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CHEBYSHEV–GRÜSS TYPE INEQUALITIES ON TIME SCALES VIA TWO LINEAR ISOTONIC FUNCTIONALS

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Abstract. We give a generalization of the Chebyshev-Grüss inequality by using the concept of derivative on time scales combined with application of the Chebyshev inequality involving two linear isotonic functionals. This approach covers integral case, discrete case, results from fractional and quantum calculus.

1. Introduction and preliminaries

The well-known classical Chebyshev inequality for Riemann integrals states that if p, f and g are integrable real functions on $[a,b] \subset \mathbb{R}$, $p \geqslant 0$, and if f and g are similarly ordered, then

$$\int_a^b p(x)dx \int_a^b p(x)f(x)g(x)dx \geqslant \int_a^b p(x)f(x)dx \int_a^b p(x)g(x)dx. \tag{1}$$

If f and g are oppositely ordered then the reverse of the inequality in (1) is valid, [12, p. 239].

There is another inequality which is also joined with the name of Chebyshev. In literature it is known as the Chebyshev-Grüss inequality. It is an inequality which gives an upper bound for the absolute value of the Chebyshev difference involving the supremum of the first derivative of functions f and g. Precisely, the Chebyshev-Grüss inequality is the following result, [12, p. 297].

THEOREM 1. *Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions. If $f', g' \in L_\infty[a,b]$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left(\frac{1}{b-a} \int_a^b f(x)dx \right) \left(\frac{1}{b-a} \int_a^b g(x)dx \right) \right| \leqslant \frac{1}{12}(b-a)^2 \|f'\|_\infty \cdot \|g'\|_\infty. \tag{2}$$

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The difference on the right hand side, under the sign of absolute value, is called the Chebyshev difference or the Chebyshev functional. Usually it is given in weighted version as follows

$$T(f,g,p)=\int_a^b p(x)dx\int_a^b p(x)f(x)g(x)dx-\int_a^b p(x)f(x)dx\int_a^b p(x)g(x)dx.$$

There exist a lot of estimations for T, but the most known is the Grüss inequality which states that

$$|T(f,g,p)|\leqslant\sqrt{T(f,f,p)T(g,g,p)}$$

and if numbers m,M,n,N are such that $m\leqslant f(x)\leqslant M$, $n\leqslant g(x)\leqslant N$ for all $x\in[a,b]$, then

$$|T(f,g,p)|\leqslant\frac{1}{4}(M-m)(N-n)\left(\int_a^b p(x)dx\right)^2.$$

Until now, Grüss type inequalities are investigated in different settings. There exist results involving sequences, functions, functionals, matrices, operators etc. In the paper [13] authors open new direction of investigation using two linear functionals instead of only one.

The lower bound for the Chebyshev difference is given in the following theorem, [14].

THEOREM 2. *Let f and g be two differentiable functions on $[a,b]$, monotonic in the same direction and $p\geqslant 0$. If $|f'(x)|\geqslant m\geqslant 0$ and $|g'(x)|\geqslant r\geqslant 0$ on $[a,b]$, then*

$$T(f,g,p)\geqslant mrT(x-a,x-a,p). \tag{3}$$

Since discrete versions of inequalities (1), (2) and (3) are also known, see for example [12, p. 240], [14], it is a natural question to ask: Does a general approach which covers integral and discrete versions of the above-mentioned inequalities exist? The answer is affirmative and it is given by a method of calculus on time scales. In this approach the Chebyshev inequality involving two isotonic linear functionals plays a main role. Let us mention some definitions and theorems related to that topic.

Let E be a non-empty set and L be a class of real-valued functions on E satisfying that a linear combination of functions from L is also in L and the function $\mathbf{1}$ belongs to L, ($\mathbf{1}(t)=1$ for $t\in E$). A functional $A:L\to\mathbb{R}$ is called an isotonic linear functional if it is linear and has a property: if $f\in L$ is non-negative, then $A(f)\geqslant 0$.

