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NEW ESTIMATES OF ESSENTIAL NORM OF AN INTEGRAL–TYPE OPERATOR BETWEEN BLOCH–TYPE SPACES ON THE UNIT BALL

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Abstract. Let $H(\mathbb{B})$ be the class of all holomorphic functions on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. For $g \in H(\mathbb{B})$ with $g(0) = 0$ and φ a holomorphic self-map of $\mathbb{B}$, the integral-type operator P_φ^g, recently introduced by S. Stević, is defined by

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \ \ z \in \mathbb{B}.$$

In this paper, we give a new characterization for the boundedness of integral-type operator P_φ^g between Bloch-type spaces on the unit ball $\mathbb{B}$. We also calculate the essential norm of the operator P_φ^g, which leads to a new condition for the compactness of the operator P_φ^g.

1. Introduction

Let $\mathbb{B}$ be the open unit ball in $\mathbb{C}^n$ with boundary $\partial\mathbb{B}$, where $\mathbb{C}^n$ is the n-dimensional complex vector space. Let $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$. For $f \in H(\mathbb{B})$, let

$$\mathscr{R}f(z) = \langle \bigtriangledown f(z), \overline{z} \rangle = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z)$$

represent the radial derivative of f, where $\bigtriangledown f(z) = (\frac{\partial f}{\partial z_1}, \cdots, \frac{\partial f}{\partial z_n})$.

For $\alpha > 0$, recall that the $\alpha-$Bloch space $\mathscr{B}^\alpha$ is the space that consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_\alpha = \sup_{z\in\mathbb{B}}(1-|z|^2)^\alpha|\mathscr{R}f(z)| < \infty.$$

It is well known that $\mathscr{B}^\alpha$ is a Banach space under the following norm

$$\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \|f\|_\alpha.$$

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Particularly, when $\alpha = 1$, $\mathscr{B}^\alpha = \mathscr{B}$, the classical Bloch space; when $0 < \alpha < 1$, $\mathscr{B}^\alpha = Lip_{1-\alpha}$, the analytic Lipschitz space; when $\alpha > 1$, $\mathscr{B}^\alpha = H^\infty_{\alpha-1}$, the weighted-type space that contains of all $f \in H(\mathbb{B})$ satisfying

$$\|f\|_\alpha = \sup_{z\in\mathbb{B}} (1-|z|^2)^{\alpha-1}|f(z)| < \infty.$$

The little α-Bloch space $\mathscr{B}^\alpha_0$ is the subspace of $\mathscr{B}^\alpha$ consisting of all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} (1-|z|^2)^\alpha |\mathscr{R}f(z)| = 0.$$

The essential norm of a continuous linear operator T is the distance from T to the family of compact operators K, that is $\|T\|_e = \inf\{\|T-K\| : K \ \ is \ \ compact\}$. Notice that $\|T\|_e = 0$ if and only if the operator T is compact, so the estimate on $\|T\|_e$ will lead to a condition for the operator T to be compact. For some results in the area, see, for example: [2, 3, 4, 5, 6, 7, 9, 16, 17, 18, 19, 21, 24, 25, 28, 35, 36]

Let φ be a holomorphic self-map of $\mathbb{B}$. Recall that the composition operator with symbol φ is defined by $C_\varphi(f) = f\circ\varphi$ for any $f \in H(\mathbb{B})$. It is important to provide function-theoretic description of the case when φ induces a bounded or compact composition operator on various function spaces. For general references on the theory of composition operators, we refer the interested readers to the book [2].

In [32], Wulan, Zheng and Zhu obtained a new result about the compactness of the composition operator on the Bloch space in terms of the n-th power of the analytic self-map φ on the unit disc $\mathbb{D}$, which is stated as follows:

THEOREM 1. *Let φ be an analytic self-map of the unit disk. Then C_φ is compact on the Bloch space $\mathscr{B}$ if and only if*

$$\lim_{m\to\infty} \|C_\varphi z^m\|_{\mathscr{B}} = 0.$$

In [34], Zhao obtained an essential norm formula for the composition operator from $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$ for any $0 < \alpha, \beta < \infty$ in terms of φ^m. We state it as follows:

THEOREM 2. *Let $0 < \alpha, \beta < \infty$ and φ be an analytic self-map of unit disk $\mathbb{D}$. Then the essential norm of $C_\varphi : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is*

$$\|C_\varphi\|_e = \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{m\to\infty} m^{\alpha-1}\|\varphi^m\|_\beta,$$

where φ^m is the m-th power of φ.

After that, some authors have extended this result to differentiation composition operators, which had been previously considerably studied mostly by S. Li and S. Stević (see, e.g. [15, 24, 26, 30] and the references therein), and weighted composition operators on the unit disk (see, e.g. [5, 9]).

For the higher dimensional case, in 2012, the authors in [6] generalized Zhao's results in [34] in the setting of the polydisk. Unlike the case of composition operators on the unit disk, the essential norms are different for the cases $\alpha \in (0,1)$ and $\alpha \geqslant 1$.

For $g \in H(\mathbb{B})$ with $g(0)=0$ and φ a holomorphic self-map of $\mathbb{B}$, the integral operator induced by g and φ on the unit ball is defined by

$$P_\varphi^g(f)(z) = \int_0^1 f(\varphi(tz))g(tz)\frac{dt}{t}, \quad f \in H(\mathbb{B}), \quad z \in \mathbb{B}. \tag{1.1}$$

When $\varphi(z)=z$ and g is replaced by $\mathscr{R}g$, the operator P_φ^g is the so called extended Cesàro operator, which was introduced in [8] and studied later, for example, in [1, 11, 12, 14].

