TRIANGLES, PARAMETERS, MODULUS OF SMOOTHNESS IN NORMED SPACES

MARCO BARONTI AND PIER LUIGI PAPINI

(Communicated by M. Sal Moslehian)

Abstract. Around 15 years ago several authors studied the parameter defined by

\[ A_2(X) = \sup \left\{ \frac{|x+y| + |x-y|}{2} : x, y \in S_X \right\}, \]

where \( S_X \) denotes the unit sphere of the real Banach space \( X \). In this paper we consider the new family of parameters that generalize \( A_2(X) \):

\[ A_{2,p}(X) = \sup \left\{ \frac{|x+y| + |x-y|}{2} : x, y \in X, \Vert\langle x, y\rangle\Vert \leq 2^{\frac{1}{p}} \right\}, 1 \leq p \leq \infty. \]

In this way, \( A_{2,\infty}(X) \) is nothing else than \( A_2(X) \) and we show how some interesting properties of real Banach spaces can be characterized by using our new constants.

1. Introduction and definitions

Let \( X \) be a normed space over \( \mathbb{R} \); we denote by \( S_X (B_X) \), or simply by \( S (B) \), its unit sphere (resp. its unit ball). The following number was considered in [2], then later in a few other papers (see [1, 9, 12, 14]):

\[ A_2(X) = \sup \left\{ \frac{|x+y| + |x-y|}{2} : x, y \in S_X \right\}. \] (1.1)

This number is connected with the largest perimeter of a triangle inscribed in a semi-circle of \( X \). Also, \( A_2(X) - 1 \) coincides with the value of the modulus of smoothness at 1 (see Section 2).

A simple use of convexity shows that

\[ A_2(X) = \sup \left\{ \frac{|x+y| + |x-y|}{2} : x, y \in B_X \right\}. \] (1.2)

Now we denote by \( \mathbb{R}^2_p \), the plane, endowed with the \( \ell_p \)-norm: if \( a, b \in \mathbb{R} \), then

\[ \Vert(a, b)\Vert_p = \left\{ \begin{array}{ll} \left(\left|a\right|^p + \left|b\right|^p\right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \max\{|a|, |b|\} & \text{if } p = \infty. \end{array} \right. \]
We can also think at $A_2(X)$ in the following way

\[ A_2(X) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} : x, y \in X, \|\|x\|, \|y\|\| \leq 1 \right\}. \tag{1.3} \]

So it can be natural to consider in general, for $1 \leq p \leq \infty$ (with obvious meaning for $p = \infty$):

\[ A_{2,p}(X) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} : x, y \in X, \|\|x\|, \|y\|\| \leq 2^{\frac{1}{p}} \right\}. \tag{1.4} \]

Thus, we shall write $A_{2,\infty}(X)$ for $A_2(X)$. We have

\[ A_{2,2}(X) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} : x, y \in X, \|x\|^2 + \|y\|^2 \leq 2 \right\}, \tag{1.5} \]

\[ A_{2,1}(X) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} : x, y \in X, \|x\| + \|y\| \leq 2 \right\}. \tag{1.6} \]

**Remark 1.1.** Note that $\sqrt{2} \leq A_{2,\infty}(X) \leq 2$ (see [2, (2.9)]) and also $A_{2,\infty}(X) = A_{2,\infty}(X^*)$ (see [2, (2.7)]).

Concerning $A_{2,1}(X)$, its value is 2 for any $X$ ($A_{2,1}(X) \leq 2$ by the triangle inequality and $A_{2,1}(X) \geq 2$ since we can take $x = \Theta$ - the origin - and $y$ of norm 2); so it is not interesting.

For the sake of completeness, we recall that different generalizations of $A_2(X)$ have been considered in [7, p. 274], in [15], in [4]. Also, it was shown in [8, p. 73] that the pairs $x, y$ with $\|x+y\| = \|x-y\|$ are not enough to estimate $A_{2,\infty}(X)$. Some others facts concerning $A_2$ were indicated in [13].

Recently, in [11], the extension of $A_2(X)$ to what we denote by $A_{2,2}(X)$ (indicated there by $A(X)$) was considered and studied (see also [10]).

In this paper, we shall be interested in $A_{2,p}(X)$ for $1 < p < \infty$. It is clear that $A_{2,p}(X) \geq 1$ (take $x = \Theta$, $y \in S$); but in fact, in Section 3, we shall show strictly larger general lower bounds, which coincide with the values of our constants for inner product spaces.