The main subject in the Chebyshev inequality is a pair of similarly or oppositely ordered functions. We say that functions f and g on E are similarly ordered (or synchronous) if for each $x,y\in E$

$$(f(x)-f(y))(g(x)-g(y))\geqslant 0.$$

If the reversed inequality holds, then we say that f and g are oppositely ordered or asynchronous.

Let us mention very recently proved result, [13].

THEOREM 3. (The Chebyshev inequality for two isotonic linear functionals) *Let A and B be two isotonic linear functionals on L and let f,g be two functions on E such that f, g, $fg \in L$. If f and g are similarly ordered functions, then*

$$A(fg)B(\mathbf{1})+A(\mathbf{1})B(fg) \geqslant A(f)B(g)+A(g)B(f). \tag{4}$$

If f and g are oppositely ordered functions, then the reverse inequality in (4) holds.

Putting $A=B$ in (4) and divided by 2 we get the Chebyshev inequality for one isotonic positive functional. As we see, in the Chebyshev-Grüss inequality the first derivative of functions f and g has appeared. In this paper we use a Δ-derivative of function defined on a time scale set $\mathbb{T}$. Let us mention here some definitions and properties from the time scale theory which we use in our research. For more details see [1, 5, 6, 11, 18].

A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the set $\mathbb{R}$. A segment $[a,b]$ in $\mathbb{T}$ is defined as $[a,b]=\{t\in\mathbb{T} : a\leqslant t\leqslant b\}$. Other kinds of intervals are defined similarly. On $\mathbb{T}$ we define two jump operators ρ and σ:

$$\rho(t)=\sup\{s\in\mathbb{T} : s<t\}, \quad \sigma(t)=\inf\{s\in\mathbb{T} : s>t\}.$$

A point $t\in\mathbb{T}$ is called left-dense if $t>\inf\mathbb{T}$ and $\rho(t)=t$, left scattered if $\rho(t)<t$, right-scattered if $\sigma(t)>t$ and right-dense if $t<\sup\mathbb{T}$ and $\sigma(t)=t$. If $\mathbb{T}$ has a left- scattered maximum M, then $\mathbb{T}^k=\mathbb{T}\backslash\{M\}$, otherwise $\mathbb{T}^k=\mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k=\mathbb{T}\backslash\{m\}$, otherwise $\mathbb{T}_k=\mathbb{T}$.

We say that $f:\mathbb{T}\to\mathbb{R}$ has the delta derivative $f^{\Delta}(t)\in\mathbb{R}$ at $t\in\mathbb{T}^k$ (provided it exists) if for each $\varepsilon>0$ there exists a neighborhood U of t in $\mathbb{T}$ such that

$$|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)|\leqslant\varepsilon|\sigma(t)-s| \text{ for all } s\in U.$$

On a similar way we define the nabla derivative (∇-derivative) $f^{\nabla}(t)$, [18]. For $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}_k$ the nabla derivative at t is the number (provided it exists) such that for each $\varepsilon>0$ there exists a neighborhood U of t in $\mathbb{T}$ such that

$$|f(\rho(t))-f(s)-f^{\nabla}(t)(\rho(t)-s)|\leqslant\varepsilon|\rho(t)-s| \text{ for all } s\in U.$$

A function $f:\mathbb{T}\to\mathbb{R}$ is called Δ-predifferentiable with region of differentiation D provided that the following conditions hold: f is continuous on $\mathbb{T}$; $D\subset\mathbb{T}^k$, $\mathbb{T}^k-D$ is countable and contains no right-scattered elements of $\mathbb{T}$ and f is Δ-differentiable at each $t\in D$, ([6, 11, 18]). Similarly, a ∇-predifferentiable function is defined in [18]. As a consequence of the mean-value theorem we have the following statement, ([6, 11]):

Let $f:\mathbb{T}\to\mathbb{R}$ be a Δ-predifferentiable function with region of differentiation D. If $f^{\Delta}(t)\geqslant 0$ for all $t\in D$, then f is increasing on $\mathbb{T}$.

Similar statement holds for ∇-predifferentiable f, [18].

The paper is organized in the following way. After this chapter with described motivation, definitions and useful properties, we follow with Chebyshev-Grüss type inequality involving two linear isotonic functionals in general settings - in time scales theory. The third section is devoted to results involving lower bounds for the Chebyshev difference and in the last chapter we give several examples.