The operator P_φ^g was first introduced by Stević in [23] (for one-dimensional analogy see [13]) and later studied, for example, in [22, 25, 27, 29, 31]. Motivated by [25], where Stević calculated the essential norm of the operator $P_\varphi^g : \mathscr{B}(\mathscr{B}_0) \to \mathscr{B}_\mu$, we will give a new characterization for the boundedness of $P_\varphi^g : \mathscr{B}^\alpha(\mathscr{B}_0^\alpha) \to \mathscr{B}^\beta$ when $\alpha > 1$. Moreover, we calculate the essential norm of $P_\varphi^g : \mathscr{B}^\alpha(\mathscr{B}_0^\alpha) \to \mathscr{B}^\beta$ in the case of $\alpha > 1$ and $0 < \alpha < 1$ and obtain a new condition for the operator P_φ^g to be compact. Our results complement the ones in [4] which studied the essential norms of generalized composition operators (introduced by Stević in [31]) between Bloch-type spaces in the unit ball.

Throughout this paper, C will denote a finite positive constant which may differ from one occurrence to another.

2. Some Lemmas

In this section, we present some lemmas which will be used in the proofs of our main results.

LEMMA 1. [25, Lemma 2] *Assume that $g \in H(\mathbb{B})$ with $g(0)=0$ and φ is a holomorphic self-map of $\mathbb{B}$. Then*

$$\mathscr{R}P_\varphi^g(f)(z) = f(\varphi(z))g(z)$$

for all $f \in H(\mathbb{B})$.

If we take $\omega(|z|) = (1-|z|^2)^\alpha$ and $\mu(|z|) = (1-|z|^2)^\beta$, then by Theorem 1 in [33], we obtain the following result:

LEMMA 2. *Assume that $g \in H(\mathbb{B})$ with $g(0)=0$ and φ is a holomorphic self-map of $\mathbb{B}$. Then operator $P_\varphi^g : \mathscr{B}^\alpha(\mathscr{B}_0^\alpha) \to \mathscr{B}^\beta$ is bounded if and only if*

(i) $0 < \alpha < 1$, $\sup_{z\in\mathbb{B}}(1-|z|^2)^\beta |g(z)| < \infty$.

(ii) $\alpha = 1$, $\sup_{z\in\mathbb{B}}(1-|z|^2)^\beta |g(z)|(1+\frac{1}{2}\ln\frac{1+|\varphi(z)|}{1-|\varphi(z)|}) < \infty$.

(iii) $\alpha > 1$, $\sup_{z\in\mathbb{B}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|g(z)| < \infty$.

LEMMA 3. [34, Lemma 2.2] *Let* $\alpha > 0$, $m \in \mathbb{N}$ *and* $0 \leqslant x \leqslant 1$. *Set* $r_m = \left(\frac{m-1}{m-1+2\alpha}\right)^{1/2}$ *for* $m \geqslant 2$ *and* $r_m = 0$ *for* $m = 1$. *Then* $H_{m,\alpha}(x) = x^{m-1}(1-x^2)^\alpha$ *has the following properties:*

(i) $\max\limits_{0\leqslant x\leqslant 1} H_{m,\alpha}(x) = H_{m,\alpha}(r_m) = \begin{cases} 1, & m=1; \\ \left(\frac{m-1}{m-1+2\alpha}\right)^{(m-1)/2}\left(\frac{2\alpha}{m-1+2\alpha}\right)^\alpha, & m \geqslant 2, \end{cases}$

so $\lim\limits_{m\to\infty} m^\alpha \max\limits_{0\leqslant x\leqslant 1} H_{m,\alpha}(x) = \left(\frac{2\alpha}{e}\right)^\alpha$.

(ii) For $m \geqslant 1$, $H_{m,\alpha}$ *is increasing on* $[0, r_m]$ *and decreasing on* $[r_m, 1]$.

(iii) For $m \geqslant 1$, $H_{m,\alpha}$ *is decreasing on* $[r_m, r_{m+1}]$, *and so*

$$\min_{x\in[r_m,r_{m+1}]} H_{m,\alpha}(x) = H_{m,\alpha}(r_{m+1}) = \left(\frac{m}{m+2\alpha}\right)^{(m-1)/2}\left(\frac{2\alpha}{m+2\alpha}\right)^\alpha.$$

Consequently,

$$\lim_{m\to\infty} m^\alpha \min_{x\in[r_m,r_{m+1}]} H_{m,\alpha}(x) = \left(\frac{2\alpha}{e}\right)^\alpha.$$

LEMMA 4. [4, Lemma 2.3] *Let* $\alpha > 0$, $m \in \mathbb{N}$. *Then for each* $\zeta \in \partial\mathbb{B}$, *we have*

$$\lim_{m\to\infty} m^{\alpha-1}\|\langle z,\zeta\rangle^m\|_\alpha = \left(\frac{2\alpha}{e}\right)^\alpha. \tag{2.1}$$

For $r \in [0,1]$, let $K_r f(z) = f(rz)$. It is easy to see that K_r is compact on the space $\mathscr{B}^\alpha$ or $\mathscr{B}_0^\alpha$ for $\alpha > 0$ and $\|K_r\| \leqslant 1$. The following lemma can be proved in a similar way as the proof of Lemma 4.3 in [34]:

LEMMA 5. *Let* $\alpha > 0$. *Then there is a sequence* $\{r_k\}_{k=1}^\infty$ *with* $0 < r_k < 1$ *tending to 1, such that the compact operator*

$$L_n = \frac{1}{n}\sum_{k=1}^n K_{r_k}$$

on $\mathscr{B}_0^\alpha$ *satisfies:*

(i) For any $s \in [0,1)$, $\lim\limits_{n\to\infty} \sup\limits_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \sup\limits_{|z|\leqslant s} |\nabla[(I-L_n)f](z)| = 0$.

(ii) For any $t \in [0,1)$, $\lim\limits_{n\to\infty} \sup\limits_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \sup\limits_{|z|\leqslant t} |[(I-L_n)f](z)| = 0$.

