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ASYMPTOTIC BEHAVIOR OF POWER MEANS

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Abstract. We consider asymptotic behavior of classical n-variable means. General expansions of these means are known in the term of Bell polynomials. Here, simple recursive algorithms are derived. The obtained coefficients are used in analysis of some inequalities between means which include the first asymptotic term.

1. Introduction

In this paper we discuss relations between some classical means, using technique of asymptotic expansions. Let us fix the notation first. A letter M will denote any of means mentioned below. For n-tuples of positive real numbers, $\mathbf{a}=(a_1,a_2,\dots,a_n)$ and $\mathbf{w}=(w_1,w_2,\dots,w_n)$, $\sum_{i=1}^n w_i=1$, let us denote weighted means:

$$Q(\mathbf{a})=\sqrt{w_1a_1^2+w_2a_2^2+\ldots+w_na_n^2}, \qquad \text{(quadratic mean)}$$

$$A(\mathbf{a})=w_1a_1+w_2a_2+\ldots+w_na_n, \qquad \text{(arithmetic mean)}$$

$$G(\mathbf{a})=a_1^{w_1}a_2^{w_2}\cdots a_n^{w_n}, \qquad \text{(geometric mean)}$$

$$H(\mathbf{a})=\frac{1}{\dfrac{w_1}{a_1}+\dfrac{w_2}{a_2}+\ldots+\dfrac{w_n}{a_n}}, \qquad \text{(harmonic mean)}$$

$$M_r(\mathbf{a})=\Big[w_1a_1^r+w_2a_2^r+\ldots+w_na_n^r\Big]^{1/r}, \quad r\neq 0. \qquad \text{(power mean)}$$

It is well known that it holds $Q\geqslant A\geqslant G\geqslant H$, and that all these means are contained in the family of power means, geometric mean corresponds to the limit case $r\to 0$.

The inequality between arithmetic and geometric mean holds for all positive values of its arguments. Note that the values of A and G can be quite different and the quotient $(A-G)/G$ can be arbitrary large.

What can be said if all arguments are taken from a subinterval I, far away from the origin? Then all variables are of the same order of magnitude and relative difference between means is in principle much smaller.

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For a fixed n-tuple $(a_1, a_2, \ldots, a_n)$ a shift by positive variable x will move this n-tuple into a bounded subinterval I^n. Let us denote by $x\mathbf{e}+\mathbf{a}$ the n-tuple with elements $(x+a_j)$, where $\mathbf{e}=(1,1,\ldots,1)$. The behaviour of $M(x\mathbf{e}+\mathbf{a})$ for large value of x will be essential in our analysis.

In the recent papers [2, 7, 8, 10], a similar problem was analyzed for two-variable means. Many of results obtained there can be translated in a setting of this paper, but here we discuss questions which are marginal in the case of two-parameter means.

Since for all x

$$A(x\mathbf{e}+\mathbf{a}) = x + A(\mathbf{a}),$$

the asymptotic expansion for this mean has only these two terms.

Since

$$\begin{aligned} H(x\mathbf{e}+\mathbf{a}) &= x \cdot \frac{1}{\sum\limits_{j=1}^{n} \dfrac{w_j}{1+a_j/x}} = x \cdot \frac{1}{\sum\limits_{j=1}^{n} w_j\left(1-\dfrac{a_j}{x}+O(1/x^2)\right)} \\ &= x \cdot \frac{1}{1-\dfrac{A(\mathbf{a})}{x}+O(1/x^2)} = x\left(1+\frac{A(\mathbf{a})}{x}+O(1/x^2)\right) \\ &= x + A(\mathbf{a}) + O(1/x) \end{aligned}$$

we can conclude that it holds

$$H(x\mathbf{e}+\mathbf{a}) - x \to A(\mathbf{a}) \qquad \text{as } x \to \infty.$$

Since $H \leqslant G \leqslant A$, the same conclusion holds also for geometric mean. It will be shown that a power mean has the same property. Therefore, the asymptotic expansion of the function $x \mapsto M(x\mathbf{e}+\mathbf{a})$ should be of the form

$$M(x\mathbf{e}+\mathbf{a}) = x + A(\mathbf{a}) + \sum_{k=2}^{\infty} c_k(\mathbf{a}) x^{-k+1}$$

for any of means from the above-mentioned list. Here c_n, $n \in \mathbb{N}_0$, are homogenous polynomials of order n which can be expressed as a function of

$$m_k = w_1 a_1^k + w_2 a_2^k + \ldots + w_n a_n^k = M_k^k(\mathbf{a}).$$

Let us denote $m_0 = 1$.

