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# A NOTE ON VECTOR–VALUED MAXIMAL MULTILINEAR OPERATORS AND THEIR COMMUTATORS

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*Abstract.* Let $T^*$ be the maximal multilinear Calderón-Zygmund operator with kernels of Dini's type and $T_q^*(\vec{f})$ be the vector-valued version of $T^*$. In this paper, we consider the weighted norm inequalities for $T_q^*(\vec{f})$. As applications, the weighted strong type and weighted end-point weak type estimates for the commutators of $T_q^*(\vec{f})$ were established respectively.

## 1. Introduction

Multilinear Calderón-Zygmund operators were introduced and first studied by Coifman and Meyer [3], [4], [5], and later on by Grafakos and Torres [10], [13] for the theory of multilinear Calderón-Zygmund operators with standard kernels. Recently, there are a number of studies concerning multilinear singular integrals which possess rough associated kernels so that they do not belong to the standard Calderón-Zygmund classes, see, for example [1, 8, 14, 18, 22]. In 2009, Maldonado and Naibo [18] established the weighted norm inequalities, with the Muckenhoupt weights, for the bilinear Calderón-Zygmund operators of type $\omega(t)$, and applied them to the study of para-products and bilinear pseudodifferential operators with mild regularity. In 2014, Lu, Zhang [16] studied the multilinear Calderón-Zygmund operators of type $\omega(t)$ and their commutators, and gave some applications to the para-products and the bilinear pseudo-differential operators with mild regularity. In this paper, we study the weighted boundedness of vector-valued maximal multilinear Calderón-Zygmund operators of type $\omega(t)$ and the weighted norm inequalities for the commutators of vector-valued maximal multilinear operators are established.

Following [16], we say that $T$ is a multilinear Calderón-Zygmund operator with kernel of type $\omega(t)$, denoted by $m$-linear $\omega$-CZO, if $T$ can be extended to a bounded multilinear operator from $L^{q_1}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for some $1<q,q_1,\cdots,$ $q_m<\infty$ with $\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac{1}{q}$, or from $L^{q_1}(\mathbb{R}^n)\times\cdots\times L^{q_m}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ for some $1<q_1,\cdots,q_m<\infty$ with $\frac{1}{q_1}+\cdots+\frac{1}{q_m}=1$, and if there exists a function $K$ defined off the diagonal $x=y_1=\cdots=y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$T\vec{f}(x)=T(f_1,\ldots,f_m)(x)=\int_{(\mathbb{R}^n)^m}K(x,y_1,\ldots,y_m)f_1(y_1)\ldots f_m(y_m)dy_1\ldots dy_m, \quad (1.1)$$

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for all $x \notin \bigcap_{j=1}^{m} \mathrm{supp} f_j$, and $f_j \in C_c^{\infty}(\mathbb{R}^n)$, $j = 1, \cdots, m$ and if there exists a constant $A > 0$ such that

$$|K(x, y_1, \cdots, y_m)| \leqslant \frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}}$$

for all $(x, y_1, \cdots, y_m) \in (\mathbb{R}^n)^{m+1}$ with $x \neq y_j$ for some $j \in \{1, 2, \cdots, m\}$, and

$$\begin{aligned} &|K(x, y_1, \cdots, y_m) - K(x', y_1, \cdots, y_m)| \\ \leqslant &\frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}} \omega\left(\frac{|x - x'|}{|x - y_1| + \cdots + |x - y_m|}\right) \end{aligned} \tag{1.2}$$

whenever $|x - x'| \leqslant \frac{1}{2} \max_{1 \leqslant j \leqslant m} |x - y_j|$, and

$$\begin{aligned} &\left|K(x, y_1, \cdots, y_j, \cdots, y_m) - K(x, y_1, \cdots, y_j', \cdots, y_m)\right| \\ \leqslant &\frac{A}{(|x - y_1| + \cdots + |x - y_m|)^{mn}} \omega\left(\frac{|y_j - y_j'|}{|x - y_1| + \cdots + |x - y_m|}\right) \end{aligned} \tag{1.3}$$

whenever $|y_j - y_j'| \leqslant \frac{1}{2} \max_{1 \leqslant j \leqslant m} |x - y_j|$.

The maximal multilinear singular integral operator is defined by

$$T^*(\vec{f})(x) = \sup_{\delta > 0} |T_\delta(f_1, \cdots, f_m)(x)|,$$

where $T_\delta$ are the smooth truncations of $T$ given by

$$T_\delta(\vec{f})(x) = \int_{\sum_{i=1}^{m} |x - y_i|^2 > \delta^2} K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) d\vec{y}.$$

Here and in the following, $d\vec{y} = dy_1 \cdots dy_m$.

The vector-valued multilinear Calderón-Zygmund operator $T_q$ and vector-valued maximal multilinear operator $T_q^*$ associated with the operator $T$ are defined and studied by Grafakos and Martell in [11].

$$\begin{aligned} T_q(\vec{f})(x) &= |T(f_1, \cdots, f_m)(x)|_q = ||T(f_{1\cdot}, \cdots, f_{m\cdot})(x)||_{l^q} \\ &= \left(\sum_{k=1}^{\infty} |T(f_{1k}, \cdots, f_{mk})(x)|^q\right)^{1/q}, \end{aligned}$$

$$\begin{aligned} T_q^*(\vec{f})(x) &= |T^*(f_1, \cdots, f_m)(x)|_q = ||T^*(f_{1\cdot}, \cdots, f_{m\cdot})(x)||_{l^q} \\ &= \left(\sum_{k=1}^{\infty} |T^*(f_{1k}, \cdots, f_{mk})(x)|^q\right)^{1/q}, \end{aligned}$$

where $f_i = \{f_{ik}\}_{k=1}^{\infty}$ for $i = 1, \cdots, m$. Grafakos and Martell [11] obtain the following results.