### 2. Preliminary results

We prove a simple lemma.

**Lemma 2.1.** For any $X$, any $p \in [1, \infty)$

\[ A_{2,p}(X) = \sup \left\{ \frac{\|x+y\| + \|x-y\|}{2} : x, y \in X, \|x\|^p + \|y\|^p = 2 \right\}. \tag{2.1} \]
**Proof.** Clearly the left term in (2.1) is not smaller then the right one; we prove the reverse inequality. Let $x, y$ be such that $||x||^p + ||y||^p \leq 2$ and also $||x + y|| + ||x - y|| \geq 2$ (otherwise $x, y$ are useless to calculate $A_{2,p}(X)$). Now we consider the convex function $f(\lambda) = ||\lambda x + y|| + ||\lambda x - y||$; we have: $f(0) = 2||y||$; $f(1) \geq 2||y||$, thus $f(\lambda)$ is non decreasing for $\lambda \geq 1$. Take $\lambda_0 \geq 1$ so that $||\lambda_0 x||^p + ||y||^p = 2$; then $f(\lambda_0) \geq ||x + y|| + ||x - y||$, which concludes the proof □

Note that the formula (2.1), for $\rho \to \infty$, reduces to the following one (which is true according to (1.2)):

$$A_{2,\infty}(X) = \sup \left\{ \frac{||x + y|| + ||x - y||}{2} : x, y \in X, \max\{||x||, ||y||\} = 1 \right\}.$$

**REMARK 2.2.** In (2.1) $x$ and $y$ play a symmetric role, so they can be exchanged; moreover $||x||^p + ||y||^p = 2$ implies that one of the addend is at least 1 and the other one is not larger than 1. Therefore we also have:

$$A_{2,p}(X) = \sup \left\{ \frac{||x + y|| + ||x - y||}{2} : x, y \in X, ||x||^p + ||y||^p = 2, ||x|| \geq 1 \right\},$$

(2.2)

$$A_{2,p}(X) = \sup \left\{ \frac{||x + y|| + ||x - y||}{2} : x, y \in X, ||x||^p + ||y||^p = 2, ||x|| \leq 1 \right\}.$$  

(2.3)

**REMARK 2.3.** Since the functions involved are continuous in $x, y$, in (2.2) we can limit to consider points $x$ with $1 \leq ||x|| < 2^\frac{1}{p}$; in (2.3) those with $0 < ||x|| \leq 1$.

**PROPOSITION 2.4.** The values of $A_{2,p}(X)$, $1 \leq p \leq \infty$, are non increasing in $p$.

**Proof.** Let $1 \leq q \leq p \leq \infty$. If $p < \infty$, then according to the definitions (see (1.4)), to compute $A_{2,p}(X)$ we consider all pairs $x, y$ such that $||(|x||, ||y||)||_p \leq 2^\frac{1}{p}$. Let $B_p$ (resp.: $B_q$) denote the unit ball in $\mathbb{R}^2_p$ (resp. $\mathbb{R}^2_q$), with the $\ell_p$ (resp. the $\ell_q$)-norm. Then $2^\frac{1}{p}B_q \supset 2^\frac{1}{p}B_p$ (for $p \neq q$, they touch at the four points $(\pm 1, \pm 1)$), and the set $2^\frac{1}{p}B_q$ is the one used to compute $A_{2,q}(X)$; thus $A_{2,p}(X) \leq A_{2,q}(X)$. Moreover $B_{\infty}$ is contained in all balls $2^\frac{1}{p}B_p$ (touching them at $(\pm 1, \pm 1)$), which concludes the proof. □

Let $p = \infty$; then (see (1.2)):

$$A_{2,\infty}(X) = \sup \left\{ \frac{||x + \tau y|| + ||x - \tau y||}{2} : x, y \in S, \ 0 < \tau \leq 1 \right\}.$$

The next proposition extends this to all $p < \infty$.

