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ON POINTWISE APPROXIMATION OF FUNCTIONS BY SOME MATRIX MEANS OF FOURIER SERIES

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Abstract. The results corresponding to some theorems of S. Lal [Appl. Math. and Comput. 209 (2009), 346–350] and the results of the authors [Banach Center Publ., 95, (2011), 339–351] are shown. The same and sometimes better degrees of pointwise approximation as in mentioned papers by weaker assumptions on considered functions and examined summability methods are obtained. From presented pointwise results the estimation on norm approximation are derived. Some special cases as corollaries for iteration of the Nörlund or the Riesz method with the Euler one are also formulated.

1. Introduction

Let L^p $(1 \leqslant p < \infty)$ [respectively L^∞] be the class of all 2π–periodic real–valued functions integrable in the Lebesgue sense with p–th power [essentially bounded] over $Q = [-\pi,\pi]$ with the norm

$$\|f\| := \|f(\cdot)\|_{L^p} = \begin{cases} \left(\frac{1}{2\pi}\int_Q |f(t)|^p\,dt\right)^{1/p} & \text{when } 1 \leqslant p < \infty, \\ ess\sup\limits_{t\in Q} |f(t)| & \text{when } p = \infty \end{cases}$$

and consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty}(a_\nu(f)\cos\nu x + b_\nu(f)\sin\nu x)$$

with the partial sums $S_k f$ [8, Th.(3.1)IV].

Let $A := (a_{n,k})$ and $B := (b_{n,k})$ be infinite lower triangular matrices of real numbers such that

$$a_{n,k} \geqslant 0 \text{ and } b_{n,k} \geqslant 0 \text{ when } k = 0,1,2,\ldots,n,$$
$$a_{n,k} = 0 \text{ and } b_{n,k} = 0 \text{ when } k > n,$$

$$\sum_{k=0}^{n} a_{n,k} = 1 \text{ and } \sum_{k=0}^{n} b_{n,k} = 1, \text{ where } n = 0,1,2,\ldots,$$

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and let, for $m=0,1,2,...,n$,

$$A_{n,m}=\sum_{k=0}^{m}a_{n,k} \quad \text{and} \quad \overline{A}_{n,m}=\sum_{k=m}^{n}a_{n,k}.$$

Let the $AB-$transformation of $(S_k f)$ be given by

$$T_{n,A,B}f(x):=\sum_{r=0}^{n}\sum_{k=0}^{r}a_{n,r}b_{r,k}S_k f(x) \quad (n=0,1,2,...).$$

We define two classes of sequences.

Following by L. Leindler [2], a sequence $c:=(c_r)$ of nonnegative numbers tending to zero is called the Mean Rest Bounded Variation Sequence, or briefly $c\in MRBVS$, if it has the property

$$\sum_{r=m}^{\infty}|c_r-c_{r+1}|\leqslant K(c)\frac{1}{m+1}\sum_{r\geqslant m/2}^{m}c_r,$$

for all positive integer m.

Analogously as in [4], a sequence $c:=(c_r)$ of nonnegative numbers will be called the Mean Head Bounded Variation Sequence, or briefly $c\in MHBVS$, if it has the property

$$\sum_{r=0}^{n-m-1}|c_r-c_{r+1}|\leqslant K(c)\frac{1}{m+1}\sum_{r=n-m}^{n}c_r,$$

for all positive integers $m<n$, where the sequence c has only finite nonzero terms and the last nonzero term is c_n. Consequently, we assume that the sequence $(K(\alpha_n))_{n=0}^{\infty}$ is bounded, that is, that there exists a constant K such that

$$0\leqslant K(\alpha_n)\leqslant K$$

holds for all n, where $K(\alpha_n)$ denote the constants appearing in the before inequalities for the sequences $\alpha_n=(a_{n,r})_{r=0}^{n}$, $n=0,1,2,\ldots$.

Now we can give the conditions to be used later on. We assume that for all n and $0\leqslant m<n$

$$\sum_{k=m}^{n-1}|a_{n,r}-a_{n,r+1}|\leqslant K\frac{1}{m+1}\sum_{r\geqslant m/2}^{m}a_{n,r}$$

and

$$\sum_{r=0}^{n-m-1}|a_{n,r}-a_{n,r+1}|\leqslant K\frac{1}{m+1}\sum_{r=n-m}^{n}a_{n,r}$$

hold if $(a_{n,r})_{r=0}^{n}$ belongs to $MRBVS$ and $MHBVS$, for $n=1,2,...,$ respectively.

