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# INEQUALITIES AND BOUNDS FOR A CERTAIN BIVARIATE ELLIPTIC MEAN

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*Abstract.* This paper deals with a new mean introduced recently by this author. This mean is a degenerate case of the completely symmetric elliptic integral of the second kind. In particular inequalities involving mean under discussion are obtained. Also, bounds in the mean in question are obtained. Bounding expressions are convex combinations of some quantities depending on variables of the mean.

## 1. Introduction and notation

In recent years certain bivariate means have been investigated extensively by several researchers. A complete list of research papers which deal with this subject is too long to be included here even if we would restrict our attention to papers published in the last ten years. The goal of this paper is to obtain inequalities and optimal bounds for the particular mean introduced recently by this author (see [15]). Its definition is included below (see (2)). In what follows the letters $a$ and $b$ will always stand for positive and unequal numbers.

First we recall definition of the Schwab-Borchardt mean of $a$ and $b$:

$$SB(a,b) \equiv SB = \begin{cases} \dfrac{\sqrt{b^2-a^2}}{\cos^{-1}(a/b)} & \text{if } a<b, \\ \dfrac{\sqrt{a^2-b^2}}{\cosh^{-1}(a/b)} & \text{if } a>b \end{cases} \tag{1}$$

(see, e.g., [2], [3]). This mean has been studied extensively in [19], [20] and in [8]. It is well known that the mean $SB$ is strict, nonsymmetric and homogeneous of degree one in its variables.

Mean $SB$ can also be expressed in terms of the degenerated completely symmetric elliptic integral of the first kind (see, e.g., [15]). It has been pointed out in [19] that some well known bivariate means such as logarithmic mean and two Seiffert means (see [23, 24]) can be represented by the Schwab-Borchardt mean of two simpler means such as geometric and arithmetic means or as the Schwab-Borchardt mean of arithmetic and the square - mean root mean. This idea was utilized lately by this author and other

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researchers as well. For more details the interested reader is referred to [4, 5, 6, 7, 8, 9, 10, 13, 14, 18, 21, 22, 25, 26]

The mean studied in this paper is defined as follows:

$$N(a,b) \equiv N = \frac{1}{2}\left(a + \frac{b^2}{SB(a,b)}\right) \tag{2}$$

(see [15]). It's easy to see that mean $N$ is also strict, nonsymmetric and homogeneous of degree one in its variables. Some authors call this mean, Neuman mean of the second kind (see, e.g., [5, 7, 21, 22, 25, 26]). Mean $N$ can also be represented in terms of the degenerated completely symmetric elliptic integral of the second kind (see, e.g., [15]). By taking the $N$-mean of two other means one can generate several new bivariate means. This idea was utilized in [15].

This paper can be regarded as continuation of investigations initiated in author's earlier papers [18, 8, 15, 11, 10, 9, 13, 16, 14, 12, 17] and is organized as follows. Some preliminary results and formulas needed in this paper are given in Section 2. Inequalities involving mean $N$ are derived in Section 3. Bounds for the mean under discussion are obtained in Section 4. The Ky Fan type inequalities are established in Section 5.

## 2. Preliminary results and formulas needed in this paper

First of all let us record another formulas for means $SB$ and $N$. Those will be utilized frequently in susequent sections of this paper.

One can easily verify that (1) implies

$$SB(a,b) \equiv SB = \begin{cases} b\dfrac{\sin r}{r} = a\dfrac{\tan r}{r} & \text{if } a < b, \\ b\dfrac{\sinh s}{s} = a\dfrac{\tanh s}{s} & \text{if } b < a, \end{cases} \tag{3}$$

where

$$\cos r = a/b \quad \text{if} \quad a < b \quad \text{and} \quad \cosh s = a/b \quad \text{if} \quad a > b. \tag{4}$$

Clearly

$$0 < r < \frac{\pi}{2} \tag{5}$$

and

$$s > 0. \tag{6}$$

Corresponding formulas for the mean $N$, obtained with the aid of (2) and (3), read as follows:

$$N(a,b) \equiv N = \frac{1}{2}b\left(\cos r + \frac{r}{\sin r}\right) = \frac{1}{2}a\left(1 + \frac{r}{\sin r \cos r}\right) \tag{7}$$

provided $a<b$. Similarly, if $a>b$, then

$$N(a,b) \equiv N = \frac{1}{2}b\left(\cosh s + \frac{s}{\sinh s}\right) = \frac{1}{2}a\left(1 + \frac{s}{\sinh s \cosh s}\right). \tag{8}$$

Here the domains for $r$ and $s$ are the same as these in (5) and (6).