(iii) $\limsup\limits_{n\to\infty} \|I-L_n\| \leqslant 1$.

Furthermore, these statements hold as well for the sequence of biadjoints L_n^{**} *on* $\mathscr{B}^\alpha$.

The following result is a crucial criterion for the compactness of $P_\varphi^g : \mathscr{B}^\alpha(\mathscr{B}_0^\alpha) \to \mathscr{B}^\beta$ (for the proof, see, for example, the corresponding lemma in [20]).

LEMMA 6. *Assume that* $\alpha,\beta>0$, $g\in H(\mathbb{B})$ *with* $g(0)=0$ *and* φ *is an analytic self-map of* $\mathbb{B}$. *Then* $P_\varphi^g:\mathscr{B}^\alpha(\mathscr{B}_0^\alpha)\to\mathscr{B}^\beta$ *is compact if and only if* $P_\varphi^g:\mathscr{B}^\alpha(\mathscr{B}_0^\alpha)\to\mathscr{B}^\beta$ *is bounded and for any bounded sequence* $\{f_m\}_{m\in\mathbb{N}}$ *in* $\mathscr{B}^\alpha$*(or* $\mathscr{B}_0^\alpha$*) which converges to zero uniformly on any compact subsets of* $\mathbb{B}$, *we have that*

$$\|P_\varphi^g f_m\|_{\mathscr{B}^\beta}\to 0,\quad m\to\infty.$$

3. Main results

First, we give a new characterization for the boundedness of $P_\varphi^g:\mathscr{B}^\alpha(\mathscr{B}_0^\alpha)\to\mathscr{B}^\beta$, in which we will use Lemma 2.

THEOREM 3. *Suppose* $\alpha>1$, $\beta>0$, $g\in H(\mathbb{B})$ *with* $g(0)=0$ *and* φ *is a holomorphic self-map of* $\mathbb{B}$. *Then the following statements are equivalent.*

(i) $P_\varphi^g:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ *is bounded;*

(ii) $P_\varphi^g:\mathscr{B}_0^\alpha\to\mathscr{B}^\beta$ *is bounded;*

(iii)

$$\sup_{m\in\mathbb{N}}\sup_{\zeta\in\partial\mathbb{B}} m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}<\infty. \tag{3.1}$$

Proof. $(i)\Rightarrow(ii)$. This implication is obvious.

$(ii)\Rightarrow(iii)$. Suppose that $P_\varphi^g:\mathscr{B}_0^\alpha\to\mathscr{B}^\beta$ is bounded. For any $m\in\mathbb{N}$ and $\zeta\in\partial\mathbb{B}$, consider the function $h_{m,\zeta}(z)=\frac{\langle z,\zeta\rangle^m}{\|\langle z,\zeta\rangle^m\|_\alpha}$, then it is easy to check that $h_{m,\zeta}\in\mathscr{B}_0^\alpha$ with $\|h_{m,\zeta}\|_{\mathscr{B}^\alpha}=1$. Note that from Lemma 4, there is a constant $C>0$ independent of m and ζ such that $\|\langle z,\zeta\rangle^m\|_\alpha\leqslant Cm^{1-\alpha}$. Combining with the boundedness of $P_\varphi^g:\mathscr{B}_0^\alpha\to\mathscr{B}^\beta$, it follows that for any $m\in\mathbb{N}$ and $\zeta\in\partial\mathbb{B}$,

$$\begin{aligned}\infty>\|P_\varphi^g\|_{\mathscr{B}_0^\alpha\to\mathscr{B}^\beta}&\geqslant\|P_\varphi^g h_{m,\zeta}\|_{\mathscr{B}^\beta}=\frac{\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_\beta}{\|\langle z,\zeta\rangle^m\|_\alpha}\\&\geqslant\frac{1}{C}m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_\beta=\frac{1}{C}m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}.\end{aligned}$$

Thus (3.1) is true.

$(iii)\Rightarrow(i)$. Consider the function $G_{m,\alpha}(x)=H_{m+1,\alpha-1}=x^m(1-x^2)^{\alpha-1}$. By Lemma 3,

$$\max_{0\leqslant x\leqslant 1}G_{m,\alpha}(x)=G_{m,\alpha}(s_m)=\begin{cases}1, & m=0;\\ \left(\frac{m}{m+2(\alpha-1)}\right)^{m/2}\left(\frac{2(\alpha-1)}{m+2(\alpha-1)}\right)^{\alpha-1}, & m\geqslant 1,\end{cases} \tag{3.2}$$

where $s_m=\left(\frac{m}{m+2(\alpha-1)}\right)^{1/2}$ for $m\geqslant 1$ and $s_m=0$ for $m=0$. For any integer $m\geqslant 1$, let

$$E_m=\{z\in\mathbb{B}:s_m\leqslant|\varphi(z)|\leqslant s_{m+1}\} \tag{3.3}$$

Let j and k be the smallest and largest positive integers respectively such that $E_j \neq \emptyset$ and $E_k \neq \emptyset$ (k could be ∞). Thus $\mathbb{B} = \bigcup_{m=j}^{k} E_m$. From (iii) of Lemma 3, we know that there exists a constant $\delta > 0$, independent of m, such that

$$\min_{z \in E_m} m^{\alpha-1} |\varphi(z)|^m (1 - |\varphi(z)|^2)^{\alpha-1} \geqslant \delta.$$