As a measure of approximation of f by $T_{n,A,B}f$ we use the pointwise moduli of continuity of f in the space L^p defined by the formulas

$$w_x^p f(\delta)_\beta=\begin{cases}\left\{\frac{1}{\delta}\int_0^\delta\left|\varphi_x(u)\sin^\beta\frac{u}{2}\right|^p du\right\}^{\frac{1}{p}} & \text{when } 1\leqslant p<\infty,\\ ess\sup\limits_{0<u\leqslant\delta}|\varphi_x(u)\sin^\beta\frac{u}{2}| & \text{when } p=\infty,\end{cases}$$

$$\overline{w}_x^p f(\delta)_\beta = \begin{cases} \sup\limits_{0<t\leqslant\delta} \left\{\frac{1}{t}\int_0^t \left|\varphi_x(u)\sin^\beta \frac{u}{2}\right|^p du\right\}^{\frac{1}{p}} & \text{when } 1\leqslant p<\infty, \\ ess \sup\limits_{0<u\leqslant\delta} |\varphi_x(u)\sin^\beta \frac{u}{2}| & \text{when } p=\infty, \end{cases}$$

and the classical ones

$$\omega_\beta f(\delta)_{L^p} = \sup_{0<t\leqslant\delta} \|\sin^\beta \frac{t}{2}\varphi_\cdot(t)\|_{L^p},$$

where

$$\varphi_x(t) := f(x+t)+f(x-t)-2f(x).$$

The deviation $T_{n,A,B}f-f$ with the lower triangular infinite matrix A, defined by $a_{n,r}=\frac{1}{n+1}$ when $r=0,1,2,...,n$ and $a_{n,r}=0$ when $r>n$, and with the lower triangular infinite matrix B, defined by $b_{r,k}=p_{r-k}/\sum_{\nu=0}^{r}p_\nu$ when $k=0,1,2,...,r$ and $b_{r,k}=0$ when $k>r$, was estimated by S. Lal [1, Theorem 2]. The deviation $T_{n,A,B}f-f$ in general form was estimated at the point as well as in the norm of L^p in [3]. The pointwise estimates from this paper are following:

THEOREM. *Let $f\in L^p$ $(1<p\leqslant\infty)$, and let a modulus type function ω satisfy*

$$\left\{\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \left(\frac{|\varphi_x(t)|}{\omega(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{\frac{1}{p}} = O_x\left((n+1)^{-\frac{2}{p}}\right), \textit{when } 1<p<\infty,$$

$$ess \sup_{t\in[\frac{\pi}{n+1},\frac{\pi}{n}]} \left|\frac{|\varphi_x(t)|}{\omega(t)}\sin^\beta\frac{t}{2}\right| = O_x(1), \textit{ when } p=\infty$$

and

$$\left\{\int_0^{\frac{\pi}{n+1}} \left(\frac{|\varphi_x(t)|}{\omega(t)}\right)^p \sin^{\beta p}\frac{t}{2}dt\right\}^{\frac{1}{p}} = O_x\left((n+1)^{-\frac{1}{p}}\right), \textit{ when } 1<p<\infty,$$

$$ess \sup_{t\in[0,\frac{\pi}{n+1}]} \left|\frac{|\varphi_x(t)|}{\omega(t)}\sin^\beta\frac{t}{2}\right| = O_x(1), \textit{ when } p=\infty,$$

with $0\leqslant\beta<1-\frac{1}{p}$. If the entries of our matrices satisfy conditions

$$a_{n,n} \ll \frac{1}{n+1}$$

and

$$\left|a_{n,r}b_{r,r-l}-a_{n,r+1}b_{r+1,r+1-l}\right| \ll \frac{a_{n,r}}{(r+1)^2} \quad \textit{for} \quad 0\leqslant l\leqslant r\leqslant n-1,$$

then

$$\left|T_{n,A,B}f(x)-f(x)\right| = O_x\left(\sum_{r=0}^{n}a_{n,r}\frac{1}{r+1}\sum_{k=0}^{r}(k+1)^\beta\omega\left(\frac{\pi}{k+1}\right) + \frac{1}{n+1}\sum_{k=0}^{n}(k+1)^\beta\omega\left(\frac{\pi}{k+1}\right)\right)$$

and, in the case $0<\beta<1-\frac{1}{p}$,

$$\begin{aligned}&\left|T_{n,A,B}f(x)-f(x)\right|\\&=O_x\left((n+1)^{\beta}\,\omega\left(\frac{\pi}{n+1}\right)\left[(n+1)^{1-\beta}\sum_{k=0}^{n}a_{n,k}\,(k+1)^{\beta-1}\right]\right),\end{aligned}$$

for considered x.