The complete asymptotic expansion of power mean in terms of Bell polynomials was proved in [1]:

THEOREM. *For each $p \in \mathbb{R}$, the power means $M_p(x\mathbf{e}+\mathbf{a})$ possess the complete asymptotic expansion*

$$M_r(x\mathbf{e}+\mathbf{a}) = x + \sum_{k=0}^{\infty} c_k(\mathbf{a}) x^{-k}, \quad x \to \infty.$$

In the case $p \neq 0$ *the coefficients are given by*

$$c_k(r) = \frac{1}{(k+1)!} \sum_{j=1}^{k+1} j! \binom{1/r}{j} \mathbf{B}_{k+1,j} \left[i! \binom{r}{i} M_i^i(\mathbf{a}) \right].$$

In case $p = 0$ *coefficients are given by*

$$c_k(0) = \frac{(-1)^{k+1}}{(k+1)!} \mathbf{Y}_{k+1} \left[-(i-1)! M_i^i(\mathbf{a}) \right].$$

$\mathbf{B}_{m,j}[y_i] := \mathbf{B}_{m,j}(y_1, \ldots, y_{m-j+1})$ denotes partial Bell polynomials defined by

$$\frac{1}{j!} \left(\sum_{k=1}^{\infty} y_k \frac{t^k}{k!} \right)^j = \sum_{m=j}^{\infty} \mathbf{B}_{m,j}[y_i] \frac{t^m}{m!}, \quad j = 1, 2, \ldots$$

and $\mathbf{Y}_m[y_i] = \mathbf{Y}_m(y_1, \ldots, y_n)$ are complete Bell polynomials defined by

$$\exp\left(\sum_{j=1}^{\infty} y_j \frac{t^j}{j!} \right) = 1 + \sum_{m=1}^{\infty} \mathbf{Y}_m[y_i] \frac{t^m}{m!}.$$

In this paper we derive an efficient recursive formula for such expansion and simplified verisons in the particular cases of the above-mentioned means. Our results are based on the using of the following lemmas about functional transformations of asymptotic series, see [3, 4, 8, 9].

LEMMA 1.1. *Let* $a_0 = 1$ *and* $g(x)$ *be a function with the asymptotic expansion*

$$g(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}.$$

Then for all real p *it holds*

$$g(x)^p \sim \sum_{n=0}^{\infty} P_n(p) x^{-n},$$

where $P_0 = 1$ *and*

$$P_n(p) = \frac{1}{n} \sum_{k=1}^{n} [k(1+p) - n] a_k P_{n-k}(p).$$

Especially, for $p = -1$ *we obtain formula for the reciprocal value of an asymptotic series:*

$$\frac{1}{g(x)} \sim \sum_{n=0}^{\infty} R_n x^{-n},$$

where $R_0 = 1$ *and*

$$R_n = -\sum_{k=1}^{n} a_k R_{n-k}.$$

LEMMA 1.2. *Let functions $f(x)$ and $g(x)$ have the following asymptotic expansions ($a_0 \neq 0,\ b_0 \neq 0$) as $x \to \infty$:*

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \qquad g(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}.$$

Then the asymptotic expansion of their quotient $f(x)/g(x)$ reads as

$$\frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} c_n x^{-n},$$

where coefficients c_n are defined by

$$c_n = \frac{1}{b_0}\left(a_n - \sum_{k=1}^{n} b_k c_{n-k}\right).$$

LEMMA 1.3. *Let*

$$g(x) = \sum_{n=1}^{\infty} a_n x^{-n}$$

be a given asymptotical expansion. Then the composition $\exp(g(x))$ *has asymptotic expansion of the following form*

$$\exp(g(x)) = \sum_{n=0}^{\infty} b_n x^{-n}$$

where $b_0 = 1$ and

$$b_n = \frac{1}{n}\sum_{k=1}^{n} k a_k b_{n-k}, \qquad n \geqslant 1.$$

This paper is organized as follows. In the next section an asymptotic expansion of power mean was found. Coefficients depending on a real parameter r are defined by recursive relation and some interesting properties are detected. As a consequence we obtained asymptotic expansions of some well known classical means. Section 3 begins with reminder of the results from our previous paper in which we studied inequalities related to bivariate means and continues with the discussion of the known inequalities related to n-variable classical means. In Section 4, we established some asymptotic inequalities covering arithmetic, geometric and harmonic means.