Thus

$$\begin{aligned}
&\sup_{z\in\mathbb{B}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} |g(z)| \\
&= \sup_{z\in\mathbb{B}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}} |g(z)| \frac{m^{\alpha-1}|\varphi(z)|^m (1-|\varphi(z)|^2)^{\alpha-1}}{m^{\alpha-1}|\varphi(z)|^m (1-|\varphi(z)|^2)^{\alpha-1}} \\
&\leqslant \frac{1}{\delta} \sup_{m\in\mathbb{N}} \sup_{z\in\mathbb{B}} m^{\alpha-1} (1-|z|^2)^\beta |g(z)| |\varphi(z)|^m \\
&\leqslant \frac{1}{\delta} \sup_{m\in\mathbb{N}} \sup_{\zeta\in\partial\mathbb{B}} \sup_{z\in\mathbb{B}} m^{\alpha-1} (1-|z|^2)^\beta |g(z)| |\langle \varphi(z), \zeta \rangle^m| \\
&= \frac{1}{\delta} \sup_{m\in\mathbb{N}} \sup_{\zeta\in\partial\mathbb{B}} m^{\alpha-1} \|P_\varphi^g(\langle z, \zeta\rangle^m)\|_{\mathscr{B}^\beta} < \infty.
\end{aligned}$$

From this and by Lemma 3 we have that $P_\varphi^g : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded. This completes the proof. □

Next we give an estimate for the essential norm of $P_\varphi^g : \mathscr{B}^\alpha(\mathscr{B}_0^\alpha) \to \mathscr{B}^\beta$.

THEOREM 4. *Suppose* $0 < \alpha < 1$, $\beta > 0$, $g \in H(\mathbb{B})$ *with* $g(0) = 0$ *and* φ *is a holomorphic self-map of* $\mathbb{B}$. *If* $P_\varphi^g : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ *is bounded, then*

$$\begin{aligned}
&\left(\frac{e}{2\alpha}\right)^\alpha \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^{\alpha-1} \|P_\varphi^g(\langle z, \zeta\rangle^m)\|_{\mathscr{B}^\beta} \\
&\leqslant \|P_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta} = \|P_\varphi^g\|_{e,\mathscr{B}_0^\alpha\to\mathscr{B}^\beta} \\
&\leqslant \left(\frac{e}{2\alpha}\right)^\alpha \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^{\alpha} \|P_\varphi^g(\langle z, \zeta\rangle^m)\|_{\mathscr{B}^\beta}, \qquad (3.4)
\end{aligned}$$

where $\|P_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}$ *denotes the essential norm of* P_φ^g *from* $\mathscr{B}^\alpha$ *to* $\mathscr{B}^\beta$.

Proof. First, we show that (3.4) is true when $\sup_{z\in\mathbb{B}} |\varphi(z)| < 1$. In this case, there is a number $r \in (0,1)$ such that

$$\sup_{z\in\mathbb{B}} |\varphi(z)| \leqslant r.$$

Since $P_\varphi^g : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded, for $f = 1 \in \mathscr{B}^\alpha$ we obtain that

$$\sup_{z\in\mathbb{B}} (1-|z|^2)^\beta |g(z)| < \infty. \qquad (3.5)$$

Hence, we have

$$\begin{aligned}
&\Big(\frac{e}{2\alpha}\Big)^{\alpha}\limsup_{m\to\infty}\sup_{\zeta\in\partial\mathbb{B}} m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}\\
&=\Big(\frac{e}{2\alpha}\Big)^{\alpha}\limsup_{m\to\infty}\sup_{\zeta\in\partial\mathbb{B}}\sup_{z\in\mathbb{B}} m^{\alpha-1}(1-|z|^2)^\beta|\langle\varphi(z),\zeta\rangle^m||g(z)|\\
&\leqslant\Big(\frac{e}{2\alpha}\Big)^{\alpha}\limsup_{m\to\infty}\sup_{z\in\mathbb{B}} m^{\alpha-1}|\varphi(z)|^m(1-|z|^2)^\beta|g(z)|\\
&\leqslant C\Big(\frac{e}{2\alpha}\Big)^{\alpha}\limsup_{m\to\infty} m^{\alpha-1}r^m=0.
\end{aligned}$$

Similarly it can be proved that $(\frac{e}{2\alpha})^\alpha\limsup_{m\to\infty}\sup_{\zeta\in\partial\mathbb{B}} m^{\alpha}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}=0$.

On the other hand, let $\{f_m\}$ be a sequence in $\mathscr{B}^\alpha$ with $\|f_m\|_{\mathscr{B}^\alpha}\leqslant 1$ and $f_m(z)\to 0$ uniformly on any compact subsets of $\mathbb{B}$. Then

$$\begin{aligned}
\|P_\varphi^g f_m\|_{\mathscr{B}^\beta}&=|P_\varphi^g(f_m)(0)|+\sup_{z\in\mathbb{B}}(1-|z|^2)^\beta|f_m(\varphi(z))||g(z)|\\
&\leqslant 0+\sup_{z\in\mathbb{B}}(1-|z|^2)^\beta|g(z)|\sup_{z\in\mathbb{B}}|f_m(\varphi(z))|\\
&\leqslant C\sup_{z\in\mathbb{B}}|f_m(\varphi(z))|. \qquad (3.6)
\end{aligned}$$

Since $|\varphi(z)|\leqslant r$ and $f_m(z)\to 0$ uniformly on any compact subset of $\mathbb{B}$, we get

$$\lim_{m\to\infty}\|P_\varphi^g f_m\|_{\mathscr{B}_\beta}=0.$$

So P_φ^g is compact from $\mathscr{B}^\alpha$ to $\mathscr{B}^\beta$. Thus (3.4) holds.