**PROPOSITION 2.5.** Let $1 \leq p \leq \infty$. Then

$$A_{2,p}(X) = \sup \left\{ \frac{||x + \tau y|| + ||x - \tau y||}{2^{1 - \frac{1}{p}}(1 + \tau^p)^\frac{1}{p}} : x, y \in S, \ 0 < \tau \leq 1 \right\}.$$  

(2.4)
\textbf{Proof.} For } p = \infty \text{ we know already this (in this case } (1 + \tau^p)^{1/p} \text{ means } \max \{1, \tau\} = 1\), while for } p = 1 \text{ it is almost immediate; so we assume } 1 < p < \infty. \text{ We use (2.2); take } u, v \text{ so that } ||u|| = t \geq 1 \text{ and } ||v|| = (2 - t^p)^{1/p}. \text{ According to Remark 2.3, we can assume that } t < 2^{1/p} \text{ so } v \neq \Theta. \text{ Let } x = \frac{u}{||u||} \text{ and } y = \frac{v}{||v||}; \text{ then}

\begin{align*}
\frac{||u + v|| + ||u - v||}{2} &= \frac{||tx + (2 - t^p)^{1/p}y|| + ||tx - (2 - t^p)^{1/p}y||}{2} \\
&= \frac{||x + (2 - t^p)^{1/p}y|| + ||x - (2 - t^p)^{1/p}y||}{2}.
\end{align*}

Now we set } \tau = \frac{(2 - t^p)^{1/p}}{t} \text{, which is non increasing in } t; \text{ } \tau \text{ is 1 for } t = 1 \text{ and 0 for } t = 2^{1/p}. \text{ Since } t = \left(\frac{2}{1 + t^p}\right)^{1/p}, \text{ from (2.2) we obtain (2.4).} \quad \square

\textbf{REMARK 2.6.} Similarly, let } 1 < p < \infty. \text{ Then}

\[ A_{2,p}(X) = \sup \left\{ \frac{||\tau x + y|| + ||\tau x - y||}{2^{1-\frac{p}{q}}(1 + \tau^p)^{1\frac{p}{q}}} : x, y \in S, \tau \geq 1 \right\}. \quad (2.5) \]

Now we recall the definition of the modulus of smoothness (see for example \([5, \text{ pp. 69–70}]\)):

\[ \rho_X(\tau) = \sup \left\{ \frac{||x + \tau y|| + ||x - \tau y||}{2} - 1 : x, y \in S \right\} \quad (\tau \geq 0). \quad (2.6) \]

It is known that

\[ \rho_X(\tau) = \sup \left\{ \frac{\tau e}{2} - \delta_X(e) : 0 \leq e \leq 2 \right\}, \quad (2.7) \]

\(\delta_X\) denoting the modulus of convexity of \(X\):

\[ \delta_X(e) = \inf \left\{ 1 - \frac{||x + y||}{2} : x, y \in B, ||x - y|| \geq e \right\} \quad (0 \leq e \leq 2). \quad (2.8) \]

Also, \(\frac{\rho_X(\tau)}{\tau}\) is non decreasing for \(\tau > 0\), so

\[ \rho_X(\tau) \leq \tau \rho_X(1) \text{ for } \tau \leq 1. \quad (2.9) \]

If \(H\) is an inner product space, then

\[ \rho_H(\tau) = \sqrt{1 + \tau^2} - 1, \quad (2.10) \]

while \(\rho_X(\tau) \geq \rho_H(\tau)\) for any \(X\) and any \(\tau\).

A space is called uniformly nonsquare, (UNS) for short, if \(\delta_X(e) > 0\) for some \(e \in (0,2)\). This means that for some \(e > 0\) we have, for all \(x, y \in S\), either \(||x - y|| \leq 2 - e\) or \(||x + y|| \leq 2 - e\).
According to (2.4)

\[ A_{2,p}(X) = \sup \left\{ \frac{2^{\frac{1}{p}}(1 + \rho_X(\tau))}{(1 + \tau^p)^{\frac{1}{p}}} : 0 < \tau \leq 1 \right\} \text{ for } 1 \leq p < +\infty, \] (2.11)

and clearly

\[ A_{2,\infty}(X) = 1 + \rho_X(1). \] (2.12)

Note that (2.11) generalizes Proposition 3.1 in [11].

We have \( \rho_X(1) = \rho_{X^*}(1) \) (see Remark 1.1); moreover \( A_{2,\infty}(X) < 2 \) if and only if \( X \) is (UNS) (see [2, p. 126]). Theorem 3.3 will generalize the latter fact.

Next result generalizes the first part of Theorem 3.4 in [11].