THEOREM A. ([11]) *Let $T$ be a multilinear Calderón-Zygmund operators, and let $1/m < p < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 < p_1, \cdots, p_m < \infty$, $1/m < q < \infty$ and $1/q = 1/q_1 + \cdots + 1/q_m$ with $1 < q_1, \cdots, q_m < \infty$. Then there exists a constant $C > 0$ such that*

$$||T_q^*(\vec{f})||_{L^p(\mathbb{R}^n)} \leqslant C \prod_{j=1}^m |||f_j|_{q_j}||_{L^{p_j}(\mathbb{R}^n)}.$$

Cruz-Uribe, Martell and Pérez [6] obtain a weak version of Theorem A as follows:

THEOREM B. ([6]) *Let $T$ be a multilinear Calderón-Zygmund operators, and let $1/m \leqslant p < \infty$, $1/p = 1/p_1 + \cdots + 1/p_m$ with $1 \leqslant p_1, \cdots, p_m < \infty$, $1/m < q < \infty$ and $1/q = 1/q_1 + \cdots + 1/q_m$ with $1 < q_1, \cdots, q_m < \infty$. Then there exists a constant $C > 0$ such that*

$$||T_q^*(\vec{f})||_{L^{p,\infty}(\mathbb{R}^n)} \leqslant C \prod_{j=1}^m |||f_j|_{q_j}||_{L^{p_j}(\mathbb{R}^n)}.$$

*Given a collection of locally integrable functions $\vec{b} = (b_1, \cdots, b_l)$, where $1 \leqslant l \leqslant m$. The commutators associated with $T$ and $T^*$ are defined by*

$$T_{\Pi\vec{b}}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \prod_{j=1}^l [b_j(x) - b_j(y_j)] K(x, y_1, \cdots, y_m) \prod_{j=1}^m f_j(y_j) dy_j.$$

$$T^*_{\Pi\vec{b}}(\vec{f})(x) = \sup_{\delta > 0} \left| \int_{\sum_{i=1}^m |x - y_i|^2 > \delta^2} \prod_{j=1}^l [b_j(x) - b_j(y_j)] K(x, y_1, \cdots, y_m) \prod_{j=1}^m f_j(y_j) dy_j \right|.$$

*The commutators associated with vector-valued $T_q$ and $T_q^*$ can be defined by*

$$T_{\Pi\vec{b},q}(\vec{f})(x) = |T_{\Pi\vec{b}}(\vec{f})(x)|_q = ||T_{\Pi\vec{b}}(f_{1\cdot}, \cdots, f_{m\cdot})(x)||_{l^q} = \left( \sum_{k=1}^\infty |T_{\Pi\vec{b}}(\vec{f}_k)(x)|^q \right)^{1/q},$$

$$T^*_{\Pi\vec{b},q}(\vec{f})(x) = |T^*_{\Pi\vec{b}}(\vec{f})(x)|_q = ||T^*_{\Pi\vec{b}}(f_{1\cdot}, \cdots, f_{m\cdot})(x)||_{l^q} = \left( \sum_{k=1}^\infty |T^*_{\Pi\vec{b}}(\vec{f}_k)(x)|^q \right)^{1/q},$$

*where $\vec{f} = (f_1, \cdots, f_m)$, with $f_j = \{f_{jk}\}_{k=1}^\infty$.*

There are a number of studies concerning multilinear singular integrals with standard kernels. The theory of weighted maximal multilinear Calderón-Zygmund type operators was established in [13], [2]. Xue [26] studied the weighted strong type and end-point estimates for $T^*_{\Pi\vec{b}}$ with multiple weights. Recently, the weighted strong type and end-point estimates for $T^*_{\Pi\vec{b},q}$ with multiple weights were obtained by Si and Xue [23] as follows:

THEOREM C. ([23]) *Let $T$ be a multilinear Calderón-Zygmund operators, and let $1/m < p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$, with $1 < p_1, \cdots, p_m < \infty$, $1/m < q < \infty$ and $\frac{1}{q_1} +$*

$\cdots+\frac{1}{q_m}=\frac{1}{q}$ *with* $1<q_1,\cdots,q_m<\infty$*. If* $\vec{\omega}\in A_{\vec{p}}$*,* $\nu_{\vec{\omega}}=\prod_{i=1}^m \omega_i^{\frac{p}{p_i}}$*, and* $\vec{b}\in(BMO)^l$*. Then there exists a constant* $C>0$ *such that*

$$||T^*_{\Pi\vec{b},q}(\vec{f})||_{L^p(\nu_{\vec{\omega}})}\leqslant C\prod_{j=1}^l\|b_j\|_{BMO}\prod_{j=1}^m\||f_j|_{q_j}\|_{L^{p_j}(w_j)}.$$

THEOREM D. ([23]) *Let* $T$ *be a multilinear Calderón-Zygmund operators, and let* $1/m<q<\infty$ *and* $\frac{1}{q_1}+\cdots+\frac{1}{q_m}=\frac{1}{q}$ *with* $1<q_1,\cdots,q_m<\infty$*. If* $\vec{\omega}\in A_{(1,\cdots,1)}$ *and* $\vec{b}\in(BMO)^l$*. Then there exists a constant* $C>0$ *depending on* $\vec{b}$ *such that*

$$\nu_{\vec{\omega}}\left(\left\{x\in\mathbb{R}^n: T^*_{\Pi\vec{b},q}(\vec{f})(x)>t^m\right\}\right)\leqslant C\left(\prod_{j=1}^m\int_{\mathbb{R}^n}\Phi^{(m)}\left(\frac{|f_j|_{q_j}(y_j)}{t}\right)\omega_j(y_j)dy_j\right)^{1/m},$$

*where* $\Phi(t)=t(1+\log^+t)$ *and* $\Phi^{(m)}=\overbrace{\Phi\circ\cdots\circ\Phi}^{m}$.