For later use let

$$v = \frac{a-b}{a+b}. \tag{9}$$

Clearly $0 < |v| < 1$.

The unweighted arithmetic mean $A$ of $a$ and $b$ is defined as

$$A = \frac{a+b}{2}.$$

For the reader's convenience let us recall definitions of the first and the second Seiffert means, denoted by $P$ and $T$, respectively, the Neuman-Sándor mean $M$, and the logarithmic mean $L$:

$$\begin{aligned} P &= A\frac{v}{\sin^{-1} v}, \quad & T &= A\frac{v}{\tan^{-1} v}, \\ M &= A\frac{v}{\sinh^{-1} v}, \quad & L &= A\frac{v}{\tanh^{-1} v}, \end{aligned} \tag{10}$$

(see [23], [24], [19]).

We will also utilize the l'Hôpital Monotone Rule [1]:

Let $c,d \in \mathbb{R}$ $(c<d)$ and let $f,g:[c,d] \to \mathbb{R}$ be continuous functions that are differentiable on $(c,d)$. Assume that $g'(x) \neq 0$ for each $x \in (c,d)$. If $f'/g'$ is increasing (decreasing) on $(c,d)$, then so are $\dfrac{f(x)-f(c)}{g(x)-g(c)}$ and $\dfrac{f(x)-f(d)}{g(x)-g(d)}$. If monotonicity of $f'/g'$ is strict, then so is monotonicity of two functions represented by the above quotients.

## 3. Inequalities involving mean $N$

The goal of this section is to establish two inequalities which connect the Schwab-Borchardt mean $SB$ with the mean $N$. We have the following:

THEOREM 1. *Let $a,b>0$, $a \neq b$. Then*

$$SB(a,b) < \frac{a+2b}{3} < \frac{b+N(b,a)}{2} < N(a,b). \tag{11}$$

*If $a>b$, then*

$$N(a,b) < A < SB(b,a). \tag{12}$$

*Proof.* The first inequality in (11) has been established in [19]. In the proof of the second inequality in (11) we apply the following one

$$\frac{b+2a}{3} < N(b,a)$$

(see [15]) to the third member of (11) to obtain the desired result. The third inequality in (11) appears in [15, Theorem 4.1]. It is included here for the sake of completeness. For the proof of the first inequality in (12) we use the first part of (8) together with the formula $\cosh s = a/b$ to obtain

$$N(a,b) = \frac{1}{2}\left(a + b\frac{s}{\sinh s}\right)$$

Taking into account that $(s/\sinh s) < 1$ we obtain the desired inequality $N(a,b) < A$. For the proof of the second inequality in (12) we will utilize the invariance property of the Schwab-Borchardt mean

$$SB(A, \sqrt{Aa}) = SB(b,a) \tag{13}$$

(see [2, 3]) and the inequality [19]:

$$(xy^2)^{1/3} < SB(x,y) \tag{14}$$

$(x, y > 0, x \neq y)$. In (14) we let $x = A$ and $y = \sqrt{Aa}$ and next apply (13) to obtain

$$A^{2/3}a^{1/3} < SB(b,a). \tag{15}$$

The assumption that $a > b$ yields $A < a$. This in conjunction with (15) gives the desired inequality $A < SB(b,a)$. This completes the proof of the second inequality in (12). □

## 4. Bounds for $N$

For the sake of presentation we introduce two auxiliary functions

$$\Phi_1(r) = \frac{2\sin r - \sin r \cos r - r}{2(\sin r)(1-\cos r)} \tag{16}$$

$(0 < r < \pi/2)$ and

$$\Psi_1(s) = \frac{s + \sinh s \cosh s - 2\sinh s}{2(\sinh s)(\cosh s - 1)} \tag{17}$$

$(s > 0)$. It is known [15] that the function $\Phi_1(r)$ is strictly decreasing while the function $\Psi_1(s)$ is strictly increasing. Moreover, $\Phi_1(0^+) = 1/3$ and $\Phi_1(\pi/2) = 1 - \pi/4$. Also, $\Psi_1(0^+) = 1/3$ and $\Psi_1(\infty^-) = 1/2$.