**Proposition 2.7.** Let \( 1 \leq p < \infty \) and \( 1/p + 1/q = 1 \). Then

\[ A_{2,p}(X) \leq 2^{\frac{1}{p}} \left( 1 + (\rho_X(1))^q \right)^{\frac{1}{q}} = 2^{\frac{1}{p}} (1 + (A_{2,\infty}(X) - 1)^q)^{\frac{1}{q}}. \] (2.13)

**Proof.** By (2.11) and (2.9) we have:

\[ A_{2,p}(X) \leq \sup \left\{ \frac{2^{\frac{1}{p}}(1 + \tau \rho_X(1))}{(1 + \tau^p)^{\frac{1}{p}}} : 0 < \tau \leq 1 \right\}. \] (2.14)

Now recall Hölder’s inequality. For any \( a, b \in \mathbb{R} \)

\[ 1 + |ab| \leq ||(1,a)||_p \cdot ||(1,b)||_q. \]

Apply this inequality with \( a = \tau, \ b = \rho_X(1) \). Then we obtain \( 1 + \tau \rho_X(1) \leq (1 + \tau^p)^{\frac{1}{p}} (1 + (\rho_X(1))^q)^{\frac{1}{q}} \), which gives (2.13). \( \Box \)

**Remark 2.8.** For \( p = \infty \) (2.13) gives (2.12); moreover, the inequality (2.13) is sharp: for example, it becomes an equality if \( X \) is not (UNS) (see Theorem 3.3; see also Proposition 4.2 below). By using the equality \( \rho_X(1) = \rho_{X^*}(1) \) we obtain

\[ A_{2,p}(X^*) \leq 2^{\frac{1}{p}} (1 + (\rho_{X^*}(1))^q)^{\frac{1}{q}} = 2^{\frac{1}{p}} (1 + (\rho_X(1))^q)^{\frac{1}{q}}. \]

### 3. Main results

**Theorem 3.1.** Let \( p \in [1, \infty] \). Then

\[ \max\{\sqrt{2}, 2^{\frac{1}{p}}\} \leq A_{2,p}(X) \leq A_{2,1}(X) = 2. \] (3.1)

**Proof.** According to Proposition 2.4, and by taking into account Remark 1.1, we only have to prove the first inequality; also, \( \sqrt{2} \leq A_{2,\infty}(X) \leq A_{2,p}(X) \). Now consider a pair with \( x = \Theta \) and \( y \) such that \( ||y|| = 2^{\frac{1}{p}} \); then

\[ \frac{||x + y|| + ||x - y||}{2} = 2^{\frac{1}{p}} \leq A_{2,p}(X), \]

so the conclusion. \( \Box \)
THEOREM 3.2. Let $H$ be an inner product space. Then

$$A_{2,p}(H) = \begin{cases} 2^{\frac{1}{p}} & \text{if } 1 \leq p < 2, \\ \sqrt{2} & \text{if } 2 \leq p \leq \infty. \end{cases}$$

Proof. If $p = 1$ or $p = \infty$ the result is known (see [2, p. 126] and Remark 1.1). Let $2 \leq p < \infty$. Then

$$1 + \rho_H(t) = \sqrt{1 + t^2} = \| (1,t) \|_2 \geq \| (1,t) \|_p = (1 + t^p)^{\frac{1}{p}}.$$ 

Moreover, $\max \left\{ \frac{\sqrt{1 + t^2}}{(1 + t^p)^{\frac{1}{p}}} : 0 \leq t \leq 1 \right\}$ is achieved for $t = 1$ (the minimum is obtained for $t = 0$ and is 1). Therefore, according to (2.11)

$$A_{2,p}(H) = \sup \left\{ \frac{2^{\frac{1}{p}}\sqrt{1 + t^2}}{(1 + t^p)^{\frac{1}{p}}} : 0 \leq t \leq 1 \right\} = \sqrt{2}.$$ 

If $1 < p < 2$, then $\max \left\{ \frac{\sqrt{1 + t^2}}{(1 + t^p)^{\frac{1}{p}}} : 0 \leq t \leq 1 \right\}$ is achieved for $t = 0$ and so $A_{2,p}(H) = 2^{\frac{1}{p}}$, which concludes the proof. \(\square\)

THEOREM 3.3. Let $1 < p \leq \infty$. Then $A_{2,p}(X) = 2$ if and only if $X$ is not (UNS).