Next we suppose $\sup_{z\in\mathbb{B}}|\varphi(z)|=1$. We first show that the upper estimate is true. Let $\{L_n\}$ be the sequence of operators given in Lemma 5. Since each L_n is compact on the space $\mathscr{B}^\alpha$, $P_\varphi^g L_n$ is also compact due to the boundedness of $P_\varphi^g:\mathscr{B}^\alpha\to\mathscr{B}^\beta$. Thus we have

$$\begin{aligned}
\|P_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}&\leqslant\|P_\varphi^g-P_\varphi^g L_n\|=\|P_\varphi^g(I-L_n)\|=\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\|[P_\varphi^g(I-L_n)]f\|_{\mathscr{B}^\alpha}\\
&=\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}|[P_\varphi^g(I-L_n)](f)(0)|\\
&\quad+\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{z\in\mathbb{B}}(1-|z|^2)^\beta|g(z)||(I-L_n)f(\varphi(z))|\\
&=0+J
\end{aligned}$$

For any integer $m\geqslant 1$, let E_m be the subset of $\mathbb{B}$ given in (3.3). Let j be the smallest positive integer such that $E_j\neq\emptyset$. Since $\sup_{z\in\mathbb{B}}|\varphi(z)|=1$, E_m is not empty for every integer $m \geqslant j$ and $\mathbb{B} = \bigcup_{m=j}^{\infty} E_m$. Now we divide J into two parts

$$\begin{aligned} J = & \sup_{\|f\|_{\mathscr{B}^\alpha} \leqslant 1} \sup_{j \leqslant m \leqslant N-1} \sup_{z \in E_m} (1-|z|^2)^\beta |g(z)||(I-L_n)f(\varphi(z))| \\ & + \sup_{\|f\|_{\mathscr{B}^\alpha} \leqslant 1} \sup_{m \geqslant N} \sup_{z \in E_m} (1-|z|^2)^\beta |g(z)||(I-L_n)f(\varphi(z))| \\ = & J_1 + J_2, \end{aligned}$$

were N is a positive integer determined as follow. For J_1, by (3.5) we have

$$\begin{aligned} J_1 & \leqslant \sup_{z \in \mathbb{B}} (1-|z|^2)^\beta |g(z)| \sup_{\|f\|_{\mathscr{B}^\alpha} \leqslant 1} \sup_{s_m \leqslant |\varphi(z)| \leqslant s_{N-1}} |(I-L_n)f(\varphi(z))| \\ & \leqslant C \sup_{\|f\|_{\mathscr{B}^\alpha} \leqslant 1} \sup_{s_m \leqslant |\varphi(z)| \leqslant s_{N-1}} |(I-L_n)f(\varphi(z))|. \end{aligned}$$

Combining with (ii) in Lemma 5, we obtain that

$$\limsup_{m \to \infty} J_1 = 0.$$

Write J_2 as

$$\sup_{\|f\|_{\mathscr{B}^\alpha} \leqslant 1} \sup_{m \geqslant N} \sup_{z \in E_m} \frac{m^\alpha |\varphi(z)|^m (1-|z|^2)^\beta |g(z)|}{m^\alpha |\varphi(z)|^m (1-|\varphi(z)|^2)^\alpha} (1-|\varphi(z)|^2)^\alpha |(I-L_n)f(\varphi(z))|.$$

From Lemma 3, it is easy to see that

$$\lim_{m \to \infty} (m^\alpha |\varphi(z)|^m (1-|\varphi(z)|^2)^\alpha)^{-1} = \left(\frac{e}{2\alpha}\right)^\alpha,$$

whenever $z \in E_m$. Hence, for an arbitrary fixed number $\varepsilon > 0$, there exists a positive integer N large enough such that for $m \geqslant N$,

$$(m^\alpha |\varphi(z)|^m (1-|\varphi(z)|^2)^\alpha)^{-1} < \left(\frac{e}{2\alpha}\right)^\alpha + \varepsilon,$$

whenever $z \in E_m$. For such N we have

$$\begin{aligned} J_2 \leqslant & \left[\left(\frac{e}{2\alpha}\right)^\alpha + \varepsilon\right] \sup_{\|f\|_{\mathscr{B}^\alpha} \leqslant 1} \sup_{m \geqslant N} \sup_{z \in E_m} m^\alpha |\varphi(z)|^m (1-|z|^2)^\beta |g(z)| \\ & \times (1-|\varphi(z)|^2)^\alpha |(I-L_n)f(\varphi(z))| \\ = & \left[\left(\frac{e}{2\alpha}\right)^\alpha + \varepsilon\right] \sup_{\|f\|_{\mathscr{B}^\alpha} \leqslant 1} \sup_{m \geqslant N} \sup_{z \in E_m} m^\alpha |\varphi(z)|^m (1-|z|^2)^\beta |g(z)| \\ & \times (1-|\varphi(z)|^2)^\alpha \left| \int_0^1 \langle \bigtriangledown [(I-L_n)f](t\varphi(z)), \overline{\varphi(z)} \rangle dt \right| \\ \leqslant & \left[\left(\frac{e}{2\alpha}\right)^\alpha + \varepsilon\right] \sup_{\|f\|_{\mathscr{B}^\alpha} \leqslant 1} \sup_{m \geqslant N} \sup_{z \in E_m} m^\alpha (1-|z|^2)^\beta |g(z)| \left| \langle \varphi(z), \frac{\varphi(z)}{|\varphi(z)|} \rangle^m \right| \\ & \times (1-|\varphi(z)|^2)^\alpha \left| \bigtriangledown [(I-L_n)f](t\varphi(z)) \right| \left| \overline{\varphi(z)} \right| \end{aligned}$$