Throughout this paper, we always assume that $\omega(t):[0,\infty)\mapsto[0,\infty)$ is a nondecreasing function with $0<\omega(1)<\infty$. For any $a>0$, we say that $\omega\in\mathrm{Dini}(a)$, if

$$|\omega|_{\mathrm{Dini}(a)}=\int_0^1\frac{\omega^a(t)}{t}dt<\infty.$$

In this paper, we will establish some weighted norm inequalities for vector-valued maximal multilinear Calderón-Zygmund operators of type $\omega(t)$ and its commutators. Our main results are as follows:

THEOREM 1. *Let* $T$ *be an* $m$*-linear* $\omega$*-CZO with* $\omega\in\mathrm{Dini}(1)$*, and let* $1/m<p<\infty$*,* $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ *with* $1<p_1,\cdots,p_m<\infty$*,* $1/m<q<\infty$ *and* $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$ *with* $1<q_1,\cdots,q_m<\infty$*. Suppose that* $(\omega_1^{p_1},\cdots,\omega_m^{p_m})\in(A_{p_1},\cdots,A_{p_m})$*, then there exists a constant* $C>0$ *such that*

$$||T^*_q(\vec{f})||_{L^p(\omega_1^p\cdots\omega_m^p)}\leqslant C\prod_{j=1}^m||\,|f_j|_{q_j}||_{L^{p_j}(\omega_j^{p_j})}.$$

THEOREM 2. *Let* $T$ *be an* $m$*-linear* $\omega$*-CZO with* $\omega\in\mathrm{Dini}(1)$*, and let* $1\leqslant p_1,\cdots,p_m<\infty$*,* $1<q_1,\cdots,q_m<\infty$ *and* $0<p,q<\infty$ *such that* $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$*,* $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$*. Then, we have*

*(i) If* $1<p_1,...,p_m<\infty$ *and* $\omega\in A_{p_1}\cap\cdots\cap A_{p_m}$*, then there exists a constant* $C>0$ *such that*

$$||T^*_q(\vec{f})||_{L^p(\omega)}\leqslant C\prod_{j=1}^m||\,|f_j|_{q_j}||_{L^{p_j}(\omega)}.$$

*(ii) If at least one* $p_j=1$ *and* $\omega\in A_1$*, then there exists a constant* $C>0$ *such that*

$$||T^*_q(\vec{f})||_{L^{p,\infty}(\omega)}\leqslant C\prod_{j=1}^m||\,|f_j|_{q_j}||_{L^{p_j}(\omega)}.$$

THEOREM 3. *Let $T$ be an $m$-linear $\omega$-CZO, and let $1/m < p < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 < p_1, \cdots, p_m < \infty$, $1/m < q < \infty$ and $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$ with $1 < q_1, \cdots, q_m < \infty$. If $\vec{\omega} \in A_{\vec{p}}$, $\vec{b} \in (BMO)^l$ and $\omega$ satisfies*

$$|\omega|_{\mathrm{Dini}_{LlogL^l}(\rho)} = \int_0^1 \frac{\omega^\rho(t)}{t}\left(1 + \log^l \frac{1}{t}\right) dt < \infty, \tag{1.4}$$

*where $\rho = \min\{1, q\}$, then there exists a constant $C > 0$ such that*

$$||T^*_{\Pi\vec{b},q}(\vec{f})||_{L^p(\nu_{\vec{\omega}})} \leqslant C \prod_{j=1}^{l} \|b_j\|_{BMO} \prod_{j=1}^{m} \||f_j|_{q_j}\|_{L^{p_j}(w_j)}.$$

THEOREM 4. *Let $T$ be an $m$-linear $\omega$-CZO, and $1/m < q < \infty$ and $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$ with $1 < q_1, \cdots, q_m < \infty$. If $\vec{\omega} \in A_{(1,\cdots,1)}$, $\vec{b} \in (BMO)^l$ and $\omega$ satisfies the condition (1.4), then there exists a constant $C > 0$ depending on $\vec{b}$ such that*

$$\nu_{\vec{\omega}}\left(\left\{x \in \mathbb{R}^n : T^*_{\Pi\vec{b},q}(\vec{f})(x) > t^m\right\}\right) \leqslant C\left(\prod_{j=1}^{m} \int_{\mathbb{R}^n} \Phi^{(m)}\left(\frac{|f_j|_{q_j}(y_j)}{t}\right) \omega_j(y_j) dy_j\right)^{1/m},$$

*where $\Phi(t) = t(1 + \log^+ t)$ and $\Phi^{(m)} = \overbrace{\Phi \circ \cdots \circ \Phi}^{m}$.*

REMARK 1. Theorem 1–Theorem 4 are also hold for $T$.

## 2. Proofs of Theorem 1 and Theorem 2

Let us begin with the definition of Hardy-Littlewood maximal operator, that is

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

The sharp maximal function is defined by

$$M^\sharp f(x) = \sup_{Q \ni x} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c| dy \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

For $\delta > 0$, $M_\delta f = M(|f|^\delta)^{\frac{1}{\delta}}$ and $M^\sharp_\delta f = M^\sharp(|f|^\delta)^{\frac{1}{\delta}}$.
The new maximal function $\mathscr{M}$ can be defined by

$$\mathscr{M}(\vec{f})(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)| dy_j$$

and the new vector valued maximal function can be defined by

$$\mathscr{M}(|\vec{f}|_q)(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j(y_j)|_{q_j} dy_j.$$

DEFINITION 1. [15] Let $1 \leqslant p_1, \cdots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$. Given $\vec{\omega} = (\omega_1, \cdots, \omega_m)$, set $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i^{p/p_i}$. We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if

$$\sup_Q \left( \frac{1}{|Q|} \int_Q \prod_{i=1}^m \omega_i^{\frac{p}{p_i}} \right)^{\frac{1}{p}} \prod_{i=1}^m \left( \frac{1}{|Q|} \int_Q \omega_i^{1-p_i'} \right)^{\frac{1}{p_i'}} < \infty,$$

when $p_i = 1$, $\left( \frac{1}{|Q|} \int_Q \omega_i^{1-p_i'} \right)^{\frac{1}{p_i'}}$ is understood as $(\inf_Q \omega_i)^{-1}$.

We now give a new weighted estimate of $T^*$.

THEOREM 5. *Let* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $\vec{\omega} \in A_{\vec{p}}$. *Let* $T$ *be an* $m$*-linear* $\omega$*-CZO with* $\omega \in \mathrm{Dini}(1)$.

