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TRIANGLES, PARAMETERS, MODULUS OF SMOOTHNESS IN NORMED SPACES

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Abstract. Around 15 years ago several authors studied the parameter defined by

$$A_2(X) = \sup\left\{\frac{||x+y||+||x-y||}{2} : x,y \in S_X\right\},$$

where S_X denotes the unit sphere of the real Banach space X. In this paper we consider the new family of parameters that generalize $A_2(X)$:

$$A_{2,p}(X) = \sup\left\{\frac{\|x+y\|+\|x-y\|}{2} \;:\; x,y \in X,\ ||(\|x\|,\|y\|)||_p \leqslant 2^{\frac{1}{p}}\right\}, 1 \leqslant p \leqslant \infty.$$

In this way, $A_{2,\infty}(X)$ is nothing else than $A_2(X)$ and we show how some interesting properties of real Banach spaces can be characterized by using our new constants.

1. Introduction and definitions

Let X be a normed space over $\mathbb{R}$; we denote by S_X (B_X), or simply by S (B), its unit sphere (resp. its unit ball). The following number was considered in [2], then later in a few other papers (see [1, 9, 12, 14]):

$$A_2(X) = \sup\left\{\frac{||x+y||+||x-y||}{2} \;:\; x,y \in S_X\right\}. \tag{1.1}$$

This number is connected with the largest perimeter of a triangle inscribed in a semicircle of X. Also, $A_2(X)-1$ coincides with the value of the modulus of smoothness at 1 (see Section 2).

A simple use of convexity shows that

$$A_2(X) = \sup\left\{\frac{||x+y||+||x-y||}{2} \;:\; x,y \in B_X\right\}. \tag{1.2}$$

Now we denote by $\mathbb{R}^2_p$ the plane, endowed with the ℓ_p-norm: if $a,b \in \mathbb{R}$, then

$$\|(a,b)\|_p = \begin{cases} (|a|^p+|b|^p)^{\frac{1}{p}} & if\ 1 \leqslant p < \infty \\ \max\{|a|,|b|\} & if\ p = \infty. \end{cases}$$

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We can also think at $A_2(X)$ in the following way

$$A_2(X) = \sup\left\{ \frac{||x+y|| + ||x-y||}{2} : \ x,y \in X, \ ||\,(||x||, ||y||)\,||_\infty \leqslant 1 \right\}. \tag{1.3}$$

So it can be natural to consider in general, for $1 \leqslant p \leqslant \infty$ (with obvious meaning for $p = \infty$):

$$A_{2,p}(X) = \sup\left\{ \frac{||x+y|| + ||x-y||}{2} : \ x,y \in X, \ ||\,(||x||, ||y||)\,||_p \leqslant 2^{\frac{1}{p}} \right\}. \tag{1.4}$$

Thus, we shall write $A_{2,\infty}(X)$ for $A_2(X)$. We have

$$A_{2,2}(X) = \sup\left\{ \frac{||x+y|| + ||x-y||}{2} : \ x,y \in X, \ ||x||^2 + ||y||^2 \leqslant 2 \right\}, \tag{1.5}$$

$$A_{2,1}(X) = \sup\left\{ \frac{||x+y|| + ||x-y||}{2} : \ x,y \in X, \ ||x|| + ||y|| \leqslant 2 \right\}. \tag{1.6}$$

REMARK 1.1. Note that $\sqrt{2} \leqslant A_{2,\infty}(X) \leqslant 2$ (see [2, (2.9)]) and also $A_{2,\infty}(X) = A_{2,\infty}(X^*)$ (see [2, (2.7)]).

Concerning $A_{2,1}(X)$, its value is 2 for any X ($A_{2,1}(X) \leqslant 2$ by the triangle inequality and $A_{2,1}(X) \geqslant 2$ since we can take $x = \Theta$ -the origin- and y of norm 2); so it is not interesting.

For the sake of completeness, we recall that different generalizations of $A_2(X)$ have been considered in [7, p. 274], in [15], in [4]. Also, it was shown in [8, p. 73] that the pairs x,y with $||x+y|| = ||x-y||$ are not enough to estimate $A_{2,\infty}(X)$. Some others facts concerning A_2 were indicated in [13].