2. Upper bound for the Chebyshev difference

As we say at the beginning of the paper we estimate a difference between two sides in the Chebyshev inequality for two functionals (4). For that difference we use the abbreviation $T(f,g)$, i.e.

$$T(f,g) = A(\mathbf{1})B(fg) + B(\mathbf{1})A(fg) - A(f)B(g) - A(g)B(f).$$

By linearity of functionals A and B we have that T is linear in each argument.

In this section, set E is a time scale $\mathbb{T}$ and L is a set of real functions defined on $\mathbb{T}$. This section is devoted to generalization of the classical Chebyshev-Grüss inequality (2). The main theorem is the following.

THEOREM 4. *Let A and B be two isotonic linear functionals on L. Let f, g, h_1, $h_2 : \mathbb{T} \to \mathbb{R}$ be Δ-predifferentiable functions with region of differentiation D, such that $T(f,g)$, $T(h_1,g)$, $T(f,h_2)$ and $T(h_1,h_2)$ exist and h_1^Δ, h_2^Δ don't change the sign, $h_1^\Delta(t)$, $h_2^\Delta(t) \neq 0$ for $t \in D$. Then*

$$|T(f,g)| \leqslant \left\| \frac{f^\Delta}{h_1^\Delta} \right\|_\infty \left\| \frac{g^\Delta}{h_2^\Delta} \right\|_\infty |T(h_1,h_2)|, \tag{5}$$

where $\left\| \frac{f^\Delta}{h_1^\Delta} \right\|_\infty = \sup_{t\in D} \left| \frac{f^\Delta(t)}{h_1^\Delta(t)} \right|$.

Proof. Let us suppose that $h_1^\Delta > 0$, $h_2^\Delta > 0$. Denote by

$$F = \left\| \frac{f^\Delta}{h_1^\Delta} \right\|_\infty, \quad G = \left\| \frac{g^\Delta}{h_2^\Delta} \right\|_\infty.$$

Without loss of generality we may assume that $F, G < \infty$. Then functions $Fh_1 + f$, $Gh_2 + g$ are increasing. Namely, from assumptions we get $\left| \frac{f^\Delta}{h_1^\Delta} \right| \leqslant F$, i.e. $-Fh_1^\Delta \leqslant f^\Delta \leqslant Fh_1^\Delta$, so $(Fh_1 + f)^\Delta \geqslant 0$ on D and $Fh_1 + f$ is increasing on $\mathbb{T}$. Similarly, we get $(Gh_2 + g)^\Delta \geqslant 0$. Using the same arguments we obtain that functions $Fh_1 - f$ and $Gh_2 - g$ are increasing.

So we can use the Chebyshev inequality for two functionals, i.e. we have

$$T(Fh_1 + f, Gh_2 + g) \geqslant 0 \text{ and } T(Fh_1 - f, Gh_2 - g) \geqslant 0.$$

By properties of T we get

$$T(f,g) + FG \cdot T(h_1,h_2) + F \cdot T(h_1,g) + G \cdot T(f,h_2) \geqslant 0$$
$$T(f,g) + FG \cdot T(h_1,h_2) - F \cdot T(h_1,g) - G \cdot T(f,h_2) \geqslant 0.$$

Adding these two inequalities we obtain

$$T(f,g) \geqslant -FG \cdot T(h_1,h_2).$$

Since $G = \left\| \frac{(-g)^\Delta}{h_2^\Delta} \right\|_\infty$ we can write $T(f,-g) \geqslant -FG \cdot T(h_1,h_2)$ and we get

$$T(f,g) \leqslant FG \cdot T(h_1,h_2).$$

Since h_1 and h_2 are similarly ordered, we have $T(h_1,h_2) \geqslant 0$, that means

$$|T(f,g)| \leqslant FG \cdot T(h_1,h_2) = FG \cdot |T(h_1,h_2)|,$$

which is in fact, inequality (5).

