namely, if A and B are isotonic linear functionals, then $A(pfg)B(q)+A(p)B(qfg) \geqslant A(pf)B(qg)+A(pg)B(qf)$, where p,q are non-negative weights and f,g are similarly ordered functions such that the above-mentioned terms are well-defined. If functionals are equal, i.e. $A=B$ and if $p=q$, then the above inequality becomes the Chebyshev inequality involving one isotonic linear functional: $A(p)A(pfg) \geqslant A(pf)A(pg)$ in which we recognize a generalization of the well-known classical integral and discrete Chebyshev inequalites as special cases.

We derive various properties of functionals related to the difference of the right-hand and the left-hand sides of the above-mentioned inequalities. The most remarkable results are the Grüss type inequalities for two functionals. Inequalities involving some fractional integral operators are also given.

1. Introduction

In this paper we point out how known results from theory of inequalities are applied to fractional integral operators and vice versa, how some results related to fractional integral operators caused the discovery of new results in theory of inequalities. Very recently several papers about Chebyshev and Grüss type inequalities for fractional integral operators appeared, see [3], [4], [5], [6] [7], [10], [11], [14], [20], [21], [22], [23], [24]. A careful introspection of those papers shows that some results in them are simple consequences of known results and they can be easily obtained using inequalities for isotonic linear functionals. On the other hand, known inequalites for fractional integral operators give us a clue how the known Chebyshev and Grüss inequalities can be generalized to inequalities with two isotonic linear functionals.

The paper is organized in the following way. The next section contains definition and some interesting examples of isotonic linear functionals connected with fractional integration. Also, Chebyshev type inequalities for one and two isotonic functionals are proved and properties of related functionals are investigated. Each result is followed by related consequences from theory of fractional integral operators. The third section is devoted to Grüss type inequalities for two isotonic functionals and applications to fractional integral operators.

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2. Inequalities involving isotonic linear functionals

DEFINITION 1. (*Isotonic linear functional*) Let E be a non-empty set and L be a class of real-valued functions on E having the properties:

L1. If $f,g \in L$, then $(af+bg) \in L$ for all $a,b \in \mathbb{R}$;

L2. The function $\mathbf{1}$ belongs to L. ($\mathbf{1}(t) = 1$ for $t \in E$).

A functional $A : L \to \mathbb{R}$ is called an isotonic linear functional if

A1. $A(af+bg) = aA(f)+bA(g)$ for $f,g \in L$, $a,b \in \mathbb{R}$;

A2. $f \in L$, $f(t) \geqslant 0$ on E implies $A(f) \geqslant 0$.

A lot of results involving isotonic linear functionals are given in monograph [18]. The Jensen, the Hölder, the Minkowski and related inequalities involving isotonic linear functionals are given in that book. But, there are no results about Chebyshev type inequalities for isotonic linear functionals. Some results of that type can be find in separated papers such as in [1], [2], [13], [17], [19]. In this paper we prove more general results which consequences are results given in the above mentioned papers.

EXAMPLE 1. (i) *Discrete functional.* If $E = \{1,2,\ldots,n\}$ and $f : E \to \mathbb{R}$, then $A(f) = \sum_{i=1}^{n} f(i)$ is an isotonic linear functional.

(ii) *Integral functional.* If $E = [a,b] \subset \mathbb{R}$ and $L = L(a,b)$, then

$$A(f) = \int_a^b f(t)dt.$$

If $A_1(f) = \frac{1}{b-a}A(f)$, then A_1 is a normalized isotonic linear functional.

(iii) *Fractional hypergeometric operator.* If $t > 0$, $\alpha > \max\{0, -\beta-\mu\}$, $\mu > -1$, $\beta - 1 < \eta < 0$, then

$$A(f) = I_t^{\alpha,\beta,\eta,\mu}\{f(t)\}$$

is an isotonic linear functional, ([3]), where $I_t^{\alpha,\beta,\eta,\mu}\{f(t)\}$ is a fractional hypergeometric operator, [9], i.e.

$$I_t^{\alpha,\beta,\eta,\mu}\{f(t)\} = \frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)} \int_0^t \sigma^{\mu}(t-\sigma)^{\alpha-1} \,{}_2F_1\left(\alpha+\beta+\mu, -\eta, \alpha; 1-\frac{\sigma}{t}\right) f(\sigma)\,d\sigma$$

where the function

$${}_2F_1(a,b,c,t) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{t^n}{n!}$$

is the Gaussian hypergeometric function and $(a)_n$ is the Pochhammer symbol: $(a)_n = a(a+1)\ldots(a+n-1)$, $(a)_0 = 1$.

(iv) There is a list of some particular cases of a fractional hypergeometric operator which are of particular interest in the theory of fractional integration.

The Saigo fractional integral operator

If $t > 0$, $\alpha > \max\{0, -\beta\}$, $\beta - 1 < \eta < 0$, then $A(f) = I_t^{\alpha,\beta,\eta}\{f(t)\}$ is an isotonic linear functional, where $I_t^{\alpha,\beta,\eta}\{f(t)\}$ is the Saigo fractional integral operator, i.e.

$$I^{\alpha,\beta,\eta}\{f(t)\} = I_t^{\alpha,\beta,\eta,0}\{f(t)\} = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^t (t-\sigma)_2^{\alpha-1} F_1\left(\alpha+\beta, -\eta, \alpha; 1-\frac{\sigma}{t}\right) f(\sigma)\, d\sigma.$$

The Erdélyi-Kober fractional integral operator

If $t > 0$, $\alpha > 0$, $-1 < \eta < 0$, then

$$A(f) = I_t^{\alpha,\eta}\{f(t)\}$$

is an isotonic linear functional, where $I_t^{\alpha,\eta}\{f(t)\}$ is the Erdélyi-Kober fractional integral operator, i.e.

