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ON SOME PROPERTIES OF STRICTLY CONVEX FUNCTIONS

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Abstract. We prove that some careless modification of the definition of strong convexity leads to a condition which is equivalent to that one of strict convexity.

Given a convex subset D of a real linear space a function $f : D \to \mathbb{R}$ is called *convex* if

$$\bigwedge_{x,y\in D} \bigwedge_{t\in(0,1)} f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)$$

and *strictly convex* if the above inequality is strict whenever $x \neq y$:

$$\bigwedge_{\substack{x,y\in D\\ x\neq y}} \bigwedge_{t\in(0,1)} f(tx+(1-t)y) < tf(x)+(1-t)f(y).$$

Replacing the signs " $\leqslant$ " and " $<$ " by " $\geqslant$ " and " $>$ " above we come to the notions of *concavity* and *strict concavity,* respectively. Clearly every strictly convex function is convex but the converse fails to be true: the function $\mathbb{R} \ni x \longmapsto x$ serves as an example.

If the considered space is endowed with a norm $\|\cdot\|$ we can introduce one notion related to the convexity more. Namely, a function $f : D \to \mathbb{R}$ is said to be *strongly convex with modulus* $c \in (0,+\infty)$ if

$$\bigwedge_{x,y\in D} \bigwedge_{t\in(0,1)} f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y) - ct(1-t)\|x-y\|^2 .$$

We say that f is *strongly convex* if it is strongly convex with some positive modulus. Similarly we introduce the notion of *strong concavity*.

It seems that the notion of strong convexity was introduced by Polyak [6] in 1966. Strongly convex functions play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature. Let us mention here the papers by Vial [8], Montrucchio [2], Jovanovič [1], Polovinkin [5]. Also the classical book [7] due to Roberts and Varberg contains some information on that notion. Finally let us mention the paper [4] by the second present author; that is a survey article entirely devoted to strongly convex functions.

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Evidently every strongly convex function is strictly convex. However, the converse is generally not the case: the exponential function $\mathbb{R} \ni x \longmapsto \exp x$ is strictly convex but it is not strongly convex with any modulus $c \in (0,+\infty)$; as any two norms in $\mathbb{R}$ (endowed with the usual linear operations) are equivalent, the latter does not depend on the norm considered there.

The main aim of this note is to answer the following question: what can be said if we make a typical "student" mistake and formally change the order of quantifiers in the definition of strong convexity: namely, instead of the condition

$$\bigvee_{c\in(0,+\infty)} \bigwedge_{x,y\in D} \bigwedge_{t\in(0,1)} f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)-ct(1-t)\|x-y\|^2$$

we consider a weaker one

$$\bigwedge_{x,y\in D} \bigvee_{c\in(0,+\infty)} \bigwedge_{t\in(0,1)} f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)-ct(1-t)\|x-y\|^2. \quad (1)$$

We prove that, rather unexpectedly, the following result holds.

THEOREM. *Let D be a convex subset of a real normed space and let $f : D \to \mathbb{R}$. Then condition (1) holds if and only if f is strictly convex.*

Proof. Assume that the function f is strictly convex. To prove (1) fix points $x,y \in D$. Without loss of generality we may assume that $x \neq y$. It is well known that the function $F_{x,y} : [0,1] \to \mathbb{R}$, given by

$$F_{x,y}(t) = f(tx+(1-t)y),$$

is strictly convex (see for instance [3, Prop. 3.4.2]; the reader can also verify this fact via routine calculation). Define the function $G_{x,y} : [0,1] \to \mathbb{R}$ by

$$G_{x,y}(t) = tf(x)+(1-t)f(y)-f(tx+(1-t)y).$$

Then

$$G_{x,y}(t) = tf(x)+(1-t)f(y)-F_{x,y}(t), \qquad t \in [0,1],$$

so $G_{x,y}$ is strictly concave as the difference of an affine function and a strictly convex one. In what follows to simplify the notation write G instead of $G_{x,y}$. Observe that G is continuous, $G(0) = G(1) = 0$ and, by the strict convexity of f, we have $G(t) > 0$ for every $t \in (0,1)$.