Recently, in [11], the extension of $A_2(X)$ to what we denote by $A_{2,2}(X)$ (indicated there by $A(X)$) was considered and studied (see also [10]).

In this paper, we shall be interested in $A_{2,p}(X)$ for $1 < p < \infty$. It is clear that $A_{2,p}(X) \geqslant 1$ (take $x = \Theta$, $y \in S$); but in fact, in Section 3, we shall show strictly larger general lower bounds, which coincide with the values of our constants for inner product spaces.

2. Preliminary results

We prove a simple lemma.

LEMMA 2.1. *For any X, any $p \in [1,\infty)$*

$$A_{2,p}(X) = \sup\left\{ \frac{||x+y|| + ||x-y||}{2} : \ x,y \in X, \ ||x||^p + ||y||^p = 2 \right\}. \tag{2.1}$$

Proof. Clearly the left term in (2.1) is not smaller then the right one; we prove the reverse inequality. Let x,y be such that $||x||^p+||y||^p \leqslant 2$ and also $||x+y||+||x-y|| \geqslant 2$ (otherwise x,y are useless to calculate $A_{2,p}(X)$). Now we consider the convex function $f(\lambda)=||\lambda x+y||+||\lambda x-y||$; we have: $f(0)=2||y||$; $f(1)\geqslant 2||y||$, thus $f(\lambda)$ is non decreasing for $\lambda \geqslant 1$. Take $\lambda_0 \geqslant 1$ so that $||\lambda_0 x||^p+||y||^p=2$; then $f(\lambda_0)\geqslant ||x+y||+||x-y||$, which concludes the proof $\square$

Note that the formula (2.1), for $p\to\infty$, reduces to the following one (which is true according to (1.2)):

$$A_{2,\infty}(X)=\sup\left\{\frac{||x+y||+||x-y||}{2} : x,y\in X, \max\{||x||,||y||\}=1\right\}.$$

REMARK 2.2. In (2.1) x and y play a symmetric role, so they can be exchanged; moreover $||x||^p+||y||^p=2$ implies that one of the addend is at least 1 and the other one is not larger than 1. Therefore we also have:

$$A_{2,p}(X)=\sup\left\{\frac{||x+y||+||x-y||}{2} : x,y\in X,\ ||x||^p+||y||^p=2,\ ||x||\geqslant 1\right\}, \tag{2.2}$$

$$A_{2,p}(X)=\sup\left\{\frac{||x+y||+||x-y||}{2} : x,y\in X,\ ||x||^p+||y||^p=2,\ ||x||\leqslant 1\right\}. \tag{2.3}$$

REMARK 2.3. Since the functions involved are continuous in x,y, in (2.2) we can limit to consider points x with $1\leqslant ||x|| < 2^{\frac{1}{p}}$; in (2.3) those with $0<||x||\leqslant 1$.

PROPOSITION 2.4. *The values of $A_{2,p}(X)$, $1\leqslant p\leqslant\infty$, are non increasing in p.*

Proof. Let $1\leqslant q\leqslant p\leqslant\infty$. If $p<\infty$, then according to the definitions (see (1.4)), to compute $A_{2,p}(X)$ we consider all pairs x,y such that $||(\|x\|,\|y\|)||_p\leqslant 2^{\frac{1}{p}}$. Let B_p (resp.: B_q) denote the unit ball in $\mathbb{R}^2_p$ (resp. $\mathbb{R}^2_q$), with the ℓ_p (resp. the ℓ_q)-norm. Then $2^{\frac{1}{q}}B_q\supset 2^{\frac{1}{p}}B_p$ (for $p\neq q$, they touch at the four points $(\pm1,\pm1)$), and the set $2^{\frac{1}{q}}B_q$ is the one used to compute $A_{2,q}(X)$; thus $A_{2,p}(X)\leqslant A_{2,q}(X)$. Moreover B_∞ is contained in all balls $2^{\frac{1}{p}}B_p$ (touching them at $(\pm1,\pm1)$), which concludes the proof. $\square$

Let $p=\infty$; then (see (1.2)):

$$A_{2,\infty}(X)=\sup\left\{\frac{||x+\tau y||+||x-\tau y||}{2} : x,y\in S,\ 0<\tau\leqslant 1\right\}.$$

The next proposition extends this to all $p<\infty$.