$$I^{\alpha,\eta}\{f(t)\} = I_t^{\alpha,0,\eta,0}\{f(t)\} = \frac{t^{-\alpha}}{\Gamma(\alpha)} \int_0^t (t-\sigma)^{\alpha-1}\, {}_2F_1\left(\alpha, -\eta, \alpha; 1-\frac{\sigma}{t}\right) f(\sigma)\, d\sigma.$$

The Riemann-Liouville fractional integral operator

One of the earliest defined and the most investigated fractional integral operator is the so-called Riemann-Liouville operator defined as

$$J^{\alpha} f(t) = I_t^{\alpha,-\alpha,0,0}\{f(t)\} = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\sigma)^{\alpha-1} f(\sigma)\, d\sigma, \quad \alpha > 0.$$

(v) *The Hadamard fractional integral*

The Hadamard fractional integral of order $\alpha > 0$ of function f is defined as

$${}_H J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x \left(\log\frac{x}{y}\right)^{\alpha-1} \frac{f(y)dy}{y}, \quad 1 < x.$$

For a further studying of theory of fractional calculus we recommend book [15].

2.1. Chebyshev type inequalities for isotonic linear functionals

We say that functions f and g on E are similarly ordered (or synchronous) if for each $x, y \in E$

$$(f(x) - f(y))(g(x) - g(y)) \geqslant 0.$$

If the reversed inequality holds, then we say that f and g are oppositely ordered or asynchronous.

The next result is the Chebyshev inequality for two isotonic linear functionals.

THEOREM 1. (The Chebyshev inequality for two isotonic linear functionals) *Let A and B be two isotonic linear functionals on L and let $p, q \in L$ be non-negative functions. Let f, g be two functions on E such that pf, pg, qf, qg, pfg, $qfg \in L$.*

If f and g are similarly ordered functions, then

$$A(pfg)B(q) + A(p)B(qfg) \geqslant A(pf)B(qg) + A(pg)B(qf). \tag{2.1}$$

If f and g are oppositely ordered functions, then the reverse inequality in (2.1) holds.

Proof. If f and g are similarly ordered functions, then

$$p(x)q(y)(f(x)-f(y))(g(x)-g(y)) \geqslant 0.$$

Acting on this inequality firstly by functional A with respect to x and using isotonicity and linearity we get

$$q(y)A(pfg)-q(y)g(y)A(pf)-q(y)f(y)A(pg)+q(y)f(y)g(y)A(p) \geqslant 0.$$

Acting on the last inequality by B with respect to y we get wanted inequality. In a similar manner we prove inequality for oppositely ordered functions. □

COROLLARY 1. *Let A be an isotonic linear functional on L and let $p \in L$ be a non-negative function.*

If f and g are similarly ordered functions on E such that $pf,\ pg,\ pfg \in L$, then

$$A(p)A(pfg) \geqslant A(pf)A(pg). \tag{2.2}$$

If f and g are oppositely ordered functions, then the reverse inequality in (2.2) holds.

Proof. Putting $A=B$, $p=q$ in (2.1) and divided by 2 we get inequality (2.2). □

Inequality (2.2) is the Chebyshev inequality for an isotonic positive functional, [17].

Let A, B be fixed isotonic linear functionals and let f and g be two fixed functions on E. Let us define a cone C as follows

$$C := \{p \in L \ :\ p \geqslant 0, pf, pg, pfg \in L\}.$$

We define functional R on cone C as

$$R(p)=A(p)B(pfg)+B(p)A(pfg)-A(pf)B(pg)-A(pg)B(pf).$$

THEOREM 2. *If f and g are similarly ordered functions, then functional R is superadditive, non-negative and positive homogeneous of order 2. If f and g are oppositely ordered functions, then functional R is subadditive, non-positive and positive homogeneous of order 2.*

Proof. Suppose that f and g are similarly ordered functions. If in (2.1) we put a substitution $p \leftrightarrow q$, then we get

$$A(qfg)B(p)+A(q)B(pfg) \geqslant A(qf)B(pg)+A(qg)B(pf). \tag{2.3}$$

Adding (2.1) and (2.3) we get a symmetric form of the Chebyshev inequality

$$\begin{aligned}&A(pfg)B(q)+A(p)B(qfg)+A(qfg)B(p)+A(q)B(pfg)\\&\quad-(A(pf)B(qg)+A(pg)B(qf)+A(qf)B(pg)+A(qg)B(pf)) \geqslant 0.\end{aligned}$$

Let us consider a difference $R(p+q)-R(p)-R(q)$. In fact, it is equal to the left-hand side of the previous inequality, so $R(p+q)-R(p)-R(q) \geqslant 0$. It means, that R is superadditive. Non-negativity of R follows from (2.1) putting $p=q$. Homogenity is also easy to check. If f and g are oppositely ordered functions the proof is similar. □

A non-negative superadditive and homogeneous of order 2 functional posses the following boundedness property.

THEOREM 3. *Let $p,q \in C$ be such that there exist numbers M,m such that $M \geqslant m > 0$ and $Mp \geqslant q \geqslant mp$. If f and g are similarly ordered functions, then*

$$M^2 R(p) \geqslant R(q) \geqslant m^2 R(p). \tag{2.4}$$

Furthermore, if $p \geqslant q$, then

$$R(p) \geqslant R(q) \tag{2.5}$$

i.e. functional R is non-decreasing.

If f and g are oppositely ordered functions, then inequalities (2.4) and (2.5) are reversed.

Proof. Using homogenity, superadditivity of R and the fact that $R(Mp-q) \geqslant 0$ we have

$$M^2 R(p) = R(Mp) = R(Mp-q+q) \geqslant R(Mp-q) + R(q) \geqslant R(q).$$

The second inequality in (2.4) is proved in the similar manner. Result (2.5) follows from (2.4) for $M=1$. □

In particular, if $A=B$, then a functional $\frac{1}{2}R$ becomes the so-called Chebyshev functional $T(f,g,p)$ which is defined as

$$p \mapsto T(f,g,p) = A(p)A(pfg) - A(pf)A(pg).$$

Properties of the Chebyshev functional which follow from the above results are collected in the following corollary.