*(1) If* $1 < p_1, \cdots, p_m < \infty$, *then*

$$||T^* \vec{f}||_{L^p(\nu_{\vec{\omega}})} \leqslant C \prod_{i=1}^m ||f_i||_{L^{p_i}(\omega_i)}.$$

*(2) If* $1 \leqslant p_1, \cdots, p_m < \infty$, *then*

$$||T^* \vec{f}||_{L^{p,\infty}(\nu_{\vec{\omega}})} \leqslant C \prod_{i=1}^m ||f_i||_{L^{p_i}(\omega_i)}.$$

To prove Theorem 5 we need some lemmas.

LEMMA 1. *Let* $T$ *be an* $m$*-linear* $\omega$*-CZO with* $\omega \in \mathrm{Dini}(1)$. *Then, for any* $\eta > 0$, *there is a constant* $C < \infty$ *depending on* $\eta$ *such that for all* $\vec{f}$ *in any product of* $L^{q_j}(\mathbb{R}^n)$ *spaces, with* $1 \leqslant q_j < \infty$, *the following inequality hold for all* $x \in \mathbb{R}^n$

$$T^*(\vec{f})(x) \leqslant C \Big( M_\eta(T(\vec{f}))(x) + \mathscr{M}(\vec{f})(x) \Big).$$

*Proof.* For a fixed point $x$ and a ball $Q$ centered at $x$ with radius $\delta$. Set $S_\delta = \{\vec{y} : \sup_{1 \leqslant j \leqslant m} |x - y_j| \leqslant \delta\}$ and $U_\delta = \{\vec{y} \in S_\delta : \sum_{i=1}^m |x - y_i|^2 > \delta^2\}$. It is clear that

$$\left| T^*(\vec{f})(x) \right| \leqslant \sup_{\delta > 0} \left| \int_{U_\delta} K(x, \vec{y}) \prod_{i=1}^m f_i(y_i) dy_i \right| + \sup_{\delta > 0} \left| \int_{(S_\delta)^c} K(x, \vec{y}) \prod_{i=1}^m f_i(y_i) dy_i \right|.$$

By using the size condition, we get

$$\begin{aligned} \left| \int_{U_\delta} K(x, \vec{y}) \prod_{i=1}^m f_i(y_i) dy_i \right| &\leqslant C \left| \int_{U_\delta} \frac{A}{(\sum_{i=1}^m |y_i - x|)^{mn}} \prod_{i=1}^m f_i(y_i) dy_i \right| \\ &\leqslant C \prod_{i=1}^m \frac{1}{\delta^n} \int_{Q(x,\delta)} f_i(y_i) dy_i \\ &\leqslant C \mathscr{M}(\vec{f})(x). \end{aligned}$$

We are ready to estimate the second term. For any $z \in Q(x, \frac{\delta}{2})$, we have

$$\tilde{T}_\delta(\vec{f})(z) = \int_{(S_\delta)^c} K(z,\vec{y}) \prod_{i=1}^m f_i(y_i) dy_i = T(\vec{f})(z) - T(\vec{f}^0)(z),$$

where $\vec{f}_0 = (f_1\chi_Q, \cdots, f_m\chi_Q)$.

Note that the integral $\int_{(S_\delta)^c} |K(z,\vec{y}) - K(x,\vec{y})| \prod_{i=1}^m f_i(y_i) dy_i$ can be written as a sum of integrals over sets $R_{\mathscr{I}} \in (\mathbb{R}^n)^m$ for some $\mathscr{I} := \{i_1, \cdots, i_l\} \subseteq \{1, \cdots, m\}$, such that for $\vec{y} = (y_1, \ldots, y_m) \in R_{\mathscr{I}}$ we have $i \in \mathscr{I}$ if and only if $|x - y_i| \leqslant \delta$. On $S_\delta^c$ the set $\mathscr{I}^c$ is not empty, by using a similar argument as in [16] we obtain that $\omega\left(\frac{|z-x|}{\sum_{i\in\mathscr{I}^c}|z-y_i|}\right) \leqslant \omega(2^{-k})$. Consequently, we have

$$\begin{aligned}
&\left|\tilde{T}_\delta(\vec{f})(x) - \tilde{T}_\delta(\vec{f})(z)\right| \\
&\leqslant C \int_{(S_\delta)^c} |K(z,\vec{y}) - K(x,\vec{y})| \prod_{i=1}^m f_i(y_i) dy_i \\
&\leqslant C \sum_{\vec{y}\in R_{\mathscr{I}}} \int_{Q(x,\delta)^l} \prod_{i\in\mathscr{I}} |f_i^0(y_i)dy_i| \sum_{k=1}^{\infty} \int_{(\mathscr{Q}_k)^{m-l}} \frac{A}{(\sum_{i\in\mathscr{I}^c}|z-y_i|)^{mn}} \omega\left(\frac{|z-x|}{\sum_{i\in\mathscr{I}^c}|z-y_i|}\right) \\
&\quad \times \prod_{i\in\mathscr{I}^c} |f_i^\infty(y_i)| dy_i \\
&\leqslant C \sum_{\vec{y}\in R_{\mathscr{I}}} \int_{Q(x,\delta)^l} \prod_{i\in\mathscr{I}} |f_i^0(y_i)dy_i| \sum_{k=1}^{\infty} \int_{(\mathscr{Q}_k)^{m-l}} \frac{\omega(2^{-k})}{|2^kQ|^m} \prod_{i\in\mathscr{I}^c} |f_i^\infty(y_i)| dy_i \\
&\leqslant C \sum_{\vec{y}\in R_{\mathscr{I}}} \sum_{k=1}^{\infty} \omega(2^{-k}) \frac{1}{|2^kQ|^m} \prod_{i\in\mathscr{I}} \int_{Q(x,\delta)} |f_i^0(y_i)dy_i| \prod_{i\in\mathscr{I}^c} \int_{2^kQ} |f_i^\infty(y_i)| dy_i \\
&\leqslant C|\omega|_{\text{Dini}(1)} \mathscr{M}(\vec{f})(x),
\end{aligned}$$

where $\mathscr{Q}_k = (2^kQ) \setminus (2^{k-1}Q)$, $k = 1, 2, \ldots$ and in the second inequality, we used the smooth condition (1.2).