We are in a position to establish the following:

THEOREM 2. *If $a<b$, then the simultaneous inequality*

$$\alpha a+(1-\alpha)b<N(a,b)<\beta a+(1-\beta)b \tag{18}$$

*holds true if*

$$\alpha \geqslant \frac{1}{3} \quad \textit{and} \quad \beta \leqslant 1-\frac{\pi}{4}=0.214\ldots. \tag{19}$$

*If $a>b$, then the inequality (18) is valid if*

$$\alpha \leqslant \frac{1}{3} \quad \textit{and} \quad \beta \geqslant \frac{1}{2}. \tag{20}$$

*Proof.* We shall prove first the theorem in the case when $a<b$. It follows from (18) that

$$\beta<\frac{N/b-1}{a/b-1}<\alpha. \tag{21}$$

Utilizing (7) and the first formula in (4) we can write (21) as follows

$$\beta<1-\frac{\pi}{4}\leqslant \Phi_1(r)\leqslant \frac{1}{3}<\alpha.$$

Hence the assertion follows. Assume now that $a>b$. We follow the idea used in the first part of this proof. Let us note that in this case inequalities in (21) are reversed, i.e. we have $\alpha<\Psi_1(s)<\beta$. Combining this with relevant parts of l'Hôpital Monotone Rule yields

$$\alpha<\frac{1}{3}\leqslant \Psi_1(s)\leqslant \frac{1}{2}<\beta.$$

The proof is complete. □

In the proof of the next result we will utilize the following function

$$\Phi_2(r)=\frac{r^2+r\sin r\cos r-2\sin^2 r}{2(\sin r)(r-\sin r)} \tag{22}$$

$(0<r<\pi/2)$.

Our next task is to determine all the parameters $\alpha$ and $\beta$ for which the following inequality

$$\alpha b+(1-\alpha)SB(a,b)<N(a,b)<\beta b+(1-\beta)SB(a,b), \tag{23}$$

is satisfied for positive numbers $a$ and $b$ which satisfy the condition $a<b$.

We shall prove now the following:

THEOREM 3. *Inequality (23) holds true if*

$$\alpha \leqslant 0 \quad \textit{and} \quad \beta \geqslant \frac{\pi^2-8}{4\pi-8}=0.409\ldots. \tag{24}$$

*Proof.* Let $a < b$. Making use of (3) and (7) we can easily show that the two-sided inequality (23) can be written in the form

$$\alpha < \Phi_2(r) < \beta. \tag{25}$$

Taking into account that the function $\Phi_2(r)$ is strictly increasing (see [22, Lemma 2.4]) and also that $\Phi_2(0^+) = 0$ and $\Phi_2(\pi/2) = (\pi^2 - 8)/(4\pi - 8))$ we conclude, using (25), that conditions (24) must be satisfied in order for the inequality (23) to be valid. □

We shall now illustrate the last result with the following:

EXAMPLE 1. Let $A$ and $G$ stand for the unweighted arithmetic and geometric means of two positive unequal numbers, and also let P be the first Seiffert mean of the same numbers. Writing $N_{GA}$ for $N(G,A)$ we obtain using (23) and the fact that $P = SB(G,A)$

$$\alpha A + (1-\alpha)P < N_{GA} < \beta A + (1-\beta)P,$$

where $\alpha$ and $\beta$ must to satisfy conditions (24). In particular, with $\alpha = 0$ and $\beta = \frac{1}{2}$, we obtain the inequality

$$P < N_{GA} < \frac{1}{2}(A+P)$$

which is a possibly a new one.

We shall discuss now a problem of finding bounds for $N(a,b)$ in the form of geometric means of $a$ and $b$:

$$a^{\alpha} b^{1-\alpha} < N(a,b) < a^{\beta} b^{1-\beta}. \tag{26}$$

We have the following:

THEOREM 4. *If $a < b$, then the inequality (26) is satisfied for all numbers $\alpha$ and $\beta$ such that $\alpha \geqslant 1/3$ and $\beta \leqslant 0$. Otherwise, if $a > b$, then (26) is valid if $\alpha \leqslant 1/3$ and $\beta \geqslant 1$.*

*Proof.* For the proof of the first part of the assertion we rewrite (26) $a$ as follows

$$\beta < \frac{\log(N/b)}{\log(a/b)} < \alpha, \tag{27}$$

where $N \equiv N(a,b)$. Using (4) and (7) we write the above two-sided inequality as

$$\beta < \Phi_3(r) < \alpha$$

where

$$\Phi_3(r) = \frac{\log(\sin 2r + 2r) - \log(2\sin r) - \log 2}{\log(\cos r)}$$

$(0 < r < \pi/2)$. It is known (see [6, Lemma 2.3]) that the function $\Phi_3(r)$ is strictly decreasing on $(0, \pi/2)$. Moreover, $0 \leqslant \Phi_3(r) \leqslant 1/3$ on the stated domain. Hence the assertion follows.