COROLLARY 2. *(i) The functional T is homogeneous of order 2 on C.*

(ii) If f and g are similarly ordered functions, $p \in C$, then T is superadditive and

$$T(f,g,p) \geqslant 0. \tag{2.6}$$

(ii) Let $p,q \in C$ be such that there exist numbers M,m with properties $M \geqslant m > 0$ and $Mp \geqslant q \geqslant mp$. If f and g are similarly ordered functions, then

$$M^2 T(f,g,p) \geqslant T(f,g,q) \geqslant m^2 T(f,g,p). \tag{2.7}$$

Furthermore, if $p \geqslant q$, then

$$T(f,g,p) \geqslant T(f,g,q) \tag{2.8}$$

i.e. the functional T is non-decreasing.

If f and g are oppositely ordered functions, then inequalities (2.6), (2.7) and (2.8) are reversed and T is subadditive.

Also, we have results for a composite functional. Let

$$\eta_R(p) = A(p)B(p)\Phi\left(\frac{R(p)}{A(p)B(p)}\right).$$

THEOREM 4. *Let f and g be similarly ordered functions on E, $p \in C$ such that $A(p),\ B(p) > 0$.*

If $\Phi : [0,\infty) \to [0,\infty)$ is a non-decreasing concave function, then the functional

$$\eta_R(p) = A(p)B(p)\Phi\left(\frac{R(p)}{A(p)B(p)}\right).$$

is superadditive.

Furthermore if $M \geqslant m > 0$ such that $Mp \geqslant q \geqslant mp$, then

$$M^2\eta_R(p) \geqslant \eta_R(q) \geqslant m^2\eta_R(p). \tag{2.9}$$

In particular, if $p \geqslant q$, then

$$\eta_R(p) \geqslant \eta_R(q).$$

Proof. The mapping $p \mapsto A(p)B(p)$ is superadditive since

$$A(p+q)B(p+q) - A(p)B(p) - A(q)B(q) = A(p)B(q) + A(q)B(p) > 0.$$

The superadditivity of $\eta_R(p)$ follows from Thm 2.2, [17]. Here we prove it directly. Namely, denote $v(p) = A(p)B(p)$ and $\gamma = \frac{v(p)+v(q)}{v(p+q)}$. Then we have

$$\begin{aligned}
\Phi\left(\frac{R(p+q)}{v(p+q)}\right) &\geqslant \Phi\left(\frac{R(p)+R(q)}{v(p+q)}\right) = \Phi\left(\frac{v(p)}{v(p+q)}\frac{R(p)}{v(p)} + \frac{v(q)}{v(p+q)}\frac{R(q)}{v(q)}\right)\\
&= \Phi\left(\frac{v(p)}{v(p)+v(q)}\left(\gamma\frac{R(p)}{v(p)}\right) + \frac{v(q)}{v(p)+v(q)}\left(\gamma\frac{R(q)}{v(q)}\right)\right)\\
&\geqslant \frac{v(p)}{v(p)+v(q)}\Phi\left(\gamma\frac{R(p)}{v(p)}\right) + \frac{v(q)}{v(p)+v(q)}\Phi\left(\gamma\frac{R(q)}{v(q)}\right)\\
&= \frac{v(p)}{v(p)+v(q)}\Phi\left(\gamma\frac{R(p)}{v(p)} + (1-\gamma)\cdot 0\right) + \frac{v(q)}{v(p)+v(q)}\Phi\left(\gamma\frac{R(q)}{v(q)} + (1-\gamma)\cdot 0\right)\\
&\geqslant \frac{v(p)}{v(p)+v(q)}\left[\gamma\Phi\left(\frac{R(p)}{v(p)}\right) + (1-\gamma)\Phi(0)\right]\\
&\qquad + \frac{v(q)}{v(p)+v(q)}\left[\gamma\Phi\left(\frac{R(q)}{v(q)}\right) + (1-\gamma)\Phi(0)\right]\\
&\geqslant \frac{v(p)}{v(p)+v(q)}\gamma\Phi\left(\frac{R(p)}{v(p)}\right) + \frac{v(q)}{v(p)+v(q)}\gamma\Phi\left(\frac{R(q)}{v(q)}\right)\\
&= \frac{v(p)}{v(p+q)}\Phi\left(\frac{R(p)}{v(p)}\right) + \frac{v(q)}{v(p+q)}\Phi\left(\frac{R(q)}{v(q)}\right).
\end{aligned}$$

The first inequality holds because of non-decreasing of Φ and superadditivity of R. The second and the third inequalities follow from the concavity of Φ and the fourth inequality holds because of nonnegativity of function Φ. Multiplying with $v(p+q) = A(p+q)B(p+q)$ we have

$$A(p+q)B(p+q)\Phi\left(\frac{R(p+q)}{A(p+q)B(p+q)}\right)$$
$$\geqslant A(p)B(p)\Phi\left(\frac{R(p)}{A(p)B(p)}\right) + A(q)B(q)\Phi\left(\frac{R(q)}{A(q)B(q)}\right). \quad \square$$

Consequence of the above theorem is that for a functional involving the Chebyshev functional

$$\eta_T(p) = A(p)\Phi\left(\frac{T(f,g,p)}{A(p)}\right)$$

we have analogue results. Namely, if f,g are similarly ordered and Φ is concave, then $\eta_T(p)$ is superadditive, has boundary property (2.9) and it is monotone.

Finally, let us prove the Chebyshev inequality for several functions.