Thus, we obtain

$$\left|\int_{(S_\delta)^c} K(x,\vec{y}) \prod_{i=1}^m f_i(y_i) dy_i\right| \leqslant C\mathscr{M}(\vec{f})(x) + \left|T(\vec{f})(z)\right| + \left|T(\vec{f}^0)(z)\right|.$$

The rest of the proof is the same as in [13], we are done. □

LEMMA 2. ([7]) *If $\omega \in A_p$ and $p > 1$, then $M$ maps from $L^p(\omega)$ to $L^p(\omega)$.*

LEMMA 3. ([2]) *If $\omega \in A_p$ and $p \geqslant 1$, then $M$ maps from $L^{p,\infty}(\omega)$ to $L^{p,\infty}(\omega)$.*

LEMMA 4. ([16]) *Let $T$ be an $m$-linear $\omega$-CZO with $\omega \in$ Dini$(1)$. Let $\vec{p} = (p_1, \cdots, p_m)$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{\omega} \in A_{\vec{p}}$.*

*(1) If* $1 < p_j < \infty$ *for all* $j = 1, \cdots, m,$ *then*

$$||T(\vec{f})||_{L^p(\nu_{\vec{\omega}})} \leqslant C \prod_{j=1}^{m} ||f_j||_{L^{p_j}(\omega_j)}.$$

*(2) If* $1 \leqslant p_j < \infty$ *for all* $j = 1, \cdots, m,$ *and at least one of the* $p_j = 1,$ *then*

$$||T(\vec{f})||_{L^{p,\infty}(\nu_{\vec{\omega}})} \leqslant C \prod_{j=1}^{m} ||f_j||_{L^{p_j}(\omega_j)}.$$

LEMMA 5. ([15]) *Let* $\vec{p} = (p_1, \cdots, p_m)$ *with* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $1 \leqslant p_1, \cdots, p_m$.

*(1) If* $1 < p_j < \infty$ *for all* $j = 1, \cdots, m,$ *then* $\mathscr{M}$ *is bounded from* $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ *to* $L^p(\nu_{\vec{\omega}})$ *if and only if* $\vec{\omega} = (\omega_1, \cdots, \omega_m) \in A_{\vec{p}}$.

*(2) If* $1 \leqslant p_j < \infty$ *for all* $j = 1, \cdots, m,$ *then* $\mathscr{M}$ *is bounded from* $L^{p_1}(\omega_1) \times \cdots \times L^{p_m}(\omega_m)$ *to* $L^{p,\infty}(\nu_{\vec{\omega}})$ *if and only if* $\vec{\omega} = (\omega_1, \cdots, \omega_m) \in A_{\vec{p}}$.

*Proof of Theorem* 5. Theorem 5 follows by using Lemma 1–Lemma 5.

$$\begin{aligned}
||T^*(\vec{f})||_{L^p(\nu_{\vec{\omega}})} &\leqslant C\left(||M_\eta(T(\vec{f}))||_{L^p(\nu_{\vec{\omega}})} + ||\mathscr{M}(\vec{f})||_{L^p(\nu_{\vec{\omega}})}\right) \\
&= C\left(||M(|T(\vec{f}|^\eta))||_{L^{\frac{p}{\eta}}(\nu_{\vec{\omega}})}^{\frac{1}{\eta}} + ||\mathscr{M}(\vec{f})||_{L^p(\nu_{\vec{\omega}})}\right) \\
&\leqslant C\left(|||T(\vec{f})|^\eta||_{L^{\frac{p}{\eta}}(\nu_{\vec{\omega}})}^{\frac{1}{\eta}} + ||\mathscr{M}(\vec{f})||_{L^p(\nu_{\vec{\omega}})}\right) \\
&\leqslant C\left(||T(\vec{f})||_{L^p(\nu_{\vec{\omega}})} + ||\mathscr{M}(\vec{f})||_{L^p(\nu_{\vec{\omega}})}\right) \\
&\leqslant C \prod_{j=1}^{m} ||f_j||_{L^{p_j}(\omega_j)}.
\end{aligned}$$

In the first inequality we used Lemma 1. The second inequality is due to Lemma 2 and the fact $\nu_{\vec{\omega}} \in A_{mp}$ for all $\vec{\omega} \in A_{\vec{p}}$. The last inequality is follows by Lemma 4 and Lemma 5. If we use Lemma 3 instead of using Lemma 2, by using the same arguments, we can get the weak type estimates. □

Note that if each $\omega_j \in A_{p_j}$, then $\prod_{j=1}^{m} A_{p_j} \subset A_{\vec{p}}$ and this inclusion is strict (see [15] for details). This fact together with Theorem 5 and Lemma 4 yields the following weighted estimates.

LEMMA 6. *Let* $T$ *be an* $m$*-linear* $\omega$*-CZO with* $\omega \in \mathrm{Dini}(1)$. *Consider an* $m$*-tuple* $(\omega_1^{p_1}, \cdots, \omega_m^{p_m}) \in (A_{p_1}, \cdots, A_{p_m})$, *where* $1 < p_1, \cdots, p_m < \infty$ *and* $\frac{1}{m} < p < \infty$ *satisfy* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. *Then there exists a constant* $C$ *such that*

$$||T(\vec{f})||_{L^p(\omega_1^p \cdots \omega_m^p)} \leqslant C \prod_{j=1}^{m} ||f_j||_{L^{p_j}(\omega_j^{p_j})},$$

*and*

$$||T^*(\vec{f})||_{L^p(\omega_1^p\cdots\omega_m^p)} \leqslant C\prod_{j=1}^m ||f_j||_{L^{p_j}(\omega_j^{p_j})}.$$

LEMMA 7. ([11]) *Let* $1 < q_1,\cdots,q_m < \infty$ *and* $\frac{1}{m} < q < \infty$ *be fixed indices such that* $\frac{1}{q} = \frac{1}{q_1}+\cdots+\frac{1}{q_m}$. *If for all functions* $f_j \in L^{q_j}(\omega_j^{q_j})$ *and* $(\omega_1^{q_1},\cdots,\omega_m^{q_m}) \in (A_{q_1},\cdots, A_{q_m})$, *the following estimate holds*

$$||\mathscr{T}(\vec{f})||_{L^q(\omega_1^q\cdots\omega_m^q)} \leqslant C\prod_{j=1}^m ||f_j||_{L^{q_j}(\omega_j^{q_j})}.$$