PROPOSITION 2.5. *Let $1\leqslant p\leqslant\infty$. Then*

$$A_{2,p}(X)=\sup\left\{\frac{||x+\tau y||+||x-\tau y||}{2^{1-\frac{1}{p}}(1+\tau^p)^{\frac{1}{p}}} : x,y\in S,\ 0<\tau\leqslant 1\right\}. \tag{2.4}$$

Proof. For $p=\infty$ we know already this (in this case $(1+\tau^p)^{\frac{1}{p}}$ means $\max\{1,\tau\}$ $=1$), while for $p=1$ it is almost immediate; so we assume $1<p<\infty$. We use (2.2); take u,v so that $||u||=t\geqslant 1$ and $||v||=(2-t^p)^{\frac{1}{p}}$. According to Remark 2.3, we can assume that $t<2^{\frac{1}{p}}$ so $v\neq\Theta$. Let $x=\frac{u}{||u||}$ and $y=\frac{v}{||v||}$; then

$$\frac{||u+v||+||u-v||}{2}=\frac{||tx+(2-t^p)^{\frac{1}{p}}y||+||tx-(2-t^p)^{\frac{1}{p}}y||}{2}$$
$$=\frac{||x+\frac{(2-t^p)^{\frac{1}{p}}}{t}y||+||x-\frac{(2-t^p)^{\frac{1}{p}}}{t}y||}{\frac{2}{t}}.$$

Now we set $\tau=\frac{(2-t^p)^{1/p}}{t}$, which is non increasing in t; τ is 1 for $t=1$ and 0 for $t=2^{\frac{1}{p}}$. Since $t=(\frac{2}{1+\tau^p})^{1/p}$, from (2.2) we obtain (2.4). □

REMARK 2.6. Similarly, let $1<p<\infty$. Then

$$A_{2,p}(X)=\sup\left\{\frac{||\tau x+y||+||\tau x-y||}{2^{1-\frac{1}{p}}(1+\tau^p)^{\frac{1}{p}}}:\ x,y\in S,\ \tau\geqslant 1\right\}. \tag{2.5}$$

Now we recall the definition of the modulus of smoothness (see for example [5, pp. 69–70]):

$$\rho_X(\tau)=\sup\left\{\frac{||x+\tau y||+||x-\tau y||}{2}-1:\ x,y\in S\right\}\ (\tau\geqslant 0). \tag{2.6}$$

It is known that

$$\rho_X(\tau)=\sup\left\{\frac{\tau\varepsilon}{2}-\delta_{X^*}(\varepsilon):\ 0\leqslant\varepsilon\leqslant 2\right\}, \tag{2.7}$$

δ_X denoting the modulus of convexity of X:

$$\delta_X(\varepsilon)=\inf\left\{1-\frac{||x+y||}{2}:\ x,y\in B,\ ||x-y||\geqslant\varepsilon\right\}\ (0\leqslant\varepsilon\leqslant 2). \tag{2.8}$$

Also, $\dfrac{\rho_X(\tau)}{\tau}$ is non decreasing for $\tau>0$, so

$$\rho_X(\tau)\leqslant\tau\rho_X(1)\ \mathit{for}\ \tau\leqslant 1. \tag{2.9}$$

If H is an inner product space, then

$$\rho_H(\tau)=\sqrt{1+\tau^2}-1, \tag{2.10}$$

while $\rho_X(\tau)\geqslant\rho_H(\tau)$ for any X and any τ.

A space is called uniformly nonsquare, (UNS) for short, if $\delta_X(\varepsilon)>0$ for some $\varepsilon\in(0,2)$. This means that for some $\varepsilon>0$ we have, for all $x,y\in S$, either $||x-y||\leqslant 2-\varepsilon$ or $||x+y||\leqslant 2-\varepsilon$.

According to (2.4)

$$A_{2,p}(X) = \sup\left\{ \frac{2^{\frac{1}{p}}(1+\rho_X(\tau))}{(1+\tau^p)^{\frac{1}{p}}} : 0 < \tau \leqslant 1 \right\} \; for \; 1 \leqslant p < +\infty, \tag{2.11}$$

and clearly

$$A_{2,\infty}(X) = 1 + \rho_X(1). \tag{2.12}$$

Note that (2.11) generalizes Proposition 3.1 in [11].