For any $c \in (0,+\infty)$ define a function $R_c : (0,1) \to \mathbb{R}$ by

$$R_c(t) = ct(1-t)\|x-y\|^2.$$

To get (1) it is enough to prove the existence of a $c \in (0,+\infty)$ satisfying

$$R_c(t) \leqslant G(t), \quad t \in (0,1).$$

Suppose on the contrary that this is not the case. Then there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of numbers from $(0,1)$ such that

$$R_{1/n}(t_n) > G(t_n), \quad n\in\mathbb{N}.$$

Choose a subsequence $(t_{k_n})_{n\in\mathbb{N}}$ of $(t_n)_{n\in\mathbb{N}}$ converging to a $t_0\in[0,1]$. Then, since $t(1-t)\leqslant 1/4$ for all $t\in\mathbb{R}$, we have

$$G(t_{k_n}) < R_{1/k_n}(t_{k_n}) = \frac{1}{k_n} t_{k_n}(1-t_{k_n})\|x-y\|^2 \leqslant \frac{\|x-y\|^2}{4k_n}$$

for each $n\in\mathbb{N}$. Therefore, by the continuity of G, we get $G(t_0)\leqslant 0$, and thus $G(t_0) = 0$, whence either $t_0=0$, or $t_0=1$. Assume, for instance, that $t_0=0$. Since $G(0) = 0 < G(t)$ for $t\in(0,1)$ and G is concave, it follows that G has a positive derivative at 0. On the other hand

$$\frac{G(t_{k_n})}{t_{k_n}} < \frac{R_{1/k_n}(t_{k_n})}{t_{k_n}} = \frac{\|x-y\|^2}{k_n}(1-t_{k_n}) < \frac{\|x-y\|^2}{k_n}, \quad n\in\mathbb{N},$$

whence

$$G'(0) = \lim_{n\to\infty}\frac{G(t_{k_n})}{t_{k_n}} \leqslant 0,$$

a contradiction. Consequently, condition (1) holds.

The converse implication is obvious. □

It turns out that if we additionally assume the continuity of a strictly convex function $f: D\to\mathbb{R}$, then also $c: D\times D\to(0,+\infty)$ provided by condition (1) can be chosen regular in a sense.

COROLLARY. *Let D be a convex subset of a real normed space and let $f: D\to\mathbb{R}$. If the function f is continuous and strictly convex, then there exists an upper semicontinuous function $c_0: D\times D\setminus\Delta\to(0,+\infty)$, where $\Delta=\{(x,y)\in D\times D:\ x=y\}$, such that*

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y) - c_0(x,y)t(1-t)\|x-y\|^2$$

for all $x,y\in D$ with $x\neq y$ and $t\in(0,1)$.

Proof. Since f is continuous it follows that for every $t\in(0,1)$ the function

$$D\times D\ni(x,y)\longmapsto G_{x,y}(t) = tf(x)+(1-t)f(y) - f(tx+(1-t)y)$$

is continuous. Now putting

$$c_0(x,y) := \inf\left\{\frac{G_{x,y}(t)}{t(1-t)\|x-y\|^2} : t\in(0,1)\right\}$$

for each $x,y\in D$, $x\neq y$, we see that the function $c_0: D\times D\setminus\Delta\to\mathbb{R}$ is upper semicontinuous.

On account of the Theorem, for all $x,y \in D$ with $x \neq y$, there exists a $c(x,y) \in (0,+\infty)$ such that

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y)-c(x,y)t(1-t)\|x-y\|^2, \quad t \in (0,1),$$

that is

$$0 < c(x,y) \leqslant \frac{G_{x,y}(t)}{t(1-t)\|x-y\|^2}, \qquad t \in (0,1).$$

Thus

$$0 < c(x,y) \leqslant c_0(x,y) \leqslant \frac{G_{x,y}(t)}{t(1-t)\|x-y\|^2}, \qquad t \in (0,1),$$

whenever $x,y \in D$ and $x \neq y$, which proves that the function c_0 takes only positive values and satisfies the inequality stated in the assertion. □

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