THEOREM 5. *Let A be an isotonic linear functional, p be a non-negative function from L and let $f_1, f_2, \ldots, f_n$ be non-negative increasing functions such that $pf_1, pf_2, \ldots, pf_n$, $pf_1 \cdot \ldots \cdot, f_n \in L$. Then*

$$\prod_{i=1}^{n} A(pf_i) \leqslant A^{n-1}(p)A\left(p\prod_{i=1}^{n} f_i\right). \tag{2.10}$$

Proof. If $n = 2$, then inequality (2.10) becomes inequality (2.2). Suppose that (2.10) is valid for n functions. Then

$$A^n(p)A(pf_1 \ldots f_n f_{n+1}) = A^{n-1}(p)A(p)A(p(f_1 \ldots f_n)f_{n+1})$$
$$\geqslant A^{n-1}(p)A(pf_1 \ldots f_n)A(pf_{n+1}) \geqslant A(pf_1) \cdot \ldots \cdot A(pf_{n+1}),$$

and by induction the statement is valid for all positive integer n. $\square$

2.2. Applications on fractional integral operators

Let us consider a fractional hypergeometric operator.

THEOREM 6. *Let f and g be two similarly ordered functions on $[0,\infty)$, and let p,q be two non-negative functions on $[0,\infty)$. Then*

$$I_t^{\alpha,\beta,\eta,\mu}\{p(t)f(t)g(t)\}I_t^{\gamma,\delta,\zeta,\nu}\{q(t)\} + I_t^{\alpha,\beta,\eta,\mu}\{p(t)\}I_t^{\gamma,\delta,\zeta,\nu}\{q(t)f(t)g(t)\} \tag{2.11}$$
$$\geqslant I_t^{\alpha,\beta,\eta,\mu}\{p(t)f(t)\}I_t^{\gamma,\delta,\zeta,\nu}\{q(t)g(t)\} + I_t^{\alpha,\beta,\eta,\mu}\{p(t)g(t)\}I_t^{\gamma,\delta,\zeta,\nu}\{q(t)f(t)\}$$

for $t > 0$, $\alpha > \max\{0, -\beta-\mu\}$, $\gamma > \max\{0, -\delta-\nu\}$, μ, $\nu > -1$, β, $\delta < 1$, $\beta - 1 < \eta < 0$, $\delta - 1 < \zeta < 0$.

Proof. Putting $A(f) = I_t^{\alpha,\beta,\eta,\mu}\{f(t)\}$ and $B(f) = I_t^{\gamma,\delta,\zeta,\nu}\{f(t)\}$ in inequality (2.1) we get inequality (2.11). □

REMARK 1. If $p = q = \mathbf{1}$, then (see [3])

$$A(\mathbf{1}) = I_t^{\alpha,\beta,\eta,\mu}\{\mathbf{1}\} = \frac{\Gamma(\mu+1)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} t^{-\beta-\mu}$$

and

$$B(\mathbf{1}) = I_t^{\gamma,\delta,\zeta,\nu}\{\mathbf{1}\} = \frac{\Gamma(\nu+1)\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\nu+\gamma+\zeta)} t^{-\delta-\nu}.$$

So, inequality (2.11) becomes

$$\begin{aligned} &\frac{\Gamma(\mu+1)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)} t^{-\beta-\mu} I_t^{\gamma,\delta,\zeta,\nu}\{f(t)g(t)\} \\ &\quad + \frac{\Gamma(\nu+1)\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\nu+\gamma+\zeta)} t^{-\delta-\nu} I_t^{\alpha,\beta,\eta,\mu}\{f(t)g(t)\} \\ &\geqslant I_t^{\alpha,\beta,\eta,\mu}\{f(t)\} I_t^{\gamma,\delta,\zeta,\nu}\{g(t)\} + I_t^{\alpha,\beta,\eta,\mu}\{g(t)\} I_t^{\gamma,\delta,\zeta,\nu}\{f(t)\}. \end{aligned} \tag{2.12}$$

This result is given in paper [3] as Theorem 6. If $A = B$, i.e. if $\alpha = \gamma$, $\beta = \delta$, $\eta = \zeta$ and $\mu = \nu$, then (2.12) becomes inequality given in [3, Theorem 5]. In the same paper [3] in Theorem 9 the following inequality occurs:

$$I_t^{\alpha,\beta,\eta,\mu}\{\prod_{i=1}^{n} f_i(t)\} \geqslant \left(\frac{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)t^{\beta+\mu}}{\Gamma(\mu+1)\Gamma(1-\beta+\eta)} \right)^{n-1} \prod_{i=1}^{n} I_t^{\alpha,\beta,\eta,\mu}\{f_i(t)\}$$

which is, in fact, inequality (2.10) when $p = \mathbf{1}$.

If in Theorem 6 we put $\mu = \eta = 0$, then we get result for the Saigo integral operator. Case $p = q = \mathbf{1}$ for the Saigo operator is given in [20, Theorem 2], while the case $p = q = \mathbf{1}$ with $A = B$ is given in [20, Theorem 1]. In the same paper [20], inequalites for q-fractional operators are given.

REMARK 2. Putting $A(f) = J^{\alpha} f(t)$, $B(f) = J^{\beta} f(t)$ and $p = q$, i.e. A and B are the Riemann-Liouville fractional integrals, then the Chebyshev inequality from Theorem 1 has a form:

$$J^{\alpha} p(t) J^{\beta} pfg(t) + J^{\alpha} pfg(t) J^{\beta} p(t) \geqslant J^{\alpha} pf(t) J^{\beta} pg(t) + J^{\alpha} pg(t) J^{\beta} pf(t).$$

This inequality is independetly proved in [10, Theorem 5]. In the same paper in Theorem 2, author proved the Chebyshev inequality for the Riemann-Liouville fractional integral operator if $\alpha = \beta$, while the case $p = q = \mathbf{1}$ is given as Theorem 3.1 in [4]. In the same paper the Chebyshev inequality involving the Riemann-Liouville fractional integral operator for n functions is given.