Suppose that $h_1^\Delta > 0$, $h_2^\Delta < 0$. We apply the above-proved result replacing the function h_2 by $-h_2$. Since

$$\left|\frac{g^\Delta(t)}{(-h_2)^\Delta(t)}\right| = \left|\frac{g^\Delta(t)}{h_2^\Delta(t)}\right|$$

we get (5) also in this case. The other two possibilities for h_1 and h_2 can be regarded in the same way. □

In the previous theorem we can substitute Δ with ∇. Then the previous theorem becomes as the following:

THEOREM 5. *Let A and B be two isotonic linear functionals. Let f, g, h_1, $h_2 : \mathbb{T} \to \mathbb{R}$ be ∇-predifferentiable functions with region of differentiation D, such that $T(f,g)$, $T(h_1,g)$, $T(f,h_2)$ and $T(h_1,h_2)$ exist and h_1^∇, h_2^∇ don't change the sign, $h_1^\nabla(t)$, $h_2^\nabla(t) \neq 0$ for $t \in D$. Then*

$$|T(f,g)| \leqslant \left\|\frac{f^\nabla}{h_1^\nabla}\right\|_\infty \left\|\frac{g^\nabla}{h_2^\nabla}\right\|_\infty |T(h_1,h_2)|. \tag{6}$$

3. Additional results for bounds of $T(f,g)$

The following theorem is a generalization of (3). In fact this result gives us additional information about bounds for $T(f,g)$ together with the Chebyshev-Grüss inequality.

THEOREM 6. *Let A and B be two isotonic linear functionals. Let f, g, h_1, $h_2 : \mathbb{T} \to \mathbb{R}$ be Δ-predifferentiable functions with region of differentiation D, such that $T(f,g), T(f,h_2)$, $T(h_1,g)$ and $T(h_1,h_2)$ exist.*

(i) If $h_1^\Delta, h_2^\Delta \underset{(\leqslant)}{\geqslant} 0$ and if

$$\left(f^\Delta \underset{(\leqslant)}{\geqslant} mh_1^\Delta \text{ and } g^\Delta \underset{(\leqslant)}{\geqslant} rh_2^\Delta\right) \text{ or } \left(f^\Delta \underset{(\geqslant)}{\leqslant} -mh_1^\Delta \text{ and } g^\Delta \underset{(\geqslant)}{\leqslant} -rh_2^\Delta\right)$$

for some non-negative m,r, then

$$T(f,g) \geqslant mr \cdot T(h_1,h_2) \geqslant 0.$$

(ii) If $h_1^\Delta \underset{(\leqslant)}{\geqslant} 0$, $h_2^\Delta \underset{(\geqslant)}{\leqslant} 0$ and if

$$\left(f^\Delta \underset{(\leqslant)}{\geqslant} mh_1^\Delta \text{ and } g^\Delta \underset{(\geqslant)}{\leqslant} rh_2^\Delta\right) \text{ or } \left(f^\Delta \underset{(\geqslant)}{\leqslant} -mh_1^\Delta \text{ and } g^\Delta \underset{(\leqslant)}{\geqslant} -rh_2^\Delta\right)$$

for some $m,r \geqslant 0$, then

$$T(f,g) \leqslant mr \cdot T(h_1,h_2) \leqslant 0.$$

(iii) If $h_1^\Delta, h_2^\Delta \underset{(\leqslant)}{\geqslant} 0$ *and if*

$$\left(f^\Delta \underset{(\leqslant)}{\geqslant} mh_1^\Delta \text{ and } g^\Delta \underset{(\geqslant)}{\leqslant} -rh_2^\Delta\right) \text{ or } \left(f^\Delta \underset{(\geqslant)}{\leqslant} -mh_1^\Delta \text{ and } g^\Delta \underset{(\leqslant)}{\geqslant} rh_2^\Delta\right),$$