We have $\rho_X(1) = \rho_{X^*}(1)$ (see Remark 1.1); moreover $A_{2,\infty}(X) < 2$ if and only if X is (UNS) (see [2, p. 126]). Theorem 3.3 will generalize the latter fact.

Next result generalizes the first part of Theorem 3.4 in [11].

PROPOSITION 2.7. *Let* $1 \leqslant p < \infty$ *and* $1/p + 1/q = 1$. *Then*

$$A_{2,p}(X) \leqslant \; 2^{\frac{1}{p}} \left(1 + (\rho_X(1))^q\right)^{\frac{1}{q}} = 2^{\frac{1}{p}} \left(1 + (A_{2,\infty}(X) - 1)^q\right)^{\frac{1}{q}}. \tag{2.13}$$

Proof. By (2.11) and (2.9) we have:

$$A_{2,p}(X) \leqslant \sup\left\{ \frac{2^{\frac{1}{p}}(1+\tau\rho_X(1))}{(1+\tau^p)^{\frac{1}{p}}} : 0 < \tau \leqslant 1 \right\}. \tag{2.14}$$

Now recall Hölder's inequality. For any $a, b \in \mathbb{R}$

$$1 + |ab| \leqslant ||(1,a)||_p \; ||(1,b)||_q.$$

Apply this inequality with $a = \tau, \quad b = \rho_X(1)$. Then we obtain $1 + \tau\rho_X(1) \leqslant (1 + \tau^p)^{\frac{1}{p}} (1 + (\rho_X(1))^q)^{\frac{1}{q}}$, which gives (2.13). $\square$

REMARK 2.8. For $p = \infty$ (2.13) gives (2.12); moreover, the inequality (2.13) is sharp: for example, it becomes an equality if X is not (UNS) (see Theorem 3.3; see also Proposition 4.2 below). By using the equality $\rho_X(1) = \rho_{X^*}(1)$ we obtain

$$A_{2,p}(X^*) \leqslant 2^{\frac{1}{p}} (1 + (\rho_{X^*}(1))^q)^{\frac{1}{q}} = 2^{\frac{1}{p}} (1 + (\rho_X(1))^q)^{\frac{1}{q}}.$$

3. Main results

THEOREM 3.1. *Let* $p \in [1, \infty]$. *Then*

$$\max\{\sqrt{2},\, 2^{\frac{1}{p}}\} \leqslant A_{2,p}(X) \leqslant A_{2,1}(X) = 2. \tag{3.1}$$

Proof. According to Proposition 2.4, and by taking into account Remark 1.1, we only have to prove the first inequality; also, $\sqrt{2} \leqslant A_{2,\infty}(X) \leqslant A_{2,p}(X)$. Now consider a pair with $x = \Theta$ and y such that $||y|| = 2^{\frac{1}{p}}$; then

$$\frac{||x+y|| + ||x-y||}{2} = 2^{\frac{1}{p}} \leqslant A_{2,p}(X),$$

so the conclusion. $\square$

THEOREM 3.2. *Let H be an inner product space. Then*

$$A_{2,p}(H)=\begin{cases}2^{\frac{1}{p}} & if\ 1\leqslant p<2,\\ \sqrt{2} & if\ 2\leqslant p\leqslant\infty.\end{cases}$$

Proof. If $p=1$ or $p=\infty$ the result is known (see [2, p. 126] and Remark 1.1). Let $2\leqslant p<\infty$. Then

$$1+\rho_H(t)=\sqrt{1+t^2}=||(1,t)||_2\geqslant||(1,t)||_p=(1+t^p)^{\frac{1}{p}}.$$

Moreover, $\max\left\{\frac{\sqrt{1+t^2}}{(1+t^p)^{\frac{1}{p}}}:\ 0\leqslant t\leqslant 1\right\}$ is achieved for $t=1$ (the minimum is obtained for $t=0$ and is 1). Therefore, according to (2.11)

$$A_{2,p}(H)=\sup\left\{\frac{2^{\frac{1}{p}}\sqrt{1+t^2}}{(1+t^p)^{\frac{1}{p}}}:\ 0\leqslant t\leqslant 1\right\}=\sqrt{2}.$$