REMARK 3. In this remark we mention results involving the Hadamard fractional integral operators. In paper [5] the following result occurs:

Let f and g be two synchronous functions on $[0,\infty)$ and $p,q : [0,\infty) \to [0,\infty)$. Then for all $x > 1$, $\alpha > 0$ we have

$$ {}_HJ^{\alpha}p(x)\,{}_HJ^{\alpha}qfg(x) + {}_HJ^{\alpha}q(x)\,{}_HJ^{\alpha}pfg(x) \geqslant {}_HJ^{\alpha}pf(x)\,{}_HJ^{\alpha}qg(x) + {}_HJ^{\alpha}qf(x)\,{}_HJ^{\alpha}pg(x). $$

It is just inequality (2.1) for $A(f) = B(f) = {}_HJ^{\alpha}f(x)$. Non-weighted result i.e. for case $p = q = \mathbf{1}$, is given in paper [6] together with a non-weighted result for two operators ${}_HJ^{\alpha}f(x)$ and ${}_HJ^{\beta}f(x)$ and with the Chebyshev inequality for n functions.

We have to point out that in the all above-mentioned papers results for differents kinds of fractional integral operators are obtained “ab ovo”. Namely, authors begin with inequality $(f(x)-f(y))(g(x)-g(y)) \geqslant 0$, multiply with appropriate weights, integrate with respect to the first and to the second variable and get the Chebyshev type inequality. Now, after proving general results (2.1), (2.2) and (2.10) we see that all the above-mentioned results about fractional integral operators are simple consequences of our basic results for one or two isotonic linear functionals.

3. Grüss type inequalities for two isotonic functionals

Estimations of the functional $T(f,g,p)$ are usually called Grüss type inequalities in honour to the result of G. Grüss who gave bounds for difference involving integrals, [18, p. 206]. The Grüss inequality for normalized isotonic linear functionals were obtained by Andrica and Badea in [1]. Here we write their result slightly modified using non-normalized isotonic linear functional.

THEOREM 7. *If $p \in L$ and f and g are functions such that pf, pg, $pfg \in L$ and*

$$ m \leqslant f \leqslant M, \quad n \leqslant g \leqslant N $$

where m, M, n, N are real numbers, then for any isotonic linear functional A one has the following inequality

$$ |T(f,g,p)| \leqslant \frac{1}{4}(M-m)(N-n)A^2(p), \tag{3.1} $$

where $T(f,g,p) = A(p)A(pfg) - A(pf)A(pg)$. The constant $\frac{1}{4}$ is the best possible.

In [2] (see also [19]) the following Grüss inequality is given:

$$ T(f,g,p) \leqslant \sqrt{T(f,f,p)}\sqrt{T(g,g,p)}. \tag{3.2} $$

In the following text we prove the Grüss type inequalities which involves two linear functionals.

Let us consider a functional D defined as follows: let A and B be isotonic functionals on L_1 and L_2 respectively, $p : E_1 \to \mathbb{R}$ and $q : E_2 \to \mathbb{R}$ be non-negative functions from L_1 and L_2 respectively. If $h: E_1 \times E_2 \to \mathbb{R}$, then we consider related function h_x defined as: $h_x : E_2 \to \mathbb{R}$, $h_x(y) = q(y)h(x,y)$ where x is a fixed element from E_1. If $h_x \in L_2$, acting by B on h_x with respect on y we get a function $B(h_x)$ depending on variable x. Then, we multiply it with $p(x)$ and if it belongs to L_1, act by A on $p(x)B(h_x)$ with respect on x. The gotten result is a value of functional D on function h, i.e. $D(h) = A(pB(h_x))$. A functional D is isotonic and linear.

THEOREM 8. (The Grüss inequality for two functionals) *Let A and B be isotonic linear functionals on L, p, q be non-negative functions from L. If f,g are functions such that pf, qf, pg, qg, pfg, $qfg \in L$, then*

$$T(A,B,p,q,f,g)^2 \leqslant T(A,B,p,q,f,f)T(A,B,p,q,g,g), \tag{3.3}$$

where

$$T(A,B,p,q,f,g) = B(q)A(pfg) + A(p)B(qfg) - A(pf)B(qg) - A(pg)B(qf).$$

Proof. Let us consider a function h defined as

$$h(x,y) = (f(x) - f(y))(g(x) - g(y)).$$

Multiplying with $q(y)$, acting by B with respect on y, multiplying with $p(x)$ and acting on it by A with respect on x we get

$$D(h) = A(p)B(qfg) + A(pfg)B(q) - A(pf)B(qg) - A(qf)B(pg).$$

Using the Cauchy inequality for an isotonic linear functional (see [18, p. 113]) we get

$$D^2(h) \leqslant D(h_1^2) \cdot D(h_2^2),$$

where $h_1(x,y) = f(x) - f(y)$ and $h_2(x,y) = g(x) - g(y)$. Since

$$D(h_1^2) = A(p)B(qf^2) + A(pf^2)B(q) - 2A(pf)B(qf)$$

and

$$D(h_2^2) = A(p)B(qg^2) + A(pg^2)B(q) - 2A(pg)B(qg)$$

we obtain inequality (3.3). □

REMARK 4. If $A = B$ and $p = q$, then inequality (3.3) becomes inequality (3.2). So, the previous theorem gives us a generalization of the Grüss inequality (3.2) to the Grüss type inequality involving two functionals.

REMARK 5. If we use notation $D(h)$ from the proof of the previous theorem, we can get a generalization of Theorem 3.2 from [12]. Namely, if f and g are differentiable on $[0,\infty)$, $f' \in L_r[0,\infty)$, $g' \in L_{r'}[0,\infty)$, $r > 1$, $\frac{1}{r} + \frac{1}{r'} = 1$, then

$$|D(h)| \leqslant \|f'\|_r \|g'\|_{r'} D(\tilde{h})$$

where $\tilde{h}(x,y) = |x-y|$. Indeed

$$h(x,y) = \int_x^y f'(s)\,ds \int_x^y g'(t)\,dt$$

and using the Hölder inequality we have

$$|h(x,y)| \leqslant |x-y| \left(\int_x^y |f'(s)|^r \,ds \right)^{\frac{1}{r}} \left(\int_x^y |g'(t)|^{r'} \,dt \right)^{\frac{1}{r'}}.$$

Since

$$h(x,y) \leqslant |h(x,y)| \leqslant \|f'\|_r \|g'\|_{r'} |x-y| = \|f'\|_r \|g'\|_{r'} \tilde{h}(x,y)$$

and D is isotonic we get

$$|D(h)| \leqslant D(\tilde{h}) \|f'\|_r \|g'\|_{r'},$$

which in the case $p = q$, $A = J^\alpha$, $B = J^\beta$ gives the result of Theorem 3.2 from [12].

