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DETERMINANTAL INEQUALITIES OF POSITIVE DEFINITE MATRICES

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Abstract. Let A_i, $i=1,\dots,m$, be positive definite matrices with diagonal blocks $A_i^{(j)}$, $1\leqslant j\leqslant k$, where $A_1^{(j)},\dots,A_m^{(j)}$ are of the same size for each j. We prove the inequality

$$\det(\sum_{i=1}^{m} A_i^{-1}) \geqslant \det(\sum_{i=1}^{m}(A_i^{(1)})^{-1})\cdots\det(\sum_{i=1}^{m}(A_i^{(k)})^{-1})$$

and more determinantal inequalities related to positive definite matrices.

1. Introduction

Notation. Throughout the paper, we will use the following notation:

- I denotes the identity matrix of a proper size. We do not specify its order.
- $A \prec B$ ($A \preccurlyeq B$) is used to imply that A and B are Hermitian matrices such that $B-A$ is positive definite (semidefinite). In particular, a positive definite (positive semidefinite) matrix A can be expressed as $A \succ 0$ ($A \succcurlyeq 0$).
- $\operatorname{diag}(D_1,\dots,D_k)$ denotes the block diagonal matrix whose diagonal blocks are $D_1,\dots,D_k$.

Fischer's inequality [1, Theorem 7.8.3] states that if A is a positive definite matrix with diagonal blocks $A_1,\dots,A_k$, then

$$\det A \leqslant \det A_1 \cdots \det A_k.$$

Let A_i, $i=1,\dots,m$, be positive definite matrices whose diagonal blocks are n_j-square matrices $A_i^{(j)}$ for $j=1,\dots,k$. Then the relation

$$\det(\sum_{i=1}^{m} A_i) \leqslant \det(\sum_{i=1}^{m} A_i^{(1)})\cdots\det(\sum_{i=1}^{m} A_i^{(k)})$$

follows directly from Fischer's inequality. The main result of the paper is to show

$$\det(\sum_{i=1}^{m} A_i^{-1}) \geqslant \det(\sum_{i=1}^{m}(A_i^{(1)})^{-1})\cdots\det(\sum_{i=1}^{m}(A_i^{(k)})^{-1}).$$

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2. Proof of the Main inequality

The following is a well-known result [1, Corollary 7.7.4].

LEMMA 1. *If* $0 \prec A \preccurlyeq B$, *then* $B^{-1} \preccurlyeq A^{-1}$ *and* $\det(A) \leqslant \det(B)$.

We expect the following is known, but we include a proof as we do not know a reference.

LEMMA 2. *Let* $P = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ *be a positive definite matrix. Then* P *can be factorized as* $P = T^*T$ *with* $T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix}$ *being conformally partitioned as* P.

Proof. Since A is positive definite, it can be factorized as $A = X^*X$ for an invertible matrix X. Since P is positive definite, the Schur complement $C - B^*A^{-1}B$ is also positive definite. Thus there exists a matrix Z such that $C - B^*A^{-1}B = Z^*Z$. If T is defined by $T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix}$, where $Y = (X^*)^{-1}B$, then a direct computation shows $P = T^*T$. $\square$

The following is in [2, Corollary 1].

LEMMA 3. *Let* $T = \begin{bmatrix} X & Y \\ O & Z \end{bmatrix}$, *where* X *and* Z *are square matrices. Then*

$$\det(I + T^*T) \geqslant \det(I + X^*X)\det(I + Z^*Z).$$

The following theorem is equivalent to Theorem 1.1 in [3]. Here we give a simple proof using Lemma 3.

THEOREM 1. *Let* $C_i \succ 0$ *and* $D_i \succcurlyeq 0$ *be* n_i*-square matrices for* $i = 1, \ldots, k$ *and* $D = diag(D_1, \ldots, D_k)$. *Then*

$$\det(I + C^{-1}D) \geqslant \det(I + C_1^{-1}D_1) \cdots \det(I + C_k^{-1}D_k). \tag{2.1}$$

Proof. By a standard continuity argument, we may assume that D_i are positive definite. In this case, it is also enough to show the inequality

$$\det(I + C^{-1}) \geqslant \det(I + C_1^{-1}) \cdots \det(I + C_k^{-1}) \tag{2.2}$$

by the following argument:

$$\begin{aligned} \det(I + C^{-1}D) &= \det(I + (D^{-\frac{1}{2}}CD^{-\frac{1}{2}})^{-1}) \\ &\geqslant \det(I + (D_1^{-\frac{1}{2}}C_1D_1^{-\frac{1}{2}})^{-1}) \cdots \det(I + (D_k^{-\frac{1}{2}}C_kD_k^{-\frac{1}{2}})^{-1}) \\ &= \det(I + C_1^{-1}D_1) \cdots \det(I + C_k^{-1}D_k). \end{aligned}$$