The second part of the thesis can be established in a similar manner. Using (26) we have

$$\alpha < \frac{\log(N/b)}{\log(a/b)} < \beta$$

Utilizing (4) and (8) we can write the above inequality in the form

$$\alpha < \Psi_3(s) < \beta,$$

where

$$\Psi_3(s) = \frac{\log(\sinh 2s + 2s) - \log(2\sinh s) - \log 2}{\log(\cosh s)}$$

$(s > 0)$. Making use of Lemma 2.4 in [6] we conclude that the function $\Psi_3(s)$ is strictly increasing provided $s > 0$ and also that $1/3 \leqslant \Psi_3(s) \leqslant 1$. The assertion now follows. The proof is complete. □

We shall now deal with problems of finding bounds for the reciprocals of the mean $N$ in terms of reciprocals of its variables $a$ and $b$. Let now $a < b$. More exactly we are looking for all numbers $\alpha$ and $\beta$ for which the inequality

$$\alpha\frac{1}{a} + (1-\alpha)\frac{1}{b} < \frac{1}{N} < \beta\frac{1}{a} + (1-\beta)\frac{1}{b} \tag{28}$$

holds true.

THEOREM 5. *If $a < b$, then the inequality (28) is satisfied provided $\alpha \leqslant 0$ and $\beta \geqslant 1/3$. Otherwise, if $a > b$, then the inequalities (28) hold true if $\alpha \geqslant 1$ and $\beta \leqslant 1/3$.*

*Proof.* It is easy to see that (28) is equivalent to

$$\alpha < \frac{\frac{a}{b}}{1-\frac{a}{b}} \, \frac{1-\frac{N}{b}}{\frac{N}{b}} < \beta$$

provided $a < b$. Let us denote the second member of the above inequality by $\Phi_4$. Then utilizing (4) and (7) we get

$$\Phi_4 \equiv \Phi_4(r) = \frac{\cos r}{1-\cos r} \, \frac{2\sin r - \sin r\cos r - r}{\sin r\cos r + r}$$

$(0 < r < \pi/2)$. Making use of Lemma 2.8 in [4] we conclude that the function $\Phi_4(r)$ is strictly decreasing on its domain and also that $\Phi_4(0^+) = 1/3$ and $\Phi_4(\pi/2) = 0$. This yields

$$\alpha < 0 \leqslant \Phi_4(r) \leqslant 1/3 < \beta \tag{29}$$

This completes the proof of the first part of the thesis of our theorem.

Assume now that $a > b$. It is easy to see that the two-sided inequality (28) can be written as

$$\beta < \frac{a/b}{1-a/b}\frac{1-N/b}{N/b} < \alpha.$$

Denote the middle member of the above inequality by $\Psi_4 \equiv \Psi_4(s)$ $(s > 0)$. Using (4) and (8) we get

$$\Psi_4(s) = \frac{\cosh(s)}{1-\cosh s}\frac{2\sinh(s) - \sinh s \cosh s - s}{\sinh s \cosh s + s}.$$

Making use of Lemma 2.6 in [4] we conclude that the function $\Psi_4(s)$ is strictly increasing for all $s > 0$ and also that $\Psi_4(0^+) = 1/3$. and $\Psi_4(\infty^-) = 1$. Hence the assertion follows. □

## 5. The Ky Fan inequalities involving mean N

Ky Fan inequalities for various pairs of means have been a subject of many research papers published in mathematical literature. The Ky Fan inequalities for the Schwab-Borchardt mean are derived in [19] while the Ky Fan inequalities for particular means derived from the $N$ mean are established in [15].