2. Power mean

The power mean can be written in a way

$$\begin{aligned} M_r(x\mathbf{e}+\mathbf{a}) &= \left[\sum_{j=1}^{n} w_j (x+a_j)^r\right]^{1/r} = x\left[\sum_{j=1}^{n} w_j \sum_{k=0}^{\infty}\binom{r}{k} a_j^k x^{-k}\right]^{1/r} \\ &= x\left[\sum_{k=0}^{\infty}\binom{r}{k}\frac{1}{n}\left(\sum_{j=1}^{n} w_j a_j^k\right)x^{-k}\right]^{1/r} = x\left[\sum_{k=0}^{\infty}\binom{r}{k} m_k x^{-k}\right]^{1/r}. \end{aligned}$$

From Lemma 1.1 we get the following result.

THEOREM 2.1. *General power mean has the following asymptotic expansion*

$$M_r(x\mathbf{e}+\mathbf{a}) = x\cdot\sum_{k=0}^{\infty} c_k(r)x^{-k},$$

where $c_0 = 1$ *and*

$$c_k(r) = \frac{1}{k}\sum_{j=1}^{k}\left[j\left(1+\frac{1}{r}\right)-k\right]\binom{r}{j}m_j c_{k-j}(r), \quad k\in\mathbb{N}.$$

The first few coefficients are

$$\begin{aligned}
c_0(r) &= 1,\\
c_1(r) &= m_1,\\
c_2(r) &= -\tfrac{1}{2}(r-1)(m_1^2-m_2),\\
c_3(r) &= \tfrac{1}{6}(r-1)\big((2r-1)m_1^3-3(r-1)m_1m_2+(r-2)m_3\big),\\
c_4(r) &= -\tfrac{1}{24}(r-1)\big((3r-1)(2r-1)m_1^4-6(r-1)(2r-1)m_1^2m_2\\
&\quad+3(r-1)^2m_2^2+4(r-2)(r-1)m_1m_3-(r-3)(r-2)m_4\big).
\end{aligned} \tag{2.1}$$

THEOREM 2.2. *Coefficients* c_k, $k\in\mathbb{N}_0$, *are homogeneous polynomials in variables* $(a_1,\dots,a_n)$ *and have the following form:*

$$c_k(r,\mathbf{a}) = \sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_k\geqslant 0\\ \alpha_1+2\alpha_2+\cdots+k\alpha_k=k}} q_{\alpha_1,\dots,\alpha_k}(r)m_1(\mathbf{a})^{\alpha_1}\cdots m_k(\mathbf{a})^{\alpha_k}, \tag{2.2}$$

where

$$\sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_k\geqslant 0\\ \alpha_1+2\alpha_2+\cdots+k\alpha_k=k}} q_{\alpha_1,\dots,\alpha_k}(r) = 0, \quad k\geqslant 2. \tag{2.3}$$

Proof. Using homogeneity of mean M_r, we have

$$\begin{aligned}
\lambda x\sum_{k=0}^{\infty} c_k(r,\mathbf{a})x^{-k} &= \lambda M_r(x\mathbf{e}+\mathbf{a}) = M_r(\lambda x\mathbf{e}+\lambda\mathbf{a})\\
&= \lambda x\sum_{k=0}^{\infty} c_k(r,\lambda\mathbf{a})(\lambda x)^{-k} = \lambda x\sum_{k=0}^{\infty}\lambda^{-k}c_k(r,\lambda\mathbf{a})x^{-k},
\end{aligned}$$

hence,

$$c_k(r,\lambda\mathbf{a}) = \lambda^k c_k(r,\mathbf{a}).$$

The form (2.2) can be deduced easily using induction. The property (2.3) follows from the special case:

$$\begin{aligned}
x+a = M_r(x\mathbf{e}+a\mathbf{e}) &= \sum_{k=0}^{\infty} c_k(r,a\mathbf{e})x^{-k}\\
&= \sum_{k=0}^{\infty} a^k\left(\sum_{\substack{\alpha_1,\alpha_2,\dots,\alpha_k\geqslant 0\\ \alpha_1+2\alpha_2+\cdots+k\alpha_k=k}} q_{\alpha_1,\dots,\alpha_k}(r)\right)x^{-k}. \quad \square
\end{aligned}$$