Using the Cauchy inequality

$$\begin{aligned} D^2(\tilde{h}) &\leqslant D((x-y)^2) \cdot D(\mathbf{1}) \\ &= \Big(A(px^2)B(q) - 2A(px)B(qy) + B(qy^2)A(p) \Big) \cdot D(\mathbf{1}) \end{aligned}$$

we write that

$$\Big| B(q)A(pfg) + A(p)B(qfg) - A(pf)B(qg) - A(qf)B(pg) \Big|$$
$$\leqslant \sqrt{\Big(A(px^2)B(q) - 2A(px)B(qy) + B(qy^2)A(p) \Big) \cdot A(p)B(q)} \|f'\|_r \|g'\|_{r'}.$$

Let us look at the case $p = q = \mathbf{1}$, $A = J^\alpha$, $B = J^\beta$ and try to estimate $D(\tilde{h})$. If $f(x) = x^k$ then

$$J^\alpha f(t) = \frac{t^{\alpha+k}\Gamma(k+1)}{\Gamma(\alpha+k+1)}$$

and we have

$$\begin{aligned} D((x-y)^2) &= 2t^{\alpha+\beta+2} \left(\frac{1}{\Gamma(\alpha+3)\Gamma(\beta+1)} - \frac{1}{\Gamma(\alpha+2)\Gamma(\beta+2)} + \frac{1}{\Gamma(\alpha+1)\Gamma(\beta+3)} \right) \\ &= 2t^{\alpha+\beta+2} \frac{1}{\Gamma(\alpha+3)\Gamma(\beta+3)} (\alpha^2 + \beta^2 + \beta + \alpha - \alpha\beta). \end{aligned}$$

When $\alpha = \beta$ then

$$D^2(\tilde{h}) \leqslant 2t^{2\alpha+2} \frac{1}{\Gamma^2(\alpha+3)} (\alpha^2 + 2\alpha) \frac{t^{2\alpha}}{\Gamma^2(\alpha+1)} = \frac{2t^{4\alpha+2}(\alpha^2+2\alpha)}{\Gamma^2(\alpha+3)\Gamma^2(\alpha+1)}.$$

In this case Theorem 3.1 from [12] says that

$$\Big| J^\alpha(\mathbf{1}) J^\alpha(fg) - J^\alpha(f) J^\alpha(g) \Big| \leqslant \frac{t^{2\alpha+1}}{2\Gamma^2(\alpha+1)} \|f'\|_r \|g'\|_{r'}.$$

Our inequality is

$$\left|J^{\alpha}(\mathbf{1})J^{\alpha}(fg)-J^{\alpha}(f)J^{\alpha}(g)\right|\leqslant\frac{t^{2\alpha+1}\sqrt{2(\alpha^2+2\alpha)}}{2\Gamma(\alpha+3)\Gamma(\alpha+1)}\|f'\|_r\|g'\|_{r'}.$$

For $\alpha>0$ one can show that

$$\frac{\sqrt{2(\alpha^2+2\alpha)}}{2\Gamma(\alpha+3)\Gamma(\alpha+1)}\leqslant\frac{1}{2\Gamma^2(\alpha+1)}$$

which is equivalent to $\alpha^3+4\alpha^2+3\alpha+2>0.$

So we see that in this case $(p=\mathbf{1})$ our estimate is better than one from Theorem 3.1 from [12].

In the following theorem we establish a generalization of inequality (3.1) using two isotonic linear functionals. Furthermore, we replace the constants which appeared as bounds of functions f and g with functions. Firstly we prove the following useful identity.

LEMMA 1. *Let $A,\ B,\ p,\ q,\ f$ satisfy assumptions of Theorem 8. Let $\phi_1,\ \phi_2\in L$ be functions such that all terms in the below identity exist. Then*

$$\begin{aligned}&A(p)B(qf^2)+A(pf^2)B(q)-2A(pf)B(qf) \qquad (3.4)\\ &=T_{A,B}(f,\phi_1,\phi_2)-A(p)B\Big(q(\phi_2-f)(f-\phi_1)\Big)-B(q)A\Big(p(\phi_2-f)(f-\phi_1)\Big),\end{aligned}$$

where

$$\begin{aligned}T_{A,B}(f,\phi_1,\phi_2)=&\Big(A(p\phi_2)-A(pf)\Big)\Big(B(qf)-B(q\phi_1)\Big)\\ &+\Big(B(q\phi_2)-B(qf)\Big)\Big(A(pf)-A(p\phi_1)\Big)\\ &+A(p\phi_2)B(q\phi_1)-A(p\phi_2)B(qf)-A(pf)B(q\phi_1)+A(p\phi_1)B(q\phi_2)\\ &-A(pf)B(q\phi_2)-A(p\phi_1)B(qf)+A(p)B(q\phi_2f)-A(p)B(q\phi_1\phi_2)\\ &+A(p)B(q\phi_1f)+A(p\phi_2f)B(q)-A(p\phi_1\phi_2)B(q)+A(p\phi_1f)B(q).\end{aligned}$$

Furthermore, if

$$\phi_1(x)\leqslant f(x)\leqslant\phi_2(x)\quad \textit{for any } x\in E,$$

then

$$A(p)B(qf^2)+A(pf^2)B(q)-2A(pf)B(qf)\leqslant T_{A,B}(f,\phi_1,\phi_2). \qquad (3.5)$$