The goal of this section is to establish Ky Fan inequalities for the means $N(a,b)$ and $N(b,a)$. Before we will state and prove the main result of this section let us introduce more notation. To this end we will assume that $0 < a,b \leqslant \frac{1}{2}$. Also, let $a' = 1-a$ and $b' = 1-b$. Research in this section is motivated by validity of the inequality [15, Theorem 4.1]:

$$b < N(a,b) < N(b,a) < a$$

provided $b < a$. It is natural to ask whether this inequality has its counterpart in the form of Ky Fan inequality? The answer is provided in the following:

THEOREM 6. *Let $0 < b < a \leqslant \frac{1}{2}$. Then the inequalities*

$$\frac{b}{b'} < \frac{N(a,b)}{N(a',b')} < \frac{N(b,a)}{N(b',a')} < \frac{a}{a'} \tag{30}$$

*are valid. Inequalities (30) are reversed if $0 < a < b \leqslant \frac{1}{2}$.*

*Proof.* It is elementary to show that assumption $0 < b < a \leqslant \frac{1}{2}$ implies the inequality

$$\frac{b}{b'} < \frac{a}{a'}. \tag{31}$$

We shall establish now inequalities (30). For, let us write the leftmost inequality in (30) as follows

$$\frac{N(a',b')}{b'} < \frac{N(a,b)}{b}$$

and also introduce $f_1$, where

$$f_1 = \frac{N(a,b)}{b}.$$

Application of (8) gives

$$f_1 \equiv f_1(s) = \frac{\sinh s \cosh s + s}{2\sinh s} =: \frac{n_1(s)}{d_1(s)},$$

where $\cosh s = a/b$. It is known that the function $f_1(s)$ is even and strictly increasing for $s > 0$ (see [15, p. 287]). Let $s'$ be defined implicitly as $\cosh s' = a'/b'$. Then (31) implies $\cosh s' < \cosh s$ and this in turn yields $s' < s$. Further, monotonicity of $f_1$ gives $f_1(s') < f_1(s)$ or what is the same that

$$\frac{N(a',b')}{b'} < \frac{N(a,b)}{b}.$$

Hence the first inequality in (30) follows. For the proof of the second inequality in (30) we introduce

$$f_2 = \frac{N(b,a)}{N(a,b)}.$$

Using (7) and (8) we obtain

$$f_2 = \frac{\sin r \cos r + r}{(\sin r)\left(1 + \dfrac{s}{\sinh s \cosh s}\right)}. \tag{32}$$

Taking into account that $\cosh s = \sec r$ we obtain $\sinh s = \tan r$ and $s = \cosh^{-1}(\sec r)$. With the aid of these formulas we can write (32) as

$$f_2(r) = \frac{\sin r \cos r + r}{\sin r + \cos^2 r \cosh^{-1}(\sec r)} =: \frac{n_2(r)}{d_2(r)}$$

$(0 < r < \pi/2)$. We shall show now that the function $f_2(r)$ is strictly increasing on its domain. Differentiating we obtain

$$\left(\frac{n_2'(r)}{d_2'(r)}\right)' = \frac{\cosh^{-1}(\sec r)}{[(\sin r)\cosh^{-1}(\sec r) - 1]^2} > 0.$$

Thus the function $\frac{n_2'(r)}{d_2'(r)}$ is strictly increasing. Using l'Hôpital Monotone Rule we conclude that the function $f_2 = \frac{n_2(r)}{d_2(r)}$ is also strictly increasing. Thus $f_2(s') < f_2(s)$. The last inequality can be written in terms of the mean $N$ as

$$\frac{N(b',a')}{N(a',b')} < \frac{N(b,a)}{N(a,b)}$$

which gives the second inequality in (30). In the proof of the third inequality in (30) we shall use quantity $f_3$, where

$$f_3 = \frac{a}{N(b,a)}.$$

Making use of (7) we obtain

$$f_3 \equiv f_3(r) = \frac{2\sin r}{\sin r \cos r + r} =: \frac{n_3(r)}{d_3(r)}$$

$(0 < r < \pi/2)$. It follows from the proof of Theorem 6.1 in [15] that $f_3(r)$ is strictly increasing on $0 < r < \pi/2$. This in turn implies that $f_3(s') < f_3(s)$. Thus the last inequality gives

$$\frac{a'}{N(b',a')} < \frac{a}{N(b,a)}.$$

The third inequality in (30) now follows. The second assertion of the Theorem 6 can be derived from the first one. It is easy to see that replacing $a$ by $b$ and $b'$ by $a'$ gives the desired result. Let us note that the assumption $0 < a < b \leqslant \frac{1}{2}$ yields

$$\frac{b}{b'} > \frac{a}{a'}.$$

The proof is complete. □

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