COROLLARY 2.3. *Harmonic mean has the following expansion*

$$H(x\mathbf{e}+\mathbf{a}) = x + m_1 + \sum_{k=2}^{\infty} c_k x^{-k+1},$$

where coefficients are given by $c_0 = 1$ *and*

$$c_k = \sum_{j=1}^{k} (-1)^{j-1} m_j c_{k-j}.$$

The first few coefficients are

$$\begin{aligned}
c_0 &= 1,\\
c_1 &= m_1,\\
c_2 &= m_1^2 - m_2,\\
c_3 &= m_1^3 - 2m_1m_2 + m_3,\\
c_4 &= m_1^4 - 3m_1^2m_2 + m_2^2 + 2m_1m_3 - m_4,\\
c_5 &= m_1^5 - 4m_1^3m_2 + 3m_1^2m_3 - 2m_2m_3 + 3m_1m_2^2 - 2m_1m_4 + m_5.
\end{aligned}$$

COROLLARY 2.4. *Quadratic mean has the following asymptotic expansion*

$$Q(x\mathbf{e}+\mathbf{a}) = x \cdot \sum_{k=0}^{\infty} c_k x^{-k},$$

where $c_0 = 1$, $c_1 = m_1$ *and*

$$c_k = \left(\frac{3}{k} - 2\right) m_1 c_{k-1} + \left(\frac{3}{k} - 1\right) m_2 c_{k-2}, \qquad k \geqslant 2.$$

The first few coefficients are

$$\begin{aligned}
c_0 &= 1,\\
c_1 &= m_1,\\
c_2 &= \tfrac{1}{2}(-m_1^2 + m_2),\\
c_3 &= \tfrac{1}{2}m_1(m_1^2 - m_2),\\
c_4 &= \tfrac{1}{8}(-5m_1^4 + 6m_1^2m_2 - m_2^2),\\
c_5 &= \tfrac{1}{8}m_1(7m_1^2 - 3m_2)(m_1^2 - m_2).
\end{aligned}$$

THEOREM 2.5. *Geometric mean has the following asymptotic expansion*

$$G(x\mathbf{e}+\mathbf{a}) = x \cdot \sum_{k=0}^{\infty} c_k x^{-k},$$

where $c_0 = 1$ *and*

$$c_k = \frac{1}{k} \sum_{j=1}^{k} (-1)^{j-1} m_j c_{k-j}, \quad k \geqslant 2.$$

Proof.

$$\log(G(x\mathbf{e}+\mathbf{a})) = \sum_{j=1}^{n} w_j \log(x+a_j) = \log x + \sum_{j=1}^{n} w_j \sum_{k=1}^{\infty} (-1)^{k-1} \frac{a_j^k}{x^k}$$
$$= \log x + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{m_k}{x^k}.$$

Now the desired result follows from

$$G(x\mathbf{e}+\mathbf{a}) = x \exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{m_k}{k} x^{-k} \right)$$

using Lemma 1.3. □

The first few coefficients are

$$\begin{aligned} c_0 &= 1, \\ c_1 &= m_1, \\ c_2 &= \tfrac{1}{2}(m_1^2 - m_2), \\ c_3 &= \tfrac{1}{6}(m_1^3 - 3m_1 m_2 + 2m_3), \\ c_4 &= \tfrac{1}{24}(m_1^4 - 6m_1^2 m_2 + 3m_2^2 + 8m_1 m_3 - 6m_4). \end{aligned}$$

Notice that in the limit $r \to 0$ coefficients (2.1) become equal to coefficients in the asymptotic expansion of geometric mean.