If $1<p<2$, then $\max\left\{\frac{\sqrt{1+t^2}}{(1+t^p)^{\frac{1}{p}}}:0\leqslant t\leqslant 1\right\}$ is achieved for $t=0$ and so $A_{2,p}(H)=2^{\frac{1}{p}}$, which concludes the proof. $\square$

THEOREM 3.3. *Let $1<p\leqslant\infty$. Then $A_{2,p}(X)=2$ if and only if X is not (UNS).*

Proof. "If" part. It is known that if X is not (UNS), then $A_{2,\infty}(X)=2$; thus in this case $A_{2,p}(X)\geqslant A_{2,\infty}(X)=2$ for all p.

"Only if" part. Let $A_{2,p}(X)=2$ $(1<p<\infty)$. Given $n\in\mathbb{N}$ let x_n and y_n be such that

$$\frac{||x_n+y_n||+||x_n-y_n||}{2}>2-\frac{1}{n};\ \ ||x_n||=t_n;\ \ ||y_n||=(2-t_n^p)^{\frac{1}{p}}.$$

Note that $|a|^p+|b|^p=2\Longleftrightarrow\|(|a|,|b|)\|_p=2^{1/p}\Longrightarrow\|(|a|,|b|)\|_1\leqslant 2.$

So we have

$$2-\frac{1}{n}<\frac{||x_n+y_n||+||x_n-y_n||}{2}\leqslant t_n+(2-t_n^p)^{1/p}\leqslant 2.$$

Let $n\to\infty$. We can suppose, by passing to a subsequence that we still denote by t_n, that $t_n\to\alpha$ and hence $(2-t_n^p)^{\frac{1}{p}}\to(2-\alpha^p)^{\frac{1}{p}}$. Then $\alpha+(2-\alpha^p)^{\frac{1}{p}}=2\Leftrightarrow 2-\alpha^p=(2-\alpha)^p\Leftrightarrow\alpha=1$. Therefore both sequences $\|x_n\|$ and $\|y_n\|$ (they are subsequences, based on the choice of the t_n's) converge to 1.

By (2.3) we can assume that $||x_n||=t_n\leqslant 1$ (hence $||y_n||\geqslant 1$). Then we have

$$2\geqslant\left\|x_n\pm\frac{y_n}{||y_n||}\right\|\geqslant\|x_n\pm y_n\|-\left(1-\frac{1}{||y_n||}\right)||y_n||.$$

So

$$4 \geqslant \left\| x_n + \frac{y_n}{\|y_n\|} \right\| + \left\| x_n - \frac{y_n}{\|y_n\|} \right\| \geqslant \|x_n + y_n\| + \|x_n - y_n\| - 2\left(1 - \frac{1}{\|y_n\|}\right) ||y_n||.$$

But the right hand side tends to 4; hence $\|x_n + y_n\| \to 2$ and $\|x_n - y_n\| \to 2$ $(n \to \infty)$.
This implies that X is not (UNS). □

The next corollary follows immediately from the above theorem.

COROLLARY 3.4. *The following are equivalent:*
a) X is not (UNS);
b) $A_{2,\infty}(X) = 2$ (according to (2.12), this is equivalent to $\rho_X(1) = 1$);
c) $A_{2,p}(X) = 2$ for all $p \in (1,\infty]$;
d) $A_{2,p}(X) = 2$ for some $p \in (1,\infty]$.

THEOREM 3.5. *If $A_{2,2}(X) = \sqrt{2}$, then X is an inner product space. The same is true if $\dim(X) \geqslant 3$ and $A_{2,p}(X) = \sqrt{2}$ for some $p > 2$.*

Proof. Let $A_{2,p}(X) = \sqrt{2}$ for some $p \geqslant 2$. Then (2.11) gives

$$1 + \rho_X(\tau) \leqslant \frac{\sqrt{2}}{2^{\frac{1}{p}}} (1 + \tau^p)^{\frac{1}{p}} \quad (0 \leqslant \tau \leqslant 1).$$

For $p = 2$ we obtain that $\rho_X(\tau) \leqslant \sqrt{1+\tau^2} - 1$ for all $\tau \leqslant 1$; it is known that this implies that X is an inner product space (see for example [3] and the references therein).