$m, r \geqslant 0$, *then*

$$T(f,g) \leqslant -mr \cdot T(h_1,h_2) \leqslant 0.$$

(iv) If $h_1^\Delta \underset{(\leqslant)}{\geqslant} 0$, $h_2^\Delta \underset{(\geqslant)}{\leqslant} 0$ *and if*

$$\left(f^\Delta \underset{(\leqslant)}{\geqslant} mh_1^\Delta \text{ and } g^\Delta \underset{(\leqslant)}{\geqslant} -rh_2^\Delta\right) \text{ or } \left(f^\Delta \underset{(\geqslant)}{\leqslant} -mh_1^\Delta \text{ and } g^\Delta \underset{(\geqslant)}{\leqslant} rh_2^\Delta\right),$$

$m, r \geqslant 0$, *then*

$$T(f,g) \geqslant -mr \cdot T(h_1,h_2) \geqslant 0.$$

Proof. Let us prove one case (among possible 16 cases) in details. Let $h_1^\Delta, h_2^\Delta \geqslant 0$ and $f^\Delta \geqslant mh_1^\Delta$, and $g^\Delta \geqslant rh_2^\Delta$. Considering Δ-derivatives of functions $f - mh_1$ and $g - rh_2$ we find that

$$(f - mh_1)^\Delta \geqslant 0 \text{ and } (g - rh_2)^\Delta \geqslant 0,$$

hence $f - mh_1$ and $g - rh_2$ are increasing. Furthermore, from assumption $f^\Delta \geqslant mh_1^\Delta \geqslant 0$ we conclude that f is increasing. Applying the Chebyshev inequality (4) on two increasing functions $f - mh_1$ and h_2 we get

$$T(f - mh_1, h_2) \geqslant 0, \text{ i.e. } T(f,h_2) \geqslant mT(h_1,h_2). \tag{7}$$

Similarly, applying the Chebyshev inequality for two functionals (4) on two increasing functions $g - rh_2$ and f we get

$$T(g - rh_2, f) \geqslant 0, \text{ i.e. } T(f,g) \geqslant rT(f,h_2). \tag{8}$$

From (7) and (8) we get

$$T(f,g) \geqslant rT(f,h_2) \geqslant mr \cdot T(h_1,h_2) \geqslant 0,$$

where the last inequality is true since h_1 and h_2 are increasing. Other cases are proved in a similar manner. □

COROLLARY 1. *Let A and B be isotonic linear functionals on L, and let $f, g : \mathbb{T} \to \mathbb{R}$ be Δ-predifferentiable functions with region of differentiation D such that $0 \leqslant m \leqslant g^\Delta(x) \leqslant M$ for $x \in D$.*

(i) If $f^\Delta \geqslant 0$, then

$$m \cdot T(f,e_1) \leqslant T(f,g) \leqslant M \cdot T(f,e_1), \tag{9}$$

where $e_1(x) = x$.

(ii) If $f^\Delta \leqslant 0$, then the reverse signs in the above inequality hold.

Proof. Putting in the previous theorem 6(i):

$$f = f, \quad g = g, \quad h_1 = f, \quad h_2 = e_1, \quad m = 1, \quad r = m$$

we get the first inequality in (9). The second inequality is a consequence of Theorem 4. Of course, this corollary can be proved directly applying the Chebyshev inequality on pairs of functions f and $g - me_1$, or f and $Me_1 - g$. □

REMARK 1. If in Theorem 6 and Corollary 1 we substitute Δ with ∇ we get corresponding results from ∇ calculus.

4. Applications

In this section we give applications of the previous theorems for some particular cases. Also, we list papers in which particular cases of results from Sections 2 and 3 appear.

4.1. Δ-integral

Let $a,b \in \mathbb{T}$ with $a<b$ and let A and B be Cauchy Δ-integrals of f, i.e. $A(f) = B(f) = \int_a^b w(x)f(x)\Delta x$, $w \geqslant 0$. Definition and properties of it are given in [5] and [11]. Using the fact that A is an isotonic linear functional (see [2]), and if assumptions of Theorem 4 are satisfied we get the following Chebyshev-Grüss inequality:

$$\begin{aligned} &\left| \int_a^b w(x)\Delta x \int_a^b w(x)f(x)g(x)\Delta x - \int_a^b w(x)f(x)\Delta x \int_a^b w(x)g(x)\Delta x \right| \\ &\leqslant \left\| \frac{f^\Delta}{h_1^\Delta} \right\|_\infty \left\| \frac{g^\Delta}{h_2^\Delta} \right\|_\infty \left| \int_a^b w(x)\Delta x \int_a^b w(x)h_1(x)h_2(x)\Delta x \right. \\ &\qquad \left. - \int_a^b w(x)h_1(x)\Delta x \int_a^b w(x)h_2(x)\Delta x \right|. \end{aligned} \tag{10}$$

Let us mention that, in general, Theorem 4 gives us a result for two different linear functionals, so we can, for example, use integrals with different weights.