2.1. Ratio of power means.

Abel and Ivan [1] proved that ratio

$$\frac{M_p(x\mathbf{e}+\mathbf{a}) - M_q(x\mathbf{e}+\mathbf{a})}{M_r(x\mathbf{e}+\mathbf{a}) - M_s(x\mathbf{e}+\mathbf{a})}$$

possesses an asymptotic expansion and the first few coefficients were written. Here, we give recursive formula for all coefficients. The proof of this theorem follows immediately using Lemma 1.2.

THEOREM 2.6. *For all real p,q,r,s with $r \neq s$, the asymptotic expansion of the ratio of power means is given by*

$$\frac{M_p(x\mathbf{e}+\mathbf{a}) - M_q(x\mathbf{e}+\mathbf{a})}{M_r(x\mathbf{e}+\mathbf{a}) - M_s(x\mathbf{e}+\mathbf{a})} \sim \sum_{k=0}^{\infty} c_k x^{-k},$$

where

$$c_k = \frac{1}{c_2(r) - c_2(s)} \Big[(c_{k+2}(p) - c_{k+2}(q)) - \sum_{j=1}^{k} (c_{j+2}(r) - c_{j+2}(s)) c_{k-j} \Big].$$

2.2. Gini mean

One natural generalization of a power mean is the Gini mean, defined as:

$$G_{p,q}(\mathbf{a}) = \begin{cases} \left[\frac{a_1^p+\cdots+a_n^p}{a_1^q+\cdots+a_n^q}\right]^{\frac{1}{p-q}}, & p \neq q, \\ \exp\left(\frac{\sum_{k=1}^n a_k^p \ln a_k}{\sum_{k=1}^n a_k^p}\right), & p = q \neq 0, \\ \sqrt{a_1 \cdots a_n}, & p = q = 0. \end{cases}$$

Using the same technique, one can prove the following theorem.

THEOREM 2.7. *The Gini mean has the following asymptotic expansion*

$$G_{p,q}(x\mathbf{e}+\mathbf{a}) = x \cdot \sum_{k=0}^{\infty} c_k x^{-k},$$

where $c_0 = 1$,

$$c_k = \frac{1}{k}\sum_{j=1}^{k}\left[j\left(1+\frac{1}{p-q}\right)-k\right] b_j c_{k-j}, \quad k \in \mathbb{N},$$

and

$$b_k = \binom{p}{k} m_k - \sum_{j=1}^{k}\binom{q}{j} m_j b_{k-j}, \quad k \geqslant 0.$$

The first few coefficients are

$$\begin{aligned} c_0 &= 1, \\ c_1 &= m_1, \\ c_2 &= -\tfrac{1}{2}(p+q-1)(m_1^2 - m_2), \\ c_3 &= \tfrac{1}{6}\big(((p+q-1)(2q-1)+2p(p-1))m_1^3 \\ &\quad - 3((p+q-1)(q-1)+p(p-1))m_1 m_2 \\ &\quad + ((p+q-1)(q-2)+p(p-1))m_3\big), \\ c_4 &= \tfrac{1}{24}\big(-((p+q-1)(6q^2-5q+6p^2-9p+1)+4p(p-1))m_1^4 \\ &\quad + 6(p+q-1)(2q^2-3q+2p^2-3p+1)m_1^2 m_2 \\ &\quad - 3(p+q-1)(p(p-2)+(q-1)^2)m_2^2 \\ &\quad - 4(p+q-2)(p(p-2)+(q-1)^2)m_1 m_3 \\ &\quad + (p+q-3)(q^2-3q+p^2-3p+2)m_4\big). \end{aligned}$$

3. Inequalities between means

3.1. Bivariate means

Let us discuss briefly inequalities between bivariate means which follow from their asymptotic expansions. Let us denote