Proof. For any $x,y\in E$ we have

$$\begin{aligned}&\Big(\phi_2(x)-f(x)\Big)\Big(f(y)-\phi_1(y)\Big)+\Big(\phi_2(y)-f(y)\Big)\Big(f(x)-\phi_1(x)\Big)\\ &\quad-\Big(\phi_2(y)-f(y)\Big)\Big(f(y)-\phi_1(y)\Big)-\Big(\phi_2(x)-f(x)\Big)\Big(f(x)-\phi_1(x)\Big)\\ &=f^2(x)+f^2(y)-2f(x)f(y)\\ &\quad+\phi_2(x)f(y)+\phi_1(y)f(x)-\phi_1(y)\phi_2(x)+\phi_2(y)f(x)+\phi_1(x)f(y)-\phi_1(x)\phi_2(y)\\ &\quad-\phi_2(y)f(y)-\phi_1(y)f(y)+\phi_1(y)\phi_2(y)-\phi_2(x)f(x)+\phi_1(x)\phi_2(x)-\phi_1(x)f(x).\end{aligned}$$

Expressing the sum $f^2(x)+f^2(y)-2f(x)f(y)$ from the above identity, then multiplying it with $q(y)$, acting by B with respect on y, multiplying with $p(x)$ and acting on it by A with respect on x we get the statement (3.4).

If $\phi_1(x)\leqslant f(x)\leqslant \phi_2(x)$, then

$$\Big(\phi_2(x)-f(x)\Big)\Big(f(x)-\phi_1(x)\Big)\geqslant 0$$

and since A and B are isotonic we obtain inequality (3.5). □

REMARK 6. If ϕ_1 and ϕ_2 are constants, i.e. $\phi_1(x)=m$ and $\phi_2(x)=M$ for all $x\in E$, then the identity from Lemma 1 becomes

$$\begin{aligned} &A(p)B(qf^2)+B(q)A(pf^2)-2A(pf)B(qf) \\ &=\Big(MA(p)-A(pf)\Big)\Big(B(qf)-mB(q)\Big)+\Big(MB(q)-B(qf)\Big)\Big(A(pf)-mA(p)\Big) \\ &\quad -A(p)B\Big(q(M\cdot\mathbf{1}-f)(f-m\cdot\mathbf{1})\Big)-B(q)A\Big(p(M\cdot\mathbf{1}-f)(f-m\cdot\mathbf{1})\Big). \end{aligned} \tag{3.6}$$

If $m\leqslant f(x)\leqslant M$, then inequality (3.5) becomes

$$\begin{aligned} &A(p)B(qf^2)+B(q)A(pf^2)-2A(pf)B(qf) \\ &\leqslant\Big(MA(p)-A(pf)\Big)\Big(B(qf)-mB(q)\Big)+\Big(MB(q)-B(qf)\Big)\Big(A(pf)-mA(p)\Big). \end{aligned}$$

If $A=B$, $p=q$, and $\phi_1(x)=m$ and $\phi_2(x)=M$ for all $x\in E$, then identity from Lemma 1 becomes

$$\begin{aligned} &A(p)A(pf^2)-A^2(pf) \\ &=\Big(MA(p)-A(pf)\Big)\Big(A(pf)-mA(p)\Big)-A(p)A\Big(p(M\cdot\mathbf{1}-f)(f-m\cdot\mathbf{1})\Big). \end{aligned} \tag{3.7}$$

This result appears in the proof of Theorem 1 from paper [1]. Furthermore, if $m\leqslant f(x)\leqslant M$, then inequality (3.5) becomes

$$A(p)A(pf^2)-A^2(pf)\leqslant\Big(MA(p)-A(pf)\Big)\Big(A(pf)-mA(p)\Big),$$

see also [16]. Using the well-known AG inequality we get

$$\begin{aligned} &\Big(MA(p)-A(pf)\Big)\Big(A(pf)-mA(p)\Big) \\ &\leqslant\left(\frac{(MA(p)-A(pf))+(A(pf)-mA(p))}{2}\right)^2=\left(\frac{M-m}{2}A(p)\right)^2. \end{aligned}$$

So, we have

$$A(p)A(pf^2)-A^2(pf)\leqslant\left(\frac{M-m}{2}A(p)\right)^2.$$

THEOREM 9. *Let A and B be isotonic linear functionals on L, p, q be non-negative functions from L. Let f,g, ϕ_1, ϕ_2, ψ_1, ψ_2 be functions such that*

$$\phi_1(x) \leqslant f(x) \leqslant \phi_2(x), \quad \psi_1(x) \leqslant g(x) \leqslant \psi_2(x), \quad x \in E.$$

Then

$$T(A,B,p,q,f,g)^2 \leqslant T_{A,B}(f,\phi_1,\phi_2)T_{A,B}(g,\psi_1,\psi_2) \tag{3.8}$$

whenever all terms in the above inequality exist.

Particularly, if $\phi_1(x) = m$, $\phi_2(x) = M$, $\psi_1(x) = n$, $\psi_2(x) = N$ for all $x \in E$, then

$$\begin{aligned} &T(A,B,p,q,f,g)^2 \\ &\leqslant \frac{1}{16}\Big\{\Big(MA(p)-A(pf)+B(qf)-mB(q)\Big)^2 + \Big(MB(q)-B(qf)+A(pf)-mA(p)\Big)^2\Big\} \\ &\quad \times \Big\{\Big(NA(p)-A(pg)+B(qg)-nB(q)\Big)^2 + \Big(NB(q)-B(qg)+A(pg)-nA(p)\Big)^2\Big\}. \end{aligned} \tag{3.9}$$

Furthermore, if $A = B$, $p = q$, $\phi_1(x) = m$, $\phi_2(x) = M$, $\psi_1(x) = n$, $\psi_2(x) = N$, then

$$|A(p)A(pfg) - A(pf)A(pg)| \leqslant \frac{1}{4}(M-m)(N-n)A^2(p), \tag{3.10}$$

where $T(A,B,p,q,f,g)$ and $T_{A,B}(f,\phi_1,\phi_2)$ are defined in Theorem 8 and Lemma 1.