Moreover, mathematical induction allows us to prove (2.2) for $k=2$. By Lemma 2, there exists a matrix $T=\begin{bmatrix} X & Y \\ O & Z \end{bmatrix}$ being conformally partitioned as C^{-1} such that $C^{-1}=T^*T$. Then we have

$$\det(I+C^{-1})=\det(I+T^*T)\geqslant \det(I+X^*X)\det(I+Z^*Z)$$

by Lemma 3. Now it is enough to show $(X^*X)^{-1}\preccurlyeq C_1$ and $(Z^*Z)^{-1}\preccurlyeq C_2$, since the relations and the above inequality imply

$$\det(I+C^{-1})\geqslant \det(I+C_1^{-1})\det(I+C_2^{-1})$$

by Lemma 1. From

$$C=(T^*T)^{-1}=\begin{bmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y+Z^*Z \end{bmatrix}^{-1},$$

we have

$$C_1=(X^*X-X^*Y(Y^*Y+Z^*Z)^{-1}Y^*X)^{-1}$$

by the block inverse theorem [1]. Thus $C_1\succcurlyeq (X^*X)^{-1}$. Similarly, we have

$$\begin{aligned} C_2 &= (Y^*Y+Z^*Z-Y^*X(X^*X)^{-1}X^*Y)^{-1} \\ &= (Y^*(I-X(X^*X)^{-1}X^*)Y+Z^*Z)^{-1} \\ &= (Z^*Z)^{-1}. \quad \square \end{aligned}$$

COROLLARY 1. *Let A be positive definite. If A_i and B_i, $i=1,\dots,k$, are the n_i-square diagonal blocks of A and A^{-1}, respectively, then*

$$\det(I+(A_iB_i)^{-1})\leqslant 2^{n_i}\leqslant \det(I+A_iB_i),\ i=1,\dots,k.$$

Proof. Fix i. If $C=A$, $D_i=A_i$, and D_j is the zero matrix for all $j\neq i$ in (2.1), then we have $\det(I+A_iB_i)\geqslant 2^{n_i}$. Similarly, if $C=A$, $D_i=B_i^{-1}$, and D_j is the zero matrix for all $j\neq i$ in (2.1), we have $2^{n_i}\geqslant \det(I+A_i^{-1}B_i^{-1})$. $\square$

We can generalize (2.1) using the following result [2, Theorem 1]:

LEMMA 4. *Let $T_i=\begin{bmatrix} X_i & Y_i \\ O & Z_i \end{bmatrix}$, $i=1,\dots,m$, be n_i-square conformally partitioned matrices. Then*

$$\det(\sum_{i=1}^m T_i^*T_i)\geqslant \det(\sum_{i=1}^m X_i^*X_i)\det(\sum_{i=1}^m Z_i^*Z_i).$$

The following is the main theorem of the paper.

THEOREM 2. (Main) *Let A_i, $i=1,\dots,m$, be positive definite matrices whose diagonal blocks are n_j-square matrices $A_i^{(j)}$ for $j=1,\dots,k$. Then*

$$\det(\sum_{i=1}^m A_i^{-1})\geqslant \det(\sum_{i=1}^m (A_i^{(1)})^{-1})\cdots\det(\sum_{i=1}^m (A_i^{(k)})^{-1}).$$

Proof. We use the same argument as we did in Theorem 1. Using mathematical induction on k, we may assume $k=2$. By Lemma 2, for each $i=1,\dots,m$ there exists a matrix $T_i = \begin{bmatrix} X_i & Y_i \\ O & Z_i \end{bmatrix}$ being conformally partitioned as A_i^{-1} such that $A_i^{-1} = T_i^* T_i$. Then

$$\det(\sum_{i=1}^{m} A_i^{-1}) \geqslant \det(\sum_{i=1}^{m} X_i^* X_i) \det(\sum_{i=1}^{m} Z_i^* Z_i)$$

by Lemma 4. Now it is enough to show $(X_i^* X_i)^{-1} \preccurlyeq A_i^{(1)}$ and $(Z_i^* Z_i)^{-1} \preccurlyeq A_i^{(2)}$ for each i, since the relations and the inequality above imply

$$\det(\sum_{i=1}^{m} A_i^{-1}) \geqslant \det(\sum_{i=1}^{m} (A_i^{(1)})^{-1}) \det(\sum_{i=1}^{m} (A_i^{(2)})^{-1})$$

by Lemma 1. From

$$A_i = (T_i^* T_i)^{-1} = \begin{bmatrix} X_i^* X_i & X_i^* Y_i \\ Y_i^* X_i & Y_i^* Y_i + Z_i^* Z_i \end{bmatrix}^{-1},$$

we have

$$A_i^{(1)} = (X_i^* X_i - X_i^* Y_i (Y_i^* Y_i + Z_i^* Z_i)^{-1} Y_i^* X_i)^{-1}$$

and thus $A_i^{(1)} \succcurlyeq (X_i^* X_i)^{-1}$. Similarly,

$$\begin{aligned} A_i^{(2)} &= (Y_i^* Y_i + Z_i^* Z_i - Y_i^* X_i (X_i^* X_i)^{-1} X_i^* Y_i)^{-1} \\ &= (Y_i^* (I - X_i (X_i^* X_i)^{-1} X_i^*) Y_i + Z_i^* Z_i)^{-1} \\ &= (Z_i^* Z_i)^{-1}. \quad \square \end{aligned}$$

3. More inequalities

Here we show more inequalities related to Theorem 1. The following will be used without proof (*See* [1, Theorem 7.7.8]).