$$\alpha = \frac{t+s}{2}, \qquad \beta = \frac{t-s}{2}.$$

It is proved in [7] that

$$\begin{aligned}
Q(x+s,x+t) &= x+\alpha+\frac{\beta^2}{2x}-\frac{\alpha\beta^2}{2x^2}+\frac{\beta^2(4\alpha^2-\beta^2)}{8x^3}+o(x^{-3}),\\
A(x+s,x+t) &= x+\alpha,\\
I(x+s,x+t) &= x+\alpha-\frac{\beta^2}{6x}+\frac{\alpha\beta^2}{6x^2}-\frac{\beta^2(60\alpha^2+13\beta^2)}{360x^3}+o(x^{-3}),\\
L(x+s,x+t) &= x+\alpha-\frac{\beta^2}{3x}+\frac{\alpha\beta^2}{3x^2}-\frac{\beta^2(15\alpha^2+4\beta^2)}{45x^3}+o(x^{-3}),\\
G(x+s,x+t) &= x+\alpha-\frac{\beta^2}{2x}+\frac{\alpha\beta^2}{2x^2}-\frac{\beta^2(4\alpha^2+\beta^2)}{8x^3}+o(x^{-3}),\\
H(x+s,x+t) &= x+\alpha-\frac{\beta^2}{x}+\frac{\alpha\beta^2}{x^2}-\frac{\beta^2\alpha^3}{8x^3}+o(x^{-3}).
\end{aligned}$$

As a consequence, the folowing inequalities are proved.

THEOREM 3.1. ([7]) *Let* $0<s<t$. *Then for all* $x>0$ *we have*

$$Q(x+s,x+t) < x+\alpha+\frac{\beta^2}{2x}, \tag{3.1}$$

$$I(x+s,x+t) > x+\alpha-\frac{\beta^2}{6x}, \tag{3.2}$$

$$L(x+s,x+t) > x+\alpha-\frac{\beta^2}{3x}, \tag{3.3}$$

$$G(x+s,x+t) > x+\alpha-\frac{\beta^2}{2x}, \tag{3.4}$$

$$H(x+s,x+t) > x+\alpha-\frac{\beta^2}{x}. \tag{3.5}$$

In the case $s<t$, $s+t<0$ *the inequalities* (3.2)–(3.5) *hold with opposite sign, for each* $x>-s$.

Here, I is identric and L logarithmic mean. We shall interpret these inequalities in different settings, which is closer to the problem of comparison of power means.

3.2. n-variable means

Let $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$ be fixed positive n-tuple. In the sequel we shall consider the equal-weight case, $w_1 = \ldots = w_n = \frac{1}{n}$. Then, all means are symmetric and as a consequence, all polynomials c_r are symmetric polynomials.

We are dealing with asymptotic behaviour of means, so it is natural to suppose that a_1 is sufficient large. Let us denote $b_i = a_i - x$, for each $1 \leqslant i \leqslant n$. Then

$$M(\mathbf{a}) = M(x\mathbf{e} + \mathbf{b})$$

and we can use asymptotic expansion for the function on the right. All means considered in this paper have asymptotic expansion of the form

$$M(\mathbf{a}) = x + A(\mathbf{b}) + \gamma_2 \frac{m_1^2 - m_2}{x} + \ldots$$

where γ_2 is a constant and both of the following terms behave well under translation:

$$x + A(\mathbf{b}) = A(\mathbf{a}),$$

$$m_2 - m_1^2 = \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} (b_i - b_j)^2 = \frac{1}{n^2} \sum_{1 \leqslant i < j \leqslant n} (a_i - a_j)^2.$$

It is natural to pose the following problem: find the best constants γ and δ such that for given two means F_1 and F_2, $F_1 \leqslant F_2$, we have

$$\frac{m_2 - m_1^2}{\gamma} < F_2(\mathbf{a}) - F_1(\mathbf{a}) < \frac{m_2 - m_1^2}{\delta}.$$

Some partial answers to this problem are already known.

For $0 \leqslant a_1 \leqslant \ldots \leqslant a_n$ the following inequalities for n-variable means A, G and H, with equal weights are given in [11, p. 39]:

$$\frac{1}{2n^2} \frac{1}{a_n} \sum_{1 \leqslant i < j \leqslant n} (a_i - a_j)^2 \leqslant A(\mathbf{a}) - G(\mathbf{a}) \leqslant \frac{1}{2n^2} \frac{1}{a_1} \sum_{1 \leqslant i < j \leqslant n} (a_i - a_j)^2 \tag{3.6}$$

and

$$\frac{1}{2n^2} \frac{a_1^3}{a_n^4} \sum_{1 \leqslant i < j \leqslant n} (a_i - a_j)^2 \leqslant G(\mathbf{a}) - H(\mathbf{a}) \leqslant \frac{1}{2n^2} \frac{a_n^3}{a_1^4} \sum_{1 \leqslant i < j \leqslant n} (a_i - a_j)^2.$$