Proof. Using inequalites from Theorem 8 and Lemma 1 we obtain (3.8). If $\phi_1(x) = m$, $\phi_2(x) = M$, $\psi_1(x) = n$, $\psi_2(x) = N$, then

$$\begin{aligned} &T_{A,B}(f,\phi_1,\phi_2) \\ &= \Big(MA(p) - A(pf)\Big)\Big(B(qf) - mB(q)\Big) + \Big(MB(q) - B(qf)\Big)\Big(A(pf) - mA(p)\Big) \\ &\leqslant \frac{1}{4}\Big(MA(p) - A(pf) + B(qf) - mB(q)\Big)^2 + \frac{1}{4}\Big(MB(q) - B(qf) + A(pf) - mA(p)\Big)^2 \end{aligned}$$

where in the last inequality the AG inequality is used. Inequality (3.9) follows from the above result and similar result for $T_{A,B}(g,\psi_1,\psi_2)$. If, furthermore, $A = B$, $p = q$, then the last inequality collapsed to the following

$$\begin{aligned} T_{A,A}(f,\phi_1,\phi_2) = T_{A,A}(f,m,M) &\leqslant \frac{1}{4}\Big(MA(p) - mA(p)\Big)^2 + \frac{1}{4}\Big(MA(p) - mA(p)\Big)^2 \\ &= \frac{1}{2}(M-m)^2A^2(p). \end{aligned}$$

Together with a similar result for function g and equality

$$T(A,A,p,p,f,g) = 2(A(p)A(pfg) - A(pf)A(pg))$$

we get inequality (3.10). $\square$

REMARK 7. If $\phi_1,\ \phi_2,\ \psi_1,\ \psi_2$ are constants and $A = B$, then inequality (3.8) becomes (3.2). Estimation (3.10) is, in fact, the Grüss inequality for isotonic linear functional described in Theorem 7. So, Theorem 9 contains Grüss type inequalities for two isotonic linear functionals and two functions with variable bounds and it is a generalization of known results of Grüss type involving one functional and two functions with constant bounds.

3.1. Applications in fractional integration

In recent literature we find several examples of Grüss type inequalites for the Hadamard integral operators. For instance, if $A = B =_H J^\alpha$ and $p = q = \mathbf{1}$, then results (3.7) and (3.10) are given in [8], while inequality (3.8) for that case is given in [22]. Inequality (3.3) for two Hadamard integral operators $A =_H J^\alpha$ and $B =_H J^\beta$ with weights $p = q = \mathbf{1}$ is given in [8].

Results involving the Riemann-Liouville fractional integrals can be found in [11] and [23]. Namely, if $A = B = J^\alpha$ and $p = q = \mathbf{1}$ then results (3.7) and (3.10) are given in [11], while inequality (3.8) for that case is given in [23]. Results (3.3) and (3.6) for two Riemann-Liouville fractional integrals $A = J^\alpha$ and $B = J^\beta$ with weights $p = q = \mathbf{1}$ are given in [11].

Results involving the Saigo fractional integrals can be found in [7] and [14] and these are results for only one functional. Similar situation is valid for generalized hypergeometric fractional integral. Only a case for the same functionals $A = B = I^{\alpha,\beta,\eta,\mu}$ with weights $p = q = \mathbf{1}$ is considered in [24]. So, here we give a general result for two different functionals $I_t^{\alpha,\beta,\eta,\mu}$ and $I_t^{\gamma,\delta,\zeta,\nu}$ with weights $p = q = \mathbf{1}$.

COROLLARY 3. *Let assumptions of Theorem 6 hold. Then*

$$
\begin{aligned}
&\Bigg[\frac{\Gamma(\mu+1)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}t^{-\beta-\mu}I_t^{\gamma,\delta,\zeta,\nu}\{f(t)g(t)\}\\
&+\frac{\Gamma(\nu+1)\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\nu+\gamma+\zeta)}t^{-\delta-\nu}I_t^{\alpha,\beta,\eta,\mu}\{f(t)g(t)\}\\
&-I_t^{\alpha,\beta,\eta,\mu}\{f(t)\}I_t^{\gamma,\delta,\zeta,\nu}\{g(t)\}-I_t^{\alpha,\beta,\eta,\mu}\{g(t)\}I_t^{\gamma,\delta,\zeta,\nu}\{f(t)\}\Bigg]^2\\
\leqslant&\Bigg[\frac{\Gamma(\mu+1)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}t^{-\beta-\mu}I_t^{\gamma,\delta,\zeta,\nu}\{f^2(t)\}\\
&+\frac{\Gamma(\nu+1)\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\nu+\gamma+\zeta)}t^{-\delta-\nu}I_t^{\alpha,\beta,\eta,\mu}\{f^2(t)\}-2I_t^{\alpha,\beta,\eta,\mu}\{f(t)\}I_t^{\gamma,\delta,\zeta,\nu}\{f(t)\}\Bigg]\\
&\times\Bigg[\frac{\Gamma(\mu+1)\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\mu+\alpha+\eta)}t^{-\beta-\mu}I_t^{\gamma,\delta,\zeta,\nu}\{g^2(t)\}\\
&+\frac{\Gamma(\nu+1)\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\nu+\gamma+\zeta)}t^{-\delta-\nu}I_t^{\alpha,\beta,\eta,\mu}\{g^2(t)\}-2I_t^{\alpha,\beta,\eta,\mu}\{g(t)\}I_t^{\gamma,\delta,\zeta,\nu}\{g(t)\}\Bigg].
\end{aligned}
$$

And finally, let us mention paper [21] where inequalities (3.8) and (3.10) for one Riemann-Liouville q-integral are occured.

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