*Then for* $1 < p_1,\cdots,p_m < \infty$ *and* $\frac{1}{m} < p < \infty$ *with* $\frac{1}{p} = \frac{1}{p_1}+\cdots+\frac{1}{p_m}$, *and* $1 < s_1,\cdots,s_m < \infty$, $\frac{1}{m} < s < \infty$ *with* $\frac{1}{s} = \frac{1}{s_1}+\cdots+\frac{1}{s_m}$, *and all* $(\omega_1^{p_1},\cdots,\omega_m^{p_m}) \in (A_{p_1},\cdots,A_{p_m})$, *the following inequality holds*

$$\left|\left|\left(\sum_k |\mathscr{T}(f_{1k},\cdots,f_{mk})|^s\right)^{\frac{1}{s}}\right|\right|_{L^p(\omega_1^p\cdots\omega_m^p)} \leqslant C\prod_{j=1}^m \left|\left|\left(\sum_k |f_{jk}|^{s_j}\right)^{\frac{1}{s_j}}\right|\right|_{L^{p_j}(\omega_j^{p_j})}.$$

*Proofs of Theorem* 1 *and Theorem* 2. As a consequence of Lemma 6 and Lemma 7, we obtain Theorem 1(see the proof of Corollary 3 in [11]). For any weight $\omega \in A_\infty$ and $p \in [\frac{1}{m},\infty)$, from Theorem 6.2 [16] we know that for all bounded functions $\vec{f} = (f_1,\cdots,f_m)$ with compact support, we have $||T\vec{f}||_{L^p(\omega)} \leqslant C||\mathscr{M}(\vec{f})||_{L^p(\omega)} \leqslant C||\prod_{j=1}^m M(f_j)||_{L^p(\omega)}$. Similarly, we obtain that $||T^*\vec{f}||_{L^p(\omega)} \leqslant C||\prod_{j=1}^m M(f_j)||_{L^p(\omega)}$. Then, Theorem 2 follows by a repetition of the same steps as in Corollary 3.3 in [6]. In fact, we apply Theorem 2.1 in [6] to the families

$$\mathscr{F}\left(T(f_1,\ldots,f_m),\prod_{j=1}^m Mf_j\right), \qquad \mathscr{F}\left(T^*(f_1,\ldots,f_m),\prod_{j=1}^m Mf_j\right).$$

Hölder's inequality and the normal inequalities for the maximal operator yield the desired results. □

## 3. Proofs of Theorem 3 and Theorem 4

Proofs of Theorem 3 and 4 follows from similar steps in [23], we omit the proof. We just give two key lemmas.

For $1 \leqslant l \leqslant m$, we define $\mathscr{M}^l_{L(\log L)}(|\vec{f}|_q)$ as follows:

$$\mathscr{M}^l_{L(\log L)}(|\vec{f}|_q)(x) = \sup_{Q\ni x}\prod_{j=1}^l |||f_j|_{q_j}||_{L(\log L),Q}\prod_{j=l+1}^m \frac{1}{|Q|}\int_Q |f_j|_{q_j},$$

$$\mathscr{M}_{L(\log L)}(|\vec{f}|_q)(x) = \sup_{Q\ni x}\prod_{j=1}^m |||f_j|_{q_j}||_{L(\log L),Q},$$

and

$$\mathscr{M}(|\vec{f}|_q)(x) = \sup_{Q \ni x} \prod_{j=1}^{m} \frac{1}{|Q|} \int_Q |f_j|_{q_j},$$

where the supremum is taken over all the cubes containing $x$.

LEMMA 8. *Let $0 < \delta < 1/m$, $1/m < q < \infty$ and $1/q = 1/q_1 + \cdots + 1/q_m$ with $1 < q_1, \cdots, q_m < \infty$. If $\omega$ satisfies the condition (1.4), then there exists a constant $C > 0$ such that*

$$M_\delta^\sharp(T_q^*(\vec{f}))(x) \leqslant C\mathscr{M}(|\vec{f}|_q)(x)$$

*for any smooth vector function $\{\vec{f}_k\}_{k=1}^\infty$ and any $x \in \mathbb{R}^n$.*

*Proof.* Fix $x \in \mathbb{R}^n$ and let $Q$ be a cube containing $x$ with side length $l(Q)$. For any smooth vector function sequence $\{\vec{f}_k\}_{k=1}^\infty$, set $\vec{f}_k^\infty = \vec{f}_k - \vec{f}_k^0$, where $\vec{f}_k^0 = \vec{f}_k \chi_{8Q} = (f_{1k}\chi_{8Q}, \cdots, f_{mk}\chi_{8Q})$. Since $0 < \delta < 1/2 < 1$, we have

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left| |\mathscr{U}_q^*(\vec{f})(y)|^\delta - |c|^\delta \right| dy\right)^{\frac{1}{\delta}} \\
&\leqslant C\left(\frac{1}{|Q|}\int_Q \left|\mathscr{U}_q^*(\vec{f}^0)(y)\right|^\delta dy\right)^{\frac{1}{\delta}} + \left(\frac{1}{|Q|}\int_Q \left|\sup_\eta |\mathscr{U}_\eta(\vec{f}\chi_{(8Q)^c})(y) - c_\eta|\right|_q^\delta dy\right)^{\frac{1}{\delta}} \\
&= U_1 + U_2,
\end{aligned}$$

where $c = \left|\sup_{\eta>0}\left|c_\eta\right|\right|_q = \left(\sum_{k\geqslant 1}\left|\sup_{\eta>0}\left|c_{\eta k}\right|\right|^q\right)^{1/q}$.