REMARK 2. In paper [16, Theorem 9] another version of the Chebyshev-Grüss inequality (10) is given. They used $h_1(t) = h_2(t) = t$ and it is proved via the generalized Montgomery identity.

Results of Theorem 6 and Corollary 1 for Δ-integral seem to be quite new.

4.2. ∇-integral

Let us define A as $A(f) = \int_a^b w(x)f(x)\nabla x = B(f)$. More about ∇-integral, especially about its properties and connections with the theory of linear functionals can be found in [2] and [5]. Applying Theorem 5 we get the following Chebyshev-Grüss inequality:

$$\begin{aligned} &\left| \int_a^b w(x)\nabla x \int_a^b w(x)f(x)g(x)\nabla x - \int_a^b w(x)f(x)\nabla x \int_a^b w(x)g(x)\nabla x \right| \\ &\leqslant \left\| \frac{f^\nabla}{h_1^\nabla} \right\|_\infty \left\| \frac{g^\nabla}{h_2^\nabla} \right\|_\infty \left| \int_a^b w(x)\nabla x \int_a^b w(x)h_1(x)h_2(x)\nabla x \right. \\ &\qquad \left. - \int_a^b w(x)h_1(x)\nabla x \int_a^b w(x)h_2(x)\nabla x \right|, \end{aligned} \tag{11}$$

where f,g,h_1,h_2 satisfy assumptions of Theorem 5. We don't find any results of that type in literature and it seems new to us. Also, a ∇-analogue of Theorem 6 containts new results.

4.3. q-integral

Let $0<q<1$, $b>0$ and $\mathbb{T}=\{0\}\cup\{bq^n : n=0,1,2,\ldots\}$. For a function $f:\mathbb{T}\to\mathbb{R}$ we have

$$f^\nabla(t)=\frac{f(qt)-f(t)}{(q-1)t},\quad t\neq 0,$$

and $f^\nabla(0)=\lim_{s\to 0}\frac{f(s)-f(0)}{s}$ if this limit exists. In q-calculus the number $f^\nabla(t)$ is usually noted as $D_q(f)(t)$ and called a q-derivative of a function f at a point t. The Jackson integral of f (or q-integral) is defined as

$$I_q(f)=\int_0^b f(x)d_q(x):=b(1-q)\sum_{n=0}^{\infty}q^n f(bq^n).$$

In particular case, when $h_1(x)=h_2(x)=x$, $A=B=I_q$ inequality (6) becomes

$$|bI_q(fg)-I_q(f)I_q(g)|\leqslant ||D_q(f)||\,||D_q(g)||\frac{qb^4}{(1+q+q^2)(1+q)^2},$$

with $||D_q(f)||=\sup_{t\in\mathbb{T}}|D_q(f)(t)|$.

In [20] one can find a similar result proved by the Montgomery identity, while in [17] an inequality of Chebyshev-Grüss type involving q-derivative and q-integral on a general interval $[a,b]$ is given. The proof of the last-mentioned result is based on the properties of Lipschitz functions.

4.4. Continuous case – isotonic functionals, the Riemann integral

If Δ-derivative coincides with classical derivative a result from Theorem 4 for one linear functional, i.e. a case $A=B$ is given in [15]. In that paper the proof is based on the Cauchy mean-value theorem. A case when $h_1=h_2=h$ is rediscovered in [10] but using different proof. In fact, that proof is based on the Chebyshev inequality and we use their idea in our proof of Theorem 4.