If $A_{2,p}(X) = \sqrt{2}$ for some $p > 2$, then $A_{2,\infty}(X) = \sqrt{2}$; this implies (see (2.12)) $\rho_X(1) = \sqrt{2} - 1$, which does not imply that X is an inner product space (see Example 4.4). But $\rho_X(1) = \sqrt{2} - 1$ implies $\delta_{X^*}(\sqrt{2}) \geqslant 1 - \frac{\sqrt{2}}{2}$ (see (2.7)); by using the fact that $\delta_X \leqslant \delta_H$ for any space X we have $\delta_{X^*}(\sqrt{2}) = 1 - \frac{\sqrt{2}}{2}$: if $\dim(X) \geqslant 3$, this implies that X^*, as well as X, is an inner product space (see [3]). □

REMARK 3.6. As we shall see later (Example 4.6) a similar result is not true for $p < 2$.

By the Theorems 3.2 and 3.5 it is obtained that "X is an inner product space if and only if $A_{2,2}(X) = \sqrt{2}$". This result is indicated in [11, Theorem 3.2] (see also [10, p. 320]).

4. Some examples

According to (2.11), the modulus of smoothness is a fundamental tool to evaluate our constant. In some cases the modulus $\rho_X(\tau)$ has a simple structure; in particular this happens when

$$\rho_X(1) = \frac{\varepsilon_0(X^*)}{2} \tag{4.1}$$

where $\varepsilon_0(X) = \sup\{\varepsilon \geqslant 0 : \delta_X(\varepsilon) = 0\}$. In fact (see for example [5, p. 70])

$$\frac{\varepsilon_0(X^*)}{2} = \lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau}. \tag{4.2}$$

So we have the following

LEMMA 4.1. *Let $\rho_X(1) = \frac{\varepsilon_0(X^*)}{2}$ hold. Then $\rho_X(\tau) = \tau\rho_X(1)$ for $0 \leqslant \tau \leqslant 1$.*

Proof. According to (2.9) and (4.2), we obtain (for $0 < \tau \leqslant 1$):

$$\frac{\varepsilon_0(X^*)}{2} = \lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} \leqslant \frac{\rho_X(\tau)}{\tau} \leqslant \rho_X(1) = \frac{\varepsilon_0(X^*)}{2}. \quad \square$$

The next proposition says that (2.13) becomes an equality under the assumption $\rho_X(1) = \frac{\varepsilon_0(X^*)}{2}$.

PROPOSITION 4.2. *Assume that $\rho_X(\tau) = \tau\rho_X(1)$ for $0 \leqslant \tau \leqslant 1$. Then, for $1 < p < \infty$:*

$$A_{2,p}(X) = 2^{\frac{1}{p}} \left(1 + (\rho_X(1))^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}. \tag{4.3}$$

Proof. Let $\rho_X(\tau) = \tau\rho_X(1)$. Then (see (2.11))

$$A_{2,p}(X) = \sup\left\{\frac{2^{\frac{1}{p}}(1+\tau\rho_X(1))}{(1+\tau^p)^{\frac{1}{p}}} : 0 \leqslant \tau \leqslant 1\right\}.$$

Setting $\varphi(\tau) = \dfrac{1+\tau\rho_X(1)}{(1+\tau^p)^{\frac{1}{p}}}$ it is easy to verify that it attains its maximum for $\tau = (\rho_X(1))^{\frac{1}{p-1}}$, and substituting we have the thesis. $\square$

Now we shall consider a few situations where $\rho_X(\tau) = \tau\rho_X(1)$ holds (as usual, we set $q = \frac{p}{p-1}$).