For $U_1$, we applying Kolmogorov's inequality and Theorem 2 to get

$$\left(\frac{1}{|Q|}\int_Q \left|\mathscr{U}_q^*(\vec{f}^0)(y)\right|^\delta dy\right)^{\frac{1}{\delta}} \leqslant C\mathscr{M}(|\vec{f}|_q)(x).$$

To estimate $U_2$, we choose $c_\eta = \sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}} c_\eta^{\vec{\alpha}}$, where $c_\eta^{\vec{\alpha}} = \mathscr{U}_\eta(f_{1k}^{\alpha_1}, \cdots, f_{mk}^{\alpha_m})(x)$. We may split $U_2 \leqslant \sum_{\vec{\alpha},\vec{\alpha}\neq\vec{0}} U_{2\vec{\alpha}}$, where

$$U_{2\vec{\alpha}} = \left(\frac{1}{|Q|}\int_Q \sup_\eta \left|\mathscr{U}_\eta(f_{1k}^{\alpha_1}, \cdots, f_{mk}^{\alpha_m})(y) - \mathscr{U}_\eta(f_{1k}^{\alpha_1}, \cdots, f_{mk}^{\alpha_m})(x)\right|_q^\delta dy\right)^{\frac{1}{\delta}}.$$

We now consider the case $\vec{\alpha} = (\infty, \cdots, \infty)$. For $x, y \in Q$ and $y_j \in 2^{s+3}Q \setminus 2^{s+2}Q$, we have $|y - y_j| \geqslant 2^s\sqrt{n}l(Q)$ and $|y - x| \leqslant \sqrt{n}l(Q)$, and $|K(y, \vec{y}) - K(x, \vec{y})| \leqslant \frac{C\omega(2^{-s})}{|2^s Q|^m}$.

Thus, we obtain that

$$
\begin{aligned}
&|\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{mk}^\infty)(y)-\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{mk}^\infty)(x)|\\
&\leqslant C\int_{(\mathbb{R}^n\setminus 8Q)^m}|K(y,\vec{y})-K(x,\vec{y})|\prod_{j=1}^m|f_{jk}^\infty(y_j)|d\vec{y}\\
&\leqslant C\sum_{s=1}^\infty\int_{(2^{s+3}Q)^m\setminus(2^{s+2}Q)^m}\frac{A}{\left(\sum_{j=1}^m|z-y_j|\right)^{mn}}\omega\left(\frac{|y-x|}{\sum_{j=1}^m|z-y_j|}\right)\prod_{j=1}^m|f_{jk}(y_j)|d\vec{y}\\
&\leqslant C\sum_{s=1}^\infty\omega(2^{-s})\prod_{j=1}^m\frac{1}{2^{(s+3)n}|Q|}\int_{2^{s+3}Q}|f_{jk}(y_j)|dy_j\\
&=C\sum_{s=1}^\infty\omega(2^{-s})G_{sk},
\end{aligned}
$$

where $g_{sjk}=\frac{1}{2^{(s+3)n}|Q|}\int_{2^{s+3}Q}|f_{jk}(y_j)|dy_j$ and $G_{sk}=\prod_{j=1}^m g_{sjk}$. Then, we get

$$
|\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{mk}^\infty)(y)-\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{mk}^\infty)(x)|_q\leqslant C\left(\sum_{s=1}^\infty\omega^\rho(2^{-s})\sum_k G_{sk}^q\right)^{1/q}.
$$

By Minkowski's inequality and Hölder's inequality, we obtain that

$$
|\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{mk}^\infty)(y)-\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{mk}^\infty)(x)|_q\leqslant C|\omega|_{\mathrm{Dini}_{LlogL^l}(\rho)}\mathscr{M}(|\vec{f}|_q)(x).
$$

Since $0<\delta<1/2$, we have

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q\sup_\eta|\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{mk}^\infty)(y)-\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{mk}^\infty)(x)|_q^\delta dy\right)^{\frac{1}{\delta}}\\
&\leqslant C\frac{1}{|Q|}\int_Q\sup_\eta|\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{mk}^\infty)(y)-\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{mk}^\infty)(x)|_q dy\\
&\leqslant C|\omega|_{\mathrm{Dini}_{LlogL^l}(\rho)}\mathscr{M}(|\vec{f}|_q)(x).
\end{aligned}
$$

We now estimate the typical term $I_{\vec{\alpha}}$ with $\vec{\alpha}=(\alpha_1,\cdots,\alpha_m)$, $\alpha_i=0$ or $\infty$ for $i=1,\cdots,m$. We may assume that $\alpha_1=\cdots=\alpha_l=\infty$ and $\alpha_{l+1}=\cdots=\alpha_m=0$. For $x,y\in Q$ and any $y_j\in 2^{s+3}Q\setminus 2^{s+2}Q$ with $j\neq l+1,\cdots,m$, one has $|z-y_j|\geqslant 2^s\sqrt{n}l(Q)$, then $|K(z,\vec{y})-K(x,\vec{y})|\leqslant\frac{C\omega(2^{-s})}{|2^sQ|^m}$. We observe the following fact

$$
\begin{aligned}
&|\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{lk}^\infty,f_{(l+1)k}^0\cdots f_{(m)k}^0)(y)-\mathscr{U}_\eta(f_{1k}^\infty,\cdots,f_{lk}^\infty,f_{(l+1)k}^0\cdots f_{(m)k}^0)(x)|\\
&\quad\leqslant C\prod_{j=l+1}^m\int_{8Q}|f_{jk}(y_j)|dy_j\int_{(\mathbb{R}^n\setminus 8Q)^l}\frac{\omega\left(\frac{|y-x|}{\sum_{j=1}^m|z-y_j|}\right)}{\left(\sum_{j=1}^m|z-y_j|\right)^{mn}}\prod_{i=1}^l|f_{jk}(y_j)|dy_1\cdots dy_l\\
&\quad\leqslant C\sum_{s=1}^\infty\omega(2^{-s})\prod_{j=l+1}^m\int_{2^{s+3}Q}|f_{1k}(y_1)|dy_1\frac{1}{|2^{(s+3)}Q|^m}\int_{(2^{s+3}Q)^l}\prod_{j=1}^l|f_{jk}(y_j)|dy_j\\
&\quad\leqslant C\sum_{s=1}^\infty\omega(2^{-s})\prod_{j=1}^m\frac{1}{2^{(s+3)n}|Q|}\int_{2^{s+3}Q}|f_{jk}(y_j)|dy_j.
\end{aligned}
$$