$$
\begin{aligned}
&= \left[\left(\frac{e}{2\alpha}\right)^{\alpha} + \varepsilon\right] \sup_{\|f\|_{\mathscr{B}^{\alpha}} \leqslant 1} \sup_{m \geqslant N} \sup_{z \in E_m} m^{\alpha} (1-|z|^2)^{\beta} |g(z)| \left| \langle \varphi(z), \frac{\varphi(z)}{|\varphi(z)|} \rangle^m \right| \\
&\quad \times (1-|t\varphi(z)|^2)^{\alpha} \left| \bigtriangledown [(I-L_n)f](t\varphi(z)) \right| \left(\frac{1-|\varphi(z)|^2}{1-|t\varphi(z)|^2} \right)^{\alpha} \\
&\leqslant \left[\left(\frac{e}{2\alpha}\right)^{\alpha} + \varepsilon\right] \sup_{\|f\|_{\mathscr{B}^{\alpha}} \leqslant 1} \|(I-L_m)f\|_{\mathscr{B}^{\alpha}} \sup_{m \geqslant N} \sup_{\zeta \in \partial\mathbb{B}} m^{\alpha} \|P_{\varphi}^{g}(\langle z, \zeta \rangle^m)\|_{\mathscr{B}^{\beta}} \\
&= \left[\left(\frac{e}{2\alpha}\right)^{\alpha} + \varepsilon\right] \|I-L_n\| \sup_{m \geqslant N} \sup_{\zeta \in \partial\mathbb{B}} m^{\alpha} \|P_{\varphi}^{g}(\langle z, \zeta \rangle^m)\|_{\mathscr{B}^{\beta}}.
\end{aligned}
$$

Since ε is an arbitrary positive number, from (iii) in Lemma 5, we obtain

$$
\limsup_{m \to \infty} J_2 = \left(\frac{e}{2\alpha}\right)^{\alpha} \limsup_{m \to \infty} \sup_{\zeta \in \partial\mathbb{B}} m^{\alpha} \|P_{\varphi}^{g}(\langle z, \zeta \rangle^m)\|_{\mathscr{B}^{\beta}}.
$$

Thus we obtain the upper estimate of $\|P_{\varphi}^{g}\|_{e, \mathscr{B}^{\alpha} \to \mathscr{B}^{\beta}}$.

On the other hand, recall that

$$
\|P_{\varphi}^{g}\|_{e, \mathscr{B}_0^{\alpha} \to \mathscr{B}^{\beta}} = \inf\{\|P_{\varphi}^{g} - K\| : K \textit{ is compact}\}.
$$

Take any compact operator $K : \mathscr{B}_0^{\alpha} \to \mathscr{B}^{\beta}$. For any $m \in \mathbb{N}$ and $\zeta \in \partial\mathbb{B}$, consider the function $h_{m,\zeta}$ defined in the proof of Theorem 3, we know that $h_{m,\zeta} \in \mathscr{B}_0^{\alpha}$ with $\|h_{m,\zeta}\|_{\mathscr{B}^{\alpha}} = 1$ and $h_{m,\zeta}$ converges to 0 uniformly on any compact subset of $\mathbb{B}$ as $m \to \infty$ for any $\zeta \in \partial\mathbb{B}$. Moreover, this convergence is also uniform respect to $\zeta \in \partial\mathbb{B}$. Actually, for each $r \in (0,1)$, denoted by $\mathbb{B}_r$ the ball with center at zero and radius equals to r. For any $z \in \mathbb{B}_r$ and $\zeta \in \partial\mathbb{B}$,

$$
\begin{aligned}
\lim_{m \to \infty} |h_{m,\zeta}(z)| &= \lim_{m \to \infty} \frac{m^{\alpha-1} |\langle z, \zeta \rangle^m|}{m^{\alpha-1} \|\langle z, \zeta \rangle^m\|_{\alpha}} \leqslant C \lim_{m \to \infty} m^{\alpha-1} |\langle z, \zeta \rangle|^m \\
&\leqslant C \lim_{m \to \infty} m^{\alpha-1} r^m = 0.
\end{aligned}
$$

Then for any compact operator $K : \mathscr{B}_0^{\alpha} \to \mathscr{B}^{\beta}$, we will prove that

$$
\liminf_{m \to \infty} \inf_{\zeta \in \partial\mathbb{B}} \|K h_{m,\zeta}\|_{\mathscr{B}^{\beta}} = 0. \tag{3.7}
$$

In fact, since for any $\zeta_1, \zeta_2 \in \partial\mathbb{B}$, we have $\|\langle z, \zeta_1 \rangle^m\|_{\alpha} = \|\langle z, \zeta_2 \rangle^m\|_{\alpha} = \|\langle z, 1 \rangle^m\|_{\alpha}$. It follows from the proof of Lemma 3 that

$$
\begin{aligned}
&\|K h_{m,\zeta_1} - K h_{m,\zeta_2}\|_{\mathscr{B}^{\beta}} \leqslant \|K\| \|h_{m,\zeta_1} - h_{m,\zeta_2}\|_{\mathscr{B}^{\alpha}} \\
&= \frac{m\|K\|}{\|\langle z, 1 \rangle^m\|_{\alpha}} \sup_{z \in \mathbb{B}} (1-|z|^2)^{\alpha} |\langle z, \zeta_1 \rangle^m - \langle z, \zeta_2 \rangle^m| \\
&\leqslant \frac{2^{\alpha} m^2 \|K\|}{\|\langle z, 1 \rangle^m\|_{\alpha}} |\zeta_1 - \zeta_2|.
\end{aligned}
$$

So $\|Kh_{m,\zeta}\|_{\mathscr{B}^\beta}$ is continuous with respect to ζ and also uniformly continuous due to the compactness of $\partial\mathbb{B}$. Then for each $m \in \mathbb{N}$, there exists $\zeta_m \in \partial\mathbb{B}$ depending on m such that

$$\|Kh_{m,\zeta_m}\|_{\mathscr{B}^\beta} = \inf_{\zeta\in\partial\mathbb{B}} \|Kh_{m,\zeta}\|_{\mathscr{B}^\beta}.$$

From this and by using the weakly convergence argument as in the proof of Theorem 3.4 in [31], we have that

$$\liminf_{m\to\infty} \inf_{\zeta\in\partial\mathbb{B}} \|Kh_{m,\zeta}\|_{\mathscr{B}^\beta} = \liminf_{m\to\infty} \|Kh_{m,\zeta_m}\|_{\mathscr{B}^\beta} = 0,$$

so we conclude that (3.7) holds.