Proof. “If” part. It is known that if $X$ is not (UNS), then $A_{2,\infty}(X) = 2$; thus in this case $A_{2,p}(X) \geq A_{2,\infty}(X) = 2$ for all $p$.

“Only if” part. Let $A_{2,p}(X) = 2$ ($1 < p < \infty$). Given $n \in \mathbb{N}$ let $x_n$ and $y_n$ be such that

$$\frac{\|x_n + y_n\| + \|x_n - y_n\|}{2} > 2 - \frac{1}{n}; \quad \|x_n\| = t_n; \quad \|y_n\| = (2 - t_n^p)^{\frac{1}{p}}.$$ 

Note that $|a|^p + |b|^p = 2 \iff \|(|a|, |b|)\|_p = 2^{1/p} \iff \|(a, b)\|_1 \leq 2$.

So we have

$$2 - \frac{1}{n} < \frac{\|x_n + y_n\| + \|x_n - y_n\|}{2} \leq t_n + (2 - t_n^p)^{1/p} \leq 2.$$ 

Let $n \to \infty$. We can suppose, by passing to a subsequence that we still denote by $t_n$, that $t_n \to \alpha$ and hence $(2 - t_n^p)^{\frac{1}{p}} \to (2 - \alpha^p)^{\frac{1}{p}}$. Then $\alpha + (2 - \alpha^p)^{\frac{1}{p}} = 2 \iff 2 - \alpha^p = (2 - \alpha)^p \iff \alpha = 1$. Therefore both sequences $\|x_n\|$ and $\|y_n\|$ (they are subsequences, based on the choice of the $t_n$’s) converge to 1.

By (2.3) we can assume that $\|x_n\| = t_n \leq 1$ (hence $\|y_n\| \geq 1$). Then we have

$$2 \geq \frac{y_n}{\|y_n\|} \geq \frac{\|x_n \pm y_n\|}{\|y_n\|} = \frac{\|x_n \pm y_n\|}{\|y_n\|} - \left(1 - \frac{1}{\|y_n\|}\right) \|y_n\|.$$
So
\[ 4 \geq \|x_n + \frac{y_n}{\|y_n\|}\| + \|x_n - \frac{y_n}{\|y_n\|}\| \geq \|x_n + y_n\| + \|x_n - y_n\| - 2\left(1 - \frac{1}{\|y_n\|}\right)\|y_n\|. \]

But the right hand side tends to 4; hence \(\|x_n + y_n\| \to 2\) and \(\|x_n - y_n\| \to 2\) \((n \to \infty)\).

This implies that \(X\) is not (UNS). □

The next corollary follows immediately from the above theorem.

**Corollary 3.4.** The following are equivalent:

a) \(X\) is not (UNS);

b) \(A_{2,\infty}(X) = 2\) (according to (2.12), this is equivalent to \(\rho_X(1) = 1\));

c) \(A_{2,p}(X) = 2\) for all \(p \in (1, \infty]\);

d) \(A_{2,p}(X) = 2\) for some \(p \in (1, \infty]\).

**Theorem 3.5.** If \(A_{2,2}(X) = \sqrt{2}\), then \(X\) is an inner product space. The same is true if \(\dim(X) \geq 3\) and \(A_{2,p}(X) = \sqrt{2}\) for some \(p > 2\).

**Proof.** Let \(A_{2,p}(X) = \sqrt{2}\) for some \(p \geq 2\). Then (2.11) gives
\[ 1 + \rho_X(\tau) \leq \frac{\sqrt{2}}{2^p} \left(1 + \tau^p\right)^{\frac{1}{p}} \quad (0 \leq \tau \leq 1). \]

For \(p = 2\) we obtain that \(\rho_X(\tau) \leq \sqrt{1 + \tau^2} - 1\) for all \(\tau \leq 1\); it is known that this implies that \(X\) is an inner product space (see for example [3] and the references therein).