Furthermore, if $A(f)=B(f)=\int_a^b w(x)f(x)dx$, $w\geqslant 0$, then inequality (5) becomes:

$$\left|\int_a^b w(x)dx\int_a^b w(x)f(x)g(x)dx-\int_a^b w(x)f(x)dx\int_a^b w(x)g(x)dx\right| \tag{12}$$

$$\leqslant\left\|\frac{f'}{h_1'}\right\|_\infty\left\|\frac{g'}{h_2'}\right\|_\infty\left|\int_a^b w(x)dx\int_a^b w(x)h_1(x)h_2(x)dx-\int_a^b w(x)h_1(x)dx\int_a^b w(x)h_2(x)dx\right|,$$

where f,g,h_1,h_2 satisfy assumptions of Theorem 4 for this particular case. The case when $h_1(x)=h_2(x)=x$ is proved in [9]. For $w=1$ we get inequality (2).

Putting in Theorem 6(i): $A(f)=B(f)=\int_a^b w(t)f(t)dt$, $h_1(x)=h_2(x)=x-a$, f and g are two differentiable, monotonic functions in the same direction with $|f'(x)|\geqslant m$ and $|g'(x)|\geqslant r$ on $[a,b]$, then we get result from [14].

If $h_1(x)=x-a$, $h_2(x)=b-x$, f and g are two differentiable, monotonic functions in the opposite direction with $|f'(x)|\geqslant m$ and $|g'(x)|\geqslant r$ on $[a,b]$, then Theorem 6(ii) gives result from [14], also.

4.5. Continuous case – fractional integral operators

Let us consider a fractional hypergeometric operator $I_t^{\alpha,\beta,\eta,\mu}$ which covers several types of well-known fractional operators: the Riemann-Liouville fractional integral operator ($\beta=-\alpha,\ \eta=\mu=0$), the Erdélyi-Kober operator ($\beta=0,\mu=0$) and Saigo operator ($\mu=0$).

If $t>0,\ \alpha>\max\{0,-\beta-\mu\},\ \mu>-1,\ \beta-1<\eta<0$, then a fractional hypergeometric operator is defined as

$$A(f)=I_t^{\alpha,\beta,\eta,\mu}\{f\}:=\frac{t^{-\alpha-\beta-2\mu}}{\Gamma(\alpha)}\int_0^t\sigma^{\mu}(t-\sigma)_2^{\alpha-1}F_1\left(\alpha+\beta+\mu,-\eta,\alpha;1-\frac{\sigma}{t}\right)f(\sigma)\,d\sigma$$

where the function ${}_2F_1(a,b,c,t)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!}$ is the Gaussian hypergeometric function and $(a)_n$ is the Pochhammer symbol: $(a)_n=a(a+1)\dots(a+n-1)$, $(a)_0=1$, [3]. A functional $A(f)=I_t^{\alpha,\beta,\eta,\mu}\{f(t)\}$ is isotonic linear, so we can write the Chebyshev-Grüss type inequality for fractional hypergeometric operators.

THEOREM 7. *Let $p,\ q,\ f,\ g,\ h_1,\ h_2$ be functions on $[0,\infty)$, $p,\ q\geqslant 0$, $f,\ g$ differentiable, $h_1,\ h_2$ differentiable, strictly monotonic in the same direction. Let $t>0,\ \alpha>\max\{0,-\beta-\mu\},\ \mu>-1,\ \beta-1<\eta<0,\ \gamma>\max\{0,-\delta-\nu\},\ \nu>-1,\ \delta-1<\zeta<0$.*

If $\left\|\frac{f'}{h_1'}\right\|_\infty,\left\|\frac{g'}{h_2'}\right\|_\infty<\infty$, then