From Lemma 3, it is easy to see that

$$\lim_{m\to\infty} (m^\alpha |z|^m (1-|z|^2)^\alpha)^{-1} = \left(\frac{e}{2\alpha}\right)^\alpha,$$

whenever $z \in E_m$. Hence, for an arbitrary fixed number $\varepsilon > 0$, there exists a positive integer N large enough such that for $k \geqslant N$,

$$(m^\alpha |z|^m (1-|z|^2)^\alpha)^{-1} > \left(\frac{e}{2\alpha}\right)^\alpha - \varepsilon,$$

whenever $z \in E_m$. Therefore

$$\begin{aligned}
\|P_\varphi^g - K\| &\geqslant \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \|(P_\varphi^g - K)(h_{m,\zeta})\|_{\mathscr{B}^\beta} \geqslant \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \|P_\varphi^g(h_{m,\zeta})\|_{\mathscr{B}^\beta} \\
&= \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \frac{m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}}{m^{\alpha-1}\|\langle z,\zeta\rangle^m\|_\alpha} \\
&= \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \frac{m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}}{\sup_{z\in\mathbb{B}} m^{\alpha-1}(1-|z|^2)^{\alpha-1}|\langle z,\zeta\rangle^m|} \\
&\geqslant \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \frac{m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}}{\sup_{z\in\mathbb{B}} m^{\alpha-1}(1-|z|^2)^{\alpha-1}|z|^m} \\
&\geqslant \left[\left(\frac{e}{2\alpha}\right)^\alpha - \varepsilon\right] \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}.
\end{aligned}$$

With the arbitrariness of ε, we obtain the lower estimate of $\|P_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}$. Thus the proof is complete. □

THEOREM 5. *Suppose $\alpha > 1$, $\beta > 0$, $g \in H(\mathbb{B})$ with $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$. If $P_\varphi^g : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded, then*

$$\begin{aligned}
\|P_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta} &= \|P_\varphi^g\|_{e,\mathscr{B}_0^\alpha\to\mathscr{B}^\beta} \\
&= \left(\frac{e}{2\alpha-1}\right)^{\alpha-1} \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}. \qquad (3.8)
\end{aligned}$$

Proof. If $\sup_{z\in\mathbb{B}}|\varphi(z)|<1$, the proof is similar to the corresponding one in Theorem 4.

Now we suppose $\sup\limits_{z\in\mathbb{B}}|\varphi(z)|=1$. First we show that the upper estimate is true. Let $\{L_n\}$ be the sequence of operators given in Lemma 5. Since each L_n is compact on the space $\mathscr{B}^\alpha$, $P_\varphi^g L_n$ is also compact due to the boundedness of $P_\varphi^g:\mathscr{B}^\alpha\to\mathscr{B}^\beta$. Thus we have

$$\begin{aligned}\|P_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta} &\leqslant \|P_\varphi^g-P_\varphi^g L_n\| = \|P_\varphi^g(I-L_n)\| = \sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\|[P_\varphi^g(I-L_n)]f\|_{\mathscr{B}^\alpha}\\ &= \sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}|[P_\varphi^g(I-L_n)](f)(0)|\\ &\quad+\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{z\in\mathbb{B}}(1-|z|^2)^\beta|g(z)||(I-L_n)f(\varphi(z))|\\ &=0+J.\end{aligned}$$

For any integer $m\geqslant 1$, let E_m be the subset of $\mathbb{B}$ given in (3.3) and let j be the smallest positive integer such that $E_j\neq\emptyset$. Since $\sup\limits_{z\in\mathbb{B}}|\varphi(z)|=1$, E_m is not empty for every integer $m\geqslant j$, and $\mathbb{B}=\bigcup_{m=j}^\infty E_m$. Now we divide J into two parts

$$\begin{aligned}J &= \sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{j\leqslant m\leqslant N-1}\sup_{z\in E_m}(1-|z|^2)^\beta|g(z)||(I-L_n)f(\varphi(z))|\\ &\quad+\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{m\geqslant N}\sup_{z\in E_m}(1-|z|^2)^\beta|g(z)||(I-L_n)f(\varphi(z))|\\ &=J_1+J_2,\end{aligned}$$

where N is a positive integer determined as follow. For J_1, by (3.5) we have

$$\begin{aligned}J_1 &\leqslant \sup_{z\in\mathbb{B}}(1-|z|^2)^\beta|g(z)|\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{s_m\leqslant|\varphi(z)|\leqslant s_{N-1}}|(I-L_n)f(\varphi(z))|\\ &\leqslant C\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{s_m\leqslant|\varphi(z)|\leqslant s_{N-1}}|(I-L_n)f(\varphi(z))|.\end{aligned}$$

Combining with (ii) in Lemma 5, we obtain that

$$\limsup_{m\to\infty}J_1=0.$$

Write J_2 as

$$\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1}\sup_{m\geqslant N}\sup_{z\in E_m}\frac{m^{\alpha-1}|\varphi(z)|^m(1-|z|^2)^\beta|g(z)|}{m^{\alpha-1}|\varphi(z)|^m(1-|\varphi(z)|^2)^{\alpha-1}}(1-|\varphi(z)|^2)^{\alpha-1}|(I-L_n)f(\varphi(z))|.$$