EXAMPLE 4.3. Let $X = \mathbb{R}^2$ with the $\ell_\infty - \ell_1$ norm (the unit ball is a regular hexagon):

$$||(a,b)|| = \begin{cases} |a| + |b| & \text{if } ab < 0 \\ \max\{|a|,|b|\} & \text{if } ab \geqslant 0. \end{cases}$$

The dual space X^* is isometric to X; moreover $\delta_X(\varepsilon) = \max\{0, \frac{\varepsilon-1}{2}\}$ ($\varepsilon \in [0,2]$) so $\varepsilon_0(X) = 1 = \varepsilon_0(X^*)$. Also

$$\rho_X(t) = \sup\left\{\frac{t\varepsilon}{2} - \max\left\{0, \frac{\varepsilon-1}{2}\right\} : 0 \leqslant \varepsilon \leqslant 2\right\} = \frac{t}{2} \ (0 \leqslant t \leqslant 1);$$

so according to (4.3)

$$A_{2,p}(X) = 2^{\frac{1}{p}}\left(1+\left(\frac{1}{2}\right)^q\right)^{\frac{1}{q}}.$$

In particular

$$A_{2,\frac{3}{2}}(X) = \sqrt[3]{\frac{9}{2}} \cong 1.65;\ A_{2,2}(X) = \sqrt{\frac{5}{2}} \cong 1.58;\ A_{2,3}(X) = \sqrt[3]{\frac{9+4\sqrt{2}}{4}} \cong 1.542,$$

and (see [2, Example 3.2]) $A_{2,\infty}(X) = 1.5 = \lim_{p\to\infty} A_{2,p}(X)$.

EXAMPLE 4.4. Consider now $X = \mathbb{R}^2$ with the norm

$$\|(a,b)\| = \max\{|a| + (\sqrt{2}-1)|b|;\ |b| + (\sqrt{2}-1)|a|\}.$$

The unit sphere is a regular octagon and the dual space X^* is isometric to X. In this case (see [2, p. 135]) $A_{2,\infty}(X) = \sqrt{2}$ but of course X is not an inner product space; it is not difficult to prove that

$$\rho_X(1) = \frac{\varepsilon_0(X^*)}{2} = \frac{\varepsilon_0(X)}{2} = \sqrt{2}-1;$$

so by Proposition 4.2 we obtain:

$$A_{2,p}(X) = 2^{\frac{1}{p}}\left(1+(\sqrt{2}-1)^q\right)^{\frac{1}{q}}.$$

In particular

$$A_{2,\frac{3}{2}}(X) = 2^{\frac{2}{3}}(1+(\sqrt{2}-1)^3)^{\frac{1}{3}} \cong 1.62;\quad A_{2,2}(X) = 2\sqrt{2-\sqrt{2}} \cong 1.53.$$

EXAMPLE 4.5. Let $X = \mathbb{R}^2$ with the $\ell_2 - \ell_\infty$ norm:

$$||(a,b)|| = \begin{cases} ||(a,b)||_\infty & if\ ab < 0 \\ ||(a,b)||_2 & if\ ab \geqslant 0. \end{cases}$$

The dual space X^* of X has the $\ell_2 - \ell_1$ norm, and so $\varepsilon_0(X^*) = \sqrt{2}$; also $\rho_X(1) = \frac{1}{\sqrt{2}}$ (see [1]). According to (4.3)

$$A_{2,p}(X) = 2^{\frac{1}{p}}\left(1+(\frac{1}{\sqrt{2}})^q\right)^{\frac{1}{q}} \text{ (in particular } \mathrm{A}_{2,2}(X) = \sqrt{3}).$$

The next example is of a different type (and the space is infinite-dimensional).

EXAMPLE 4.6. Let $X = L_s$ (an infinite-dimensional $L_s(\mu)$-space). We recall that (see [6]):

$$\rho_X(t) = \begin{cases} (1+t^s)^{\frac{1}{s}} - 1 & 1 \leqslant s \leqslant 2 \\ \left(\dfrac{(1+t)^s + (1-t^s)}{2}\right)^{\frac{1}{s}} - 1 & s > 2. \end{cases}$$

Let $1 \leqslant s \leqslant 2$; then, by (2.11)

$$A_{2,p}(L_s) = \sup\left\{ \frac{2^{\frac{1}{p}}(1+t^s)^{\frac{1}{s}}}{(1+t^p)^{\frac{1}{p}}} : t \leqslant 1 \right\}.$$

If $p = s$, then $A_{2,p}(L_p) = 2^{\frac{1}{p}}$. This shows that for no $p \in (1,2)$ the condition $A_{2,p}(X) = 2^{\frac{1}{p}} = A_{2,p}(H)$ forces the space X to be an inner product space.

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