If \(A_{2,p}(X) = \sqrt{2}\) for some \(p > 2\), then \(A_{2,\infty}(X) = \sqrt{2}\); this implies (see (2.12)) \(\rho_X(1) = \sqrt{2} - 1\), which does not imply that \(X\) is an inner product space (see Example 4.4). But \(\rho_X(1) = \sqrt{2} - 1\) implies \(\delta_X(\sqrt{2}) \geq 1 - \frac{\sqrt{2}}{2}\) (see (2.7)); by using the fact that \(\delta_X \leq \delta_H\) for any space \(X\) we have \(\delta_X(\sqrt{2}) = 1 - \frac{\sqrt{2}}{2}\); if \(\dim(X) \geq 3\), this implies that \(X^*\), as well as \(X\), is an inner product space (see [3]). □

**Remark 3.6.** As we shall see later (Example 4.6) a similar result is not true for \(p < 2\).

By the Theorems 3.2 and 3.5 it is obtained that “\(X\) is an inner product space if and only if \(A_{2,2}(X) = \sqrt{2}\)”.

This result is indicated in [11, Theorem 3.2] (see also [10, p. 320]).

**4. Some examples**

According to (2.11), the modulus of smoothness is a fundamental tool to evaluate our constant. In some cases the modulus \(\rho_X(\tau)\) has a simple structure; in particular this happens when
\[ \rho_X(1) = \frac{\varepsilon_0(X^*)}{2} \] (4.1)
where $\varepsilon_0(X) = \sup \{ \varepsilon \geq 0 : \delta_X(\varepsilon) = 0 \}$. In fact (see for example [5, p. 70])

$$\frac{\varepsilon_0(X^*)}{2} = \lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau}.$$  \hfill (4.2)

So we have the following

**Lemma 4.1.** Let $\rho_X(1) = \frac{\varepsilon_0(X^*)}{2}$ hold. Then $\rho_X(\tau) = \tau \rho_X(1)$ for $0 \leq \tau \leq 1$.

**Proof.** According to (2.9) and (4.2), we obtain (for $0 < \tau \leq 1$):

$$\frac{\varepsilon_0(X^*)}{2} = \lim_{\tau \to 0} \frac{\rho_X(\tau)}{\tau} \leq \frac{\rho_X(\tau)}{\tau} \leq \rho_X(1) = \frac{\varepsilon_0(X^*)}{2}. \quad \Box$$

The next proposition says that (2.13) becomes an equality under the assumption $\rho_X(1) = \frac{\varepsilon_0(X^*)}{2}$.

**Proposition 4.2.** Assume that $\rho_X(\tau) = \tau \rho_X(1)$ for $0 \leq \tau \leq 1$. Then, for $1 < p < \infty$:

$$A_{2,p}(X) = 2^\frac{1}{p} \left( 1 + \left( \rho_X(1) \right)^\frac{p}{p-1} \right) \frac{p-1}{p}. \quad (4.3)$$

**Proof.** Let $\rho_X(\tau) = \tau \rho_X(1)$. Then (see (2.11))

$$A_{2,p}(X) = \sup \left\{ \frac{2^\frac{1}{p} \left( 1 + \tau \rho_X(1) \right)}{(1 + \tau^p)^\frac{1}{p}} : 0 \leq \tau \leq 1 \right\}.$$

Setting $\varphi(\tau) = \frac{1 + \tau \rho_X(1)}{(1 + \tau^p)^\frac{1}{p}}$ it is easy to verify that it attains its maximum for $\tau = (\rho_X(1))^{\frac{1}{p-1}}$, and substituting we have the thesis. \hfill $\Box$

Now we shall consider a few situations where $\rho_X(\tau) = \tau \rho_X(1)$ holds (as usual, we set $q = \frac{p}{p-1}$).

**Example 4.3.** Let $X = \mathbb{R}^2$ with the $\ell_\infty - \ell_1$ norm (the unit ball is a regular hexagon):

$$||(a,b)|| = \begin{cases} |a| + |b| & \text{if } ab < 0 \\ \max\{|a|,|b|\} & \text{if } ab \geq 0. \end{cases}$$

The dual space $X^*$ is isometric to $X$; moreover $\delta_X(\varepsilon) = \max\{0, \frac{\varepsilon - 1}{2}\} \quad (\varepsilon \in [0,2])$ so $\varepsilon_0(X) = 1 = \varepsilon_0(X^*)$. Also

$$\rho_X(t) = \sup \left\{ \frac{t \varepsilon}{2} - \max\left\{0, \frac{\varepsilon - 1}{2}\right\} : 0 \leq \varepsilon \leq 2 \right\} = \frac{t}{2} \quad (0 \leq t \leq 1);$$
so according to (4.3)

\[ A_{2,p}(X) = 2^\frac{1}{p} \left( 1 + \left( \frac{1}{2} \right)^q \right)^{\frac{1}{q}}. \]

In particular

\[ A_{2,\frac{1}{2}}(X) = \frac{3}{\sqrt{2}} \approx 1.65; \quad A_{2,2}(X) = \sqrt{\frac{3}{2}} \approx 1.58; \quad A_{2,3}(X) = \frac{3}{\sqrt{\frac{9 + 4\sqrt{2}}{4}}} \approx 1.542, \]

and (see [2, Example 3.2]) \( A_{2,\infty}(X) = 1.5 = \lim_{p \to \infty} A_{2,p}(X) \).