The rest of the proof is the same as in [23], hence we proved Lemma 8. □

LEMMA 9. *Let* $0<\delta<\varepsilon<1/m$. *If* $\omega$ *satisfies the condition* (1.4), *then there exists a constant* $C>0$ *depending only on* $\delta$ *and* $\varepsilon$ *such that*

$$\begin{aligned} M_\delta^\sharp(T^*_{\Pi\vec{b},q}\vec{f})(x) \leqslant C\prod_{j=1}^{l}||b_j||_{BMO}\Big(\mathscr{M}^l_{L(\log L)}(|\vec{f}|_q)(x)+M_\varepsilon(T^*_q\vec{f})(x)\Big) \\ +C\sum_{j=1}^{l-1}\sum_{\sigma\in\mathscr{C}_j^l}\prod_{i\in\sigma}||b_i||_{BMO}M_\varepsilon(T^*_{\Pi b_{\sigma'},q}\vec{f})(x) \end{aligned} \tag{3.1}$$

*for any smooth vector function* $\{\vec{f}_k\}_{k=1}^\infty$ *and for any* $x\in\mathbb{R}^n$, *where* $\sigma'=\{1,\cdots,l\}\setminus\sigma$.

*Proof.* The proof of this part is similar to that of Lemma 2.3 in [23], we just give the part of the proof that is different. We apply Kolmogorov's estimate and Theorem 2 to get

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q\left|\mathscr{U}_q^*((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^0)(z)\right|^\delta dz\right)^{1/\delta} \\ &\leqslant C||\mathscr{U}_q^*((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^0)||_{L^{1/m,\infty}(Q,\frac{dz}{|Q|})} \\ &\leqslant C\prod_{j=1}^{l}\frac{1}{|Q|}\int_Q|b_j(z)-\lambda_j||f_j(z)|_{q_j}dz\prod_{j=l+1}^{m}\frac{1}{|Q|}\int_Q|f_j(z)|_{q_j}dz \\ &\leqslant C\prod_{j=1}^{l}||b_j||_{BMO}\mathscr{M}^l_{L(\log L)}(|\vec{f}|_q)(x). \end{aligned}$$

If all the $\alpha_j=\infty$, we have

$$\begin{aligned} &\sup_{\eta>0}\left|(\mathscr{U}_\eta((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^\alpha))(z)-(\mathscr{U}_\eta((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)\vec{f}^\alpha))(x)\right|_q \\ &\leqslant C\sum_{k=1}^\infty\int_{(\mathscr{Q}_k)^m}|K(z,\vec{y})-K(x,\vec{y})||(b_1(y_1)-\lambda_1)\cdots(b_l(y_l)-\lambda_l)||f_1(y_j)|_{q_j}\cdots|f_m(y_m)|_{q_m}d\vec{y} \\ &\leqslant C\sum_{k=1}^\infty\omega(2^{-k})\prod_{j=1}^{l}\frac{1}{2^{(k+3)n}|Q|}\int_{2^{k+3}Q}|b_j(y_j)-\lambda_j||f_j|_{q_j}dy_j\prod_{j=l+1}^{m}\frac{1}{2^{(k+3)n}|Q|}\int_{2^{k+3}Q}|f_j|_{q_j}dy_j \\ &\leqslant C\sum_{k=1}^\infty\omega(2^{-k})k^l\prod_{j=1}^{l}||b_j||_{BMO}|||f_j|_{q_j}||_{L(\log L),2^{k+3}Q}\prod_{j=l+1}^{m}\frac{1}{2^{(k+3)n}|Q|}\int_{2^{k+3}Q}|f_j|_{q_j}dy_j \\ &\leqslant C|\omega|_{\mathrm{Dini}_{LlogL^l}(\rho)}\prod_{j=1}^{l}||b_j||_{BMO}\mathscr{M}^l_{L(\log L)}(|\vec{f}|_q)(x). \end{aligned}$$

In the last inequality we used the fact that $\omega$ satisfies the condition (1.4).

If $\alpha_1 = \cdots = \alpha_i = \infty$ and $\alpha_{i+1} = \cdots = \alpha_m = 0$. Minkowski's inequality gives

$$
\begin{aligned}
&\sup_{\eta>0}\Big|\mathscr{U}_\eta((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)f_1^\infty,\cdots,f_i^\infty,f_{i+1}^0,\cdots,f_m^0)(z)\\
&-\mathscr{U}_\eta((b_1(\cdot_1)-\lambda_1)\cdots(b_l(\cdot_l)-\lambda_l)f_1^\infty,\cdots,f_i^\infty,f_{i+1}^0,\cdots,f_m^0)(x)\Big|_q\\
&\leqslant C\int_{(\mathscr{Q}_k)^l}\frac{A|(b_1(y_1)-\lambda_1)\cdots(b_l(y_l)-\lambda_l)||f_1(y_1)|_{q_1}\cdots|f_l(y_l)|_{q_l}dy_1\cdots dy_l}{(|z-y_1|+\cdots+|z-y_m|)^{mn}}\\
&\quad\times\prod_{j=l+1}^m\int_{8Q}|f_j(y_j)|_{q_j}dy_j\\
&\leqslant C\sum_{k=1}^m\frac{\omega(2^{-k})}{(2^k|Q|^{\frac{1}{n}})^{mn}}\prod_{j=1}^l\int_{2^{k+3}Q}|b_j(y_j)-\lambda_j||f_j(y_j)|_{q_j}dy_j\prod_{j=l+1}^m\int_{2^{k+3}Q}|f_j(y_j)|_{q_j}dy_j\\
&\leqslant C\sum_{k=1}^m\omega(2^{-k})k^l\prod_{j=1}^l||b_j||_{BMO}|||f_j|_{q_j}||_{L(\log L),2^{k+3}Q}\prod_{j=l+1}^m\frac{1}{2^{(k+3)n}|Q|}\int_{2^{k+3}Q}|f_j|_{q_j}dy_j\\
&\leqslant C|\omega|_{\mathrm{Dini}_{LlogL^l}(\rho)}\prod_{j=1}^l||b_j||_{BMO}\mathscr{M}^l_{L(\log L)}(|\vec{f}|_q)(x).
\end{aligned}
$$

In other cases, the estimates is similar. Then we proved Lemma 9. □

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