From Lemma 3, it is easy to see that

$$\lim_{m\to\infty}(m^{\alpha-1}|\varphi(z)|^m(1-|\varphi(z)|^2)^{\alpha-1})^{-1}=\left(\frac{e}{2(\alpha-1)}\right)^{\alpha-1},$$

whenever $z \in E_m$. Hence, for an arbitrary fixed number $\varepsilon > 0$ and for $z \in E_m$, there exists a positive integer N large enough such that for $m \geqslant N$,

$$(m^{\alpha-1}|\varphi(z)|^m(1-|\varphi(z)|^2)^{\alpha-1})^{-1} < \Big(\frac{e}{2(\alpha-1)}\Big)^{\alpha-1} + \varepsilon,$$

whenever $z \in E_m$. For such N we have

$$\begin{aligned} J_2 &\leqslant \Big[\Big(\frac{e}{2(\alpha-1)}\Big)^{\alpha-1} + \varepsilon\Big] \sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \sup_{m\geqslant N} \sup_{z\in E_m} m^{\alpha-1}|\varphi(z)|^m(1-|z|^2)^\beta |g(z)| \\ &\quad \times (1-|\varphi(z)|^2)^{\alpha-1}|(I-L_n)f(\varphi(z))| \\ &\leqslant \Big[\Big(\frac{e}{2(\alpha-1)}\Big)^{\alpha-1} + \varepsilon\Big] \\ &\quad \times \sup_{m\geqslant N} \sup_{z\in E_m} m^{\alpha-1}|\varphi(z)|^m(1-|z|^2)^\beta |g(z)| \sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant 1} \|(I-L_n)f\|_{\mathscr{B}^\alpha} \\ &= \Big[\Big(\frac{e}{2(\alpha-1)}\Big)^{\alpha-1} + \varepsilon\Big] \\ &\quad \times \|I-L_n\| \sup_{m\geqslant N} \sup_{z\in E_m} m^{\alpha-1}(1-|z|^2)^\beta |g(z)| \Big|\langle \varphi(z), \frac{\varphi(z)}{|\varphi(z)|}\rangle^m\Big| \\ &\leqslant \Big[\Big(\frac{e}{2(\alpha-1)}\Big)^{\alpha-1} + \varepsilon\Big] \|I-L_n\| \sup_{m\geqslant N} \sup_{\zeta\in\partial\mathbb{B}} m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}. \end{aligned}$$

Since ε is an arbitrary positive number, from (iii) in Lemma 5, we obtain

$$\limsup_{m\to\infty} J_2 = \Big(\frac{e}{2(\alpha-1)}\Big)^{\alpha-1} \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}.$$

Thus we obtain the upper estimate of $\|P_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}$.

On the other hand, recall that

$$\|P_\varphi^g\|_{e,\mathscr{B}_0^\alpha\to\mathscr{B}^\beta} = \inf\{\|P_\varphi^g - K\| : K \textit{ is compact}\}.$$

Take any compact operator $K : \mathscr{B}_0^\alpha \to \mathscr{B}^\beta$. For any $m \in \mathbb{N}$ and $\zeta \in \partial\mathbb{B}$, consider the function $h_{m,\zeta}$ defined in the proof of Theorem 3. From Theorem 4 we know that

$$\liminf_{m\to\infty} \inf_{\zeta\in\partial\mathbb{B}} \|Kh_{m,\zeta}\|_{\mathscr{B}^\beta} = \liminf_{m\to\infty} \|Kh_{m,\zeta_m}\|_{\mathscr{B}^\beta} = 0.$$

It is easy to see that

$$\lim_{m\to\infty} (m^{\alpha-1}|z|^m(1-|z|^2)^{\alpha-1})^{-1} = \Big(\frac{e}{2(\alpha-1)}\Big)^{\alpha-1},$$

whenever $z \in E_m$. Hence, for an arbitrary fixed number $\varepsilon > 0$, there exists a positive integer N large enough such that for $k \geqslant N$,

$$(m^{\alpha-1}|z|^m(1-|z|^2)^{\alpha-1})^{-1} > \Big(\frac{e}{2(\alpha-1)}\Big)^{\alpha-1} - \varepsilon,$$

whenever $z \in E_m$. Therefore

$$\begin{aligned}
\|P_\varphi^g - K\| &\geqslant \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \|(P_\varphi^g - K)(h_{m,\zeta})\|_{\mathscr{B}^\beta} \geqslant \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \|P_\varphi^g(h_{m,\zeta})\|_{\mathscr{B}^\beta} \\
&= \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \frac{m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}}{m^{\alpha-1}\|\langle z,\zeta\rangle^m\|_\alpha} \\
&\geqslant \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} \frac{m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}}{\sup_{z\in\mathbb{B}} m^{\alpha-1}(1-|z|^2)^{\alpha-1}|z|^m} \\
&\geqslant \left[\left(\frac{e}{2\alpha-1}\right)^{\alpha-1} - \varepsilon\right] \limsup_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta}.
\end{aligned}$$

By the arbitrariness of ε, we obtain the lower estimate of $\|P_\varphi^g\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta}$. Thus the proof is complete. □

It follows from Theorem 5 that we can directly obtain the following result about the compactness of the operator P_φ^g.

COROLLARY 1. *Suppose $\alpha > 1$, $\beta > 0$, $g \in H(\mathbb{B})$ with $g(0) = 0$ and φ is a holomorphic self-map of $\mathbb{B}$. If $P_\varphi^g : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded, then the following statements are equivalent.*

(i) $P_\varphi^g : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact;

(ii) $P_\varphi^g : \mathscr{B}_0^\alpha \to \mathscr{B}^\beta$ is compact;

(iii) $\lim_{m\to\infty} \sup_{\zeta\in\partial\mathbb{B}} m^{\alpha-1}\|P_\varphi^g(\langle z,\zeta\rangle^m)\|_{\mathscr{B}^\beta} = 0.$

We cannot deal with the cases of $0 < \alpha \leqslant 1$ in Theorem 3 and $\alpha = 1$ in Theorem 5. So we pose the following question.

Question. What about the cases of $0 < \alpha \leqslant 1$ in Theorem 3 and $\alpha = 1$ in Theorem 5?

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