**Example 4.4.** Consider now \( X = \mathbb{R}^2 \) with the norm

\[ \| (a,b) \| = \max\{ |a| + (\sqrt{2} - 1)|b|; |b| + (\sqrt{2} - 1)|a| \}. \]

The unit sphere is a regular octagon and the dual space \( X^* \) is isometric to \( X \). In this case (see [2, p. 135]) \( A_{2,\infty}(X) = \sqrt{2} \) but of course \( X \) is not an inner product space; it is not difficult to prove that

\[ \rho_X(1) = \frac{\varepsilon_0(X^*)}{2} = \frac{\varepsilon_0(X)}{2} = \sqrt{2} - 1; \]

so by Proposition 4.2 we obtain:

\[ A_{2,p}(X) = 2^\frac{1}{p} \left( 1 + (\sqrt{2} - 1)^q \right)^{\frac{1}{q}}. \]

In particular

\[ A_{2,\frac{1}{2}}(X) = 2^\frac{2}{\sqrt{2}} \left( 1 + (\sqrt{2} - 1)^3 \right)^{\frac{1}{3}} \approx 1.62; \quad A_{2,2}(X) = 2\sqrt{2} - \sqrt{2} \approx 1.53. \]

**Example 4.5.** Let \( X = \mathbb{R}^2 \) with the \( \ell_2 - \ell_\infty \) norm:

\[ \| (a,b) \| = \begin{cases} \| (a,b) \|_\infty & \text{if } ab < 0 \\ \| (a,b) \|_2 & \text{if } ab \geq 0. \end{cases} \]

The dual space \( X^* \) of \( X \) has the \( \ell_2 - \ell_1 \) norm, and so \( \varepsilon_0(X^*) = \sqrt{2} \); also \( \rho_X(1) = \frac{1}{\sqrt{2}} \) (see [1]). According to (4.3)

\[ A_{2,p}(X) = 2^\frac{1}{p} \left( 1 + \left( \frac{1}{\sqrt{2}} \right)^q \right)^{\frac{1}{q}} \quad \text{(in particular } A_{2,2}(X) = \sqrt{3} \).\]

The next example is of a different type (and the space is infinite-dimensional).
EXAMPLE 4.6. Let $X = L_s$ (an infinite-dimensional $L_s(\mu)$-space). We recall that (see [6]):

$$
\rho_X(t) = \begin{cases} 
(1 + t^s)^{\frac{1}{s}} - 1 & 1 \leq s \leq 2 \\
\frac{1}{2} \left((1+t)^{s} + (1-t)^{s}\right)^{\frac{1}{s}} - 1 & s > 2.
\end{cases}
$$

Let $1 \leq s \leq 2$; then, by (2.11)

$$A_{2,p}(L_s) = \sup \left\{ \frac{2t^{\frac{1}{s}}(1+t^s)^{\frac{1}{s}}}{(1+t^p)^{\frac{1}{p}}} : t \leq 1 \right\}.$$ 

If $p = s$, then $A_{2,p}(L^p) = 2^{\frac{1}{s}}$. This shows that for no $p \in (1, 2)$ the condition $A_{2,p}(X) = 2^{\frac{1}{s}} = A_{2,p}(H)$ forces the space $X$ to be an inner product space.

Acknowledgements. The research of the second author was partially supported by the G.N.A.M.P.A. of the Istituto Nazionale di Alta Matematica.

REFERENCES


(Received February 22, 2015)

Marco Baronti
dipartimento di Matematica, Università
Via Dodecaneso, 35, 16146 Genova, Italy
e-mail: baronti@dima.unige.it

Pier Luigi Papini
Via Martucci, 19, 40136 Bologna, Italy
e-mail: pierluigi.papini@unibo.it