Later on, Zhan, Xi and Chu [13] found improvements of these inequalities. If $n \geqslant 2$ and $0 < b \leqslant a_1 \leqslant \ldots \leqslant a_n \leqslant B$, then

$$\frac{\sum_{1 \leqslant i < j \leqslant n} (a_i - a_j)^2}{2n^2 B^{(n-1)/n} A^{1/n}(\mathbf{a})} \leqslant A(\mathbf{a}) - M_0(\mathbf{a}) \leqslant \frac{\sum_{1 \leqslant i < j \leqslant n} (a_i - a_j)^2}{2n^2 b^{(n-1)/n} A^{1/n}(\mathbf{a})}$$

and

$$\frac{b^{(n-1)/n}}{2n^2 B^{(2n-1)/n}} \sum_{1 \leqslant i < j \leqslant n} (a_i - a_j)^2 \leqslant M_0(\mathbf{a}) - M_{-1}(\mathbf{a}) \leqslant \frac{B^{(n-3)/n}}{2n^2 b^{(2n-3)/n}} \sum_{1 \leqslant i < j \leqslant n} (a_i - a_j)^2. \tag{3.7}$$

We shall show that these inequalities are not optimal. Let us begin with the A-G case for $n=2$. We have

$$\frac{a_1+a_2}{2}-\sqrt{a_1a_2}=\frac{(a_2-a_1)^2}{8\xi},$$

where

$$\xi=\left(\frac{\sqrt{a_1}+\sqrt{a_2}}{2}\right)^2.$$

Therefore, one has

$$\frac{(a_2-a_1)^2}{8\gamma}<A(\mathbf{a})-G(\mathbf{a})<\frac{(a_2-a_1)^2}{8\delta}$$

whenever $\delta<\xi<\gamma$. This is consistent with (3.6) since ξ given above is a mean and $a_1\leqslant\xi\leqslant a_n$.

Similar inequality is true also in the A-H inequality, where the critical value is $\xi=(a_1+a_2)/2$.

In the case of G-H inequality, the critical value is

$$\xi=\left(\frac{\sqrt{a_1}+\sqrt{a_2}}{2}\right)^2\left(\frac{a_1+a_2}{2}\right)\frac{1}{\sqrt{a_1a_2}}.$$

Therefore, at least for $n=2$ better simple bounds can be found, for example:

$$\delta=\frac{a_1+a_2}{2}<\xi<\frac{a_1+a_2}{2}\sqrt{\frac{a_2}{a_1}}=\gamma$$

which is better than bounds

$$\delta=\sqrt{a_1a_2},\qquad \gamma=a_2\sqrt{\frac{a_2}{a_1}}$$

which follow from (3.7).

4. Asymptotic inequalities

DEFINITION 4.1. Let $F(s,t)$ be any homogenous bivariate function such that

$$F(x+s,x+t)=c_k(t,s)x^{-k+1}+\mathscr{O}(x^{-k}).$$

If $c_k(s,t)>0$ for all s and t, we say that F is *asymptotically* greater than zero, and write

$$F\succ 0.$$

Asymptotic inequalities between means impose necessary conditions for the proper inequalities, see [4–7]. The sign of the first neglected coefficient is essential in such analysis. Let us describe the idea in the case of means G and H. It holds

$$G(x\mathbf{e}+\mathbf{b})-H(x\mathbf{e}+\mathbf{b})-\frac{m_2-m_1^2}{2x}=-\frac{1}{6x^2}(4m_3-9m_1m_2+5m_1^3)+o(x^{-2}).$$

In the general case, the sign of

$$\Delta(\mathbf{b})=4m_3-9m_1m_2+5m_1^3$$

depends on the number of variables n.

THEOREM 4.2. *Let $n\in\{1,2,3,4,5\}$. In the case $0\leqslant b_1\leqslant b_2\leqslant\ldots\leqslant b_n$ we have*

$$G(x\mathbf{e}+\mathbf{b})-H(x\mathbf{e}+\mathbf{b})-\frac{m_2-m_1^2}{2x}\prec 0. \tag{4.1}$$

In the case $b_1\leqslant b_2\leqslant\ldots\leqslant b_n\leqslant 0$ the opposite inequality holds in (4.1).