$$\begin{aligned}&|I_t^{\alpha,\beta,\eta,\mu}\{p\}I_t^{\gamma,\delta,\zeta,\nu}\{qfg\}+I_t^{\alpha,\beta,\eta,\mu}\{pfg\}I_t^{\gamma,\delta,\zeta,\nu}\{q\}-I_t^{\alpha,\beta,\eta,\mu}\{pf\}I_t^{\gamma,\delta,\zeta,\nu}\{qg\}\\&-I_t^{\alpha,\beta,\eta,\mu}\{pg\}I_t^{\gamma,\delta,\zeta,\nu}\{qf\}|\\&\leqslant\left\|\frac{f'}{h_1'}\right\|_\infty\left\|\frac{g'}{h_2'}\right\|_\infty\Big|I_t^{\alpha,\beta,\eta,\mu}\{p\}I_t^{\gamma,\delta,\zeta,\nu}\{qh_1h_2\}+I_t^{\alpha,\beta,\eta,\mu}\{ph_1h_2\}I_t^{\gamma,\delta,\zeta,\nu}\{q\}\\&-I_t^{\alpha,\beta,\eta,\mu}\{ph_1\}I_t^{\gamma,\delta,\zeta,\nu}\{qh_2\}-I_t^{\alpha,\beta,\eta,\mu}\{ph_2\}I_t^{\gamma,\delta,\zeta,\nu}\{qh_1\}\Big|.\end{aligned}$$

Proof. It is a consequence of Theorem 4 for $A(f)=I_t^{\alpha,\beta,\eta,\mu}\{pf\}$ and $B(f)=I_t^{\gamma,\delta,\zeta,\nu}\{qf\}$. □

A particular case of Theorem 7 for two Riemann-Liouville fractional operators $A(f)=J^{\alpha}pf(t)$, $B(f)=J^{\beta}qf(t)$ and $h(t)=t$ is given in [8]. An analogue result for two Riemann-Liouville q-integrals can be find in [7]. Furthermore, the Chebyshev-Grüss type inequality for the Saigo q-integral operators is given in [19].

Some results of the third section also can be found in recent literature. For example, if A and B are the same non-weighted Riemann-Liouville fractional integrals, i.e. $A(f)=B(f)=J^{\alpha}f(t)$, then results from Corollary 1 are given in [4]. Results of the same Corollary but for one fractional hypergeometric operator are given in [3].

Here we give results for only one class of fractional integral operators, i.e. for fractional hypergeometric operators. Of course, analogue results, which are consequences of Theorems 4 and 6 and Corollary 1 hold for other types of fractional integral operators which have isotonic property, for example, for the Hadamard operator, for the Katugampola operator, the Agrawal integral operator etc.

4.6. Discrete case

Let $\mathbb{T} \subseteq \mathbb{N}$. Then functions f, g, h_1, h_2 become sequences $(f_k)_k$, $(g_k)_k$, $(h_{1k})_k$, $(h_{2k})_k$, and Δ-derivative becomes a difference Δ between two consecutive elements of a sequence, i.e. $\Delta a_k = a_{k+1} - a_k$. For this particular case a part (i) of Theorem 6 has the following form.

THEOREM 8. *Let $(p_k)_k$ and $(q_k)_k$ be non-negative n-tuples. If real n-tuples $(h_{1k})_k$, $(h_{2k})_k$ are monotonic in the same direction with property: $\Delta h_{ik} \neq 0$ for all $k = 1, \dots, n$, $i = 1, 2$, and if for some non-negative m, r*

$$\left(\frac{\Delta f_k}{\Delta h_{1k}} \geqslant m \text{ and } \frac{\Delta g_k}{\Delta h_{2k}} \geqslant r \right) \text{ or } \left(\frac{\Delta f_k}{\Delta h_{1k}} \leqslant -m \text{ and } \frac{\Delta g_k}{\Delta h_{2k}} \leqslant -r \right)$$

for $k = 1, \dots, n$, then

$$\sum p_k f_k g_k \sum q_k + \sum p_k \sum q_k f_k q_k - \sum p_k f_k \sum q_k g_k - \sum p_k g_k \sum q_k f_k$$
$$\geqslant mr \Big(\sum p_k h_{1k} h_{2k} \sum q_k + \sum p_k \sum q_k h_{1k} h_{2k} - \sum p_k h_{1k} \sum q_k h_{2k} - \sum p_k h_{2k} \sum q_k h_{1k} \Big).$$

The above result for same n-tuples p and q, $h_{1k} = h_{2k} = k$, and corresponding variant of Theorem 4 is given in [14]. Other parts of Theorem 6 can also be given in discrete form.

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