LEMMA 5. *If $S \subset \{1,2,\dots,n\}$ is an index set, then $A(S)^{-1} \preccurlyeq A^{-1}(S)$, where $B(T)$ denotes the principle submatrix of B determined by deletion of the rows and columns indicated by T.*

The following presents additional inequalities of determinants. One of them is the inequality in Theorem 1. We contains it here since it is proved in a different way.

THEOREM 3. *Let $C_i \succ 0$ and $D_i \succcurlyeq 0$ be n_i-square matrices for $i=1,\dots,k$ and $D = diag(D_1,\dots,D_k)$. Then we have the following results:*

(a) $\det(I+CD) \leqslant \det(I+C_1D_1)\cdots\det(I+C_kD_k)$.[1]

[1]The inequality is equivalent to the inequality in [3, Theorem 1.2].

(b) If $D \preccurlyeq C^{-1}$*, then* $D_i \preccurlyeq C_i^{-1}$ *and*

$$\det(I - CD) \leqslant \det(I - C_1D_1)\cdots\det(I - C_kD_k).$$

(c) $\det(I + C^{-1}D) \geqslant \det(I + C_1^{-1}D_1)\cdots\det(I + C_k^{-1}D_k)$. (Theorem 1)

(d) If $D \preccurlyeq C$*, then* $D_i \preccurlyeq C_i$ *and*

$$\det(I - C^{-1}D) \leqslant \det(I - C_1^{-1}D_1)\cdots\det(I - C_k^{-1}D_k).$$

Proof. (a) follows directly from Fischer's inequality:

$$\begin{aligned}\det(I + CD) &= \det(I + \sqrt{D}C\sqrt{D}) \\ &\leqslant \prod_{i=1}^{k}\det(I + \sqrt{D_i}C_i\sqrt{D_i}) \\ &= \prod_{i=1}^{k}\det(I + C_iD_i).\end{aligned}$$

Assume $D \preccurlyeq C^{-1}$. Since $I - \sqrt{D_i}C\sqrt{D_i}$, $i = 1,\ldots,k$, are the diagonal blocks of the positive semidefinite matrix $I - \sqrt{D}C\sqrt{D}$, the relation $D_i \preccurlyeq C_i^{-1}$ holds for all i and (b) also follows from Fischer's inequality.

Now we prove (c). Let $B = (C + D)^{-1}$. Then B is a positive definite matrix such that $D \preccurlyeq B^{-1}$. By (b), $D_i \preccurlyeq B_i^{-1}$ for all i and

$$\det(I - BD) \leqslant \det(I - B_1D_1)\cdots\det(I - B_kD_k), \tag{3.1}$$

where $B_1,\ldots,B_k$ are the diagonal blocks of B. Since $(I - BD)(I + C^{-1}D) = I$, the left hand side of (3.1) is $1/\det(I + C^{-1}D)$. Meanwhile, fix i and let $S \subset \{1,2,\ldots,n\}$ be the index set such that $B_i = B(S)$ (thus $C_i = C(S)$ and $D_i = D(S)$). Then $B_i^{-1} \preccurlyeq B^{-1}(S) = C_i + D_i$ by Lemma 5 and

$$\begin{aligned}\det((I - B_iD_i)^{-1}) &= \frac{\det(B_i^{-1})}{\det(B_i^{-1} - D_i)} \\ &= \frac{\det(B_i^{-1} - D_i + D_i)}{\det(B_i^{-1} - D_i)} \\ &= \det(I + (B_i^{-1} - D_i)^{-1}D_i) \\ &\geqslant \det(I + C_i^{-1}D_i).\end{aligned}$$

Therefore (c) holds. A similar argument is applied to (d). Assume $D \preccurlyeq C$ and let $B = C - D$. Without loss of generality, we may assume $D \prec C$. By (c),

$$\det(I + B^{-1}D) \geqslant \det(I + B_1^{-1}D_1)\cdots\det(I + B_k^{-1}D_k), \tag{3.2}$$

where $B_i = C_i - D_i$ for $i = 1,\ldots,k$. Since $(I + B^{-1}D)(I - C^{-1}D) = I$, the left hand side of (3.2) is $1/\det(I - C^{-1}D)$. Moreover, since

$$\det((I + B_i^{-1}D_i)^{-1}) = \frac{\det(B_i + D_i - D_i)}{\det(B_i + D_i)} = \frac{\det(C_i - D_i)}{\det(C_i)} = \det(I - C_i^{-1}D_i),$$

inequality (d) holds. □

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