Proof. Let $0\leqslant b_1\leqslant\ldots\leqslant b_n$. According to Corollary 2.1. (1) from [12], inequality

$$\Delta(\mathbf{b})\geqslant 0$$

of degree $d=3$ holds for all $\mathbf{b}\in\mathbf{R}_+^n$ if and only if it holds for every $\mathbf{b}\in\mathbf{R}_+^n$ such that the number of non-zero distinct components of $\mathbf{b}$ is less or equal to $\max(\lfloor\frac{3}{2}\rfloor,1)=1$. Let us take

$$\mathbf{b}=(1,\ldots,1,0,\ldots,0) \tag{4.2}$$

which consists of k units and $n-k$ zeros. Since $0\leqslant\frac{k}{n}\leqslant\frac{4}{5}$ or $\frac{k}{n}=1$, it follows

$$\Delta(\mathbf{b})=5\frac{k^3}{n^3}-9\frac{k^2}{n^2}+4\frac{k}{n}=5\frac{k}{n}\left(\frac{k}{n}-\frac{4}{5}\right)\left(\frac{k}{n}-1\right)\geqslant 0. \qquad \square$$

The positivity od Δ is not true for $n\geqslant 6$. Take $k<n$ such that $\frac{4}{5}<\frac{k}{n}$. This is possible for each $n\geqslant 6$. Then, for $\mathbf{b}$ as in (4.2) we have $\Delta(\mathbf{b})<0$. Hence, the inequality

$$G(x\mathbf{e}+\mathbf{b})-H(x\mathbf{e}+\mathbf{b})-\frac{m_2-m_1^2}{2x}\prec 0$$

cannot be true under conditions $0\leqslant b_1\leqslant\ldots\leqslant b_n$, for $n\geqslant 6$.

Similar conclusions also hold for other means.

THEOREM 4.3. *In the case $0\leqslant b_1\leqslant b_2\leqslant\ldots\leqslant b_n$ we have*

$$A(x\mathbf{e}+\mathbf{b})-G(x\mathbf{e}+\mathbf{b})-\frac{m_2-m_1^2}{2x}\succ 0,$$

$$A(x\mathbf{e}+\mathbf{b})-H(x\mathbf{e}+\mathbf{b})-\frac{m_2-m_1^2}{x}\succ 0,$$

$$
Q(x\mathbf{e}+\mathbf{b})-H(x\mathbf{e}+\mathbf{b})-\frac{3(m_2-m_1^2)}{2x}\prec 0,
$$
$$
Q(x\mathbf{e}+\mathbf{b})-G(x\mathbf{e}+\mathbf{b})-\frac{m_2-m_1^2}{x}\succ 0,
$$
$$
Q(x\mathbf{e}+\mathbf{b})-A(x\mathbf{e}+\mathbf{b})-\frac{m_2-m_1^2}{x}\prec 0.
$$

In the case $b_1 \leqslant b_2 \leqslant \ldots \leqslant b_n \leqslant 0$ *the opposite inequalities hold.*

Proof. The proof follows as in the previous theorem since it holds:

$$
\begin{aligned}
A(x\mathbf{e}+\mathbf{b})-G(x\mathbf{e}+\mathbf{b})-\frac{m_2-m_1^2}{2x}&=\frac{m_1^3-3m_1m_2+2m_3}{6x^2}+\mathscr{O}(x^{-3}),\\
Q(x\mathbf{e}+\mathbf{b})-H(x\mathbf{e}+\mathbf{b})-\frac{3(m_2-m_1^2)}{2x}&=-\frac{m_1^3-3m_1m_2+2m_3}{2x^2}+\mathscr{O}(x^{-3}),\\
Q(x\mathbf{e}+\mathbf{b})-G(x\mathbf{e}+\mathbf{b})-\frac{m_2-m_1^2}{x}&=-\frac{m_1^3-m_3}{3x^2}+\mathscr{O}(x^{-3}),\\
Q(x\mathbf{e}+\mathbf{b})-A(x\mathbf{e}+\mathbf{b})-\frac{m_2-m_1^2}{x}&=-\frac{m_1(m_2-m_1^2)}{2x^2}+\mathscr{O}(x^{-3}). \quad \square
\end{aligned}
$$

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