In this paper we shall consider the deviation $T_{n,A,B}f-f$ without any special assumptions on modulus of continuity and another assumptions on summability method obtaining the same degrees of approximation as above. Finally, we also give some results on norm approximation and corollary generalizing the results of H. K. Nigam, A. Sharma [6] and H. K. Nigam, K. Sharma [7].

We shall write $I_1 \ll I_2$ if there exists a positive constant K, sometimes depending on some parameters, such that $I_1 \leqslant KI_2$.

2. Statement of the results

Let formulate our main results.

THEOREM 1. *Let* $f\in L^p$ *with* $1<p\leqslant\infty$ *and let* $0<\beta<1-\frac{1}{p}$. *If* $\left(a_{n,k}\right)_{k=0}^{n}\in$ *MRBVS and*

$$\left|\sum_{r=\mu}^{\nu}\sum_{k=0}^{r}b_{r,k}\sin\frac{(2k+1)t}{2}\right|\ll\tau \tag{1}$$

for $0\leqslant\mu\leqslant\nu$ *with* $\tau=\left[\frac{\pi}{t}\right]$, *where* $0<t\leqslant\pi$, *then*

$$\left|T_{n,A,B}f(x)-f(x)\right|=O_x\left((n+1)^{\beta}\sum_{k=0}^{n}a_{n,k}\left[w_x^pf\left(\frac{\pi}{n+1}\right)_{\beta}+\overline{w}_x^1f\left(\frac{\pi}{k+1}\right)_{\beta}\right]\right),$$

for almost all considered x.

THEOREM 2. *Let* $f\in L^p$ *with* $1<p\leqslant\infty$ *and let* $0<\beta<1-\frac{1}{p}$. *If* $\left(a_{n,k}\right)_{k=0}^{n}\in$ *MHBVS and the entries of matrix* B *satisfy the condition* (1) *for* $0\leqslant\mu\leqslant\nu$, *then*

$$\left|T_{n,A,B}f(x)-f(x)\right|=O_x\left((n+1)^{\beta}\sum_{k=0}^{n}a_{n,n-k}\left[w_x^pf\left(\frac{\pi}{n+1}\right)_{\beta}+\overline{w}_x^1f\left(\frac{\pi}{k+1}\right)_{\beta}\right]\right),$$

for almost all considered x .

Next, we formulate the results on estimates of L^p norm of the considered deviation.

THEOREM 3. *Let* $f \in L^p$ *with* $1 < p \leqslant \infty$. *If* $\left(a_{n,k}\right)_{k=0}^{n} \in MRBVS$ *and the entries of matrix* B *satisfy the condition* (1) *for* $0 \leqslant \mu \leqslant \nu$, *then*

$$\left\|T_{n,A,B}f(\cdot) - f(\cdot)\right\|_{L^p} = O\left((n+1)^{\beta} \sum_{k=0}^{n} a_{n,k} \omega_{\beta} f\left(\frac{\pi}{k+1}\right)_{L^p}\right)$$

where $0 < \beta < 1 - \frac{1}{p}$.

THEOREM 4. *Let* $f \in L^p$ *with* $1 < p \leqslant \infty$. *If* $\left(a_{n,k}\right)_{k=0}^{n} \in MHBVS$ *and the entries of matrix* B *satisfy the condition* (1) *for* $0 \leqslant \mu \leqslant \nu$, *then*

$$\left\|T_{n,A,B}f(\cdot) - f(\cdot)\right\|_{L^p} = O\left((n+1)^{\beta} \sum_{k=0}^{n} a_{n,n-k} \omega_{\beta} f\left(\frac{\pi}{k+1}\right)_{L^p}\right)$$

where $0 < \beta < 1 - \frac{1}{p}$.

Finally, we give an application of our results as a corollary and remarks.

Taking $a_{n,r} = p_{n-r} / \sum_{\nu=0}^{n} p_{\nu}$ (or $a_{n,r} = p_r / \sum_{\nu=0}^{n} p_{\nu}$), when $r = 0,1,2,\ldots,n$ and $a_{n,r} = 0$, when $r > n$ with $p_{\nu} > 0$, $p_{\nu} \leqslant p_{\nu+1}$, and $b_{r,k} = \frac{\binom{r}{k}\gamma^k}{(1+\gamma)^r}$, when $k = 0,1,2,\ldots,r$ and $b_{n,r} = 0$ when $k > r$ with $\gamma > 0$, Theorem 1 and Theorem 2 imply:

COROLLARY 1. *If* $f \in L^p$ *with* $1 < p \leqslant \infty$ *and* $0 < \beta < 1 - \frac{1}{p}$, *then*

$$\left|\frac{1}{\sum_{\nu=0}^{n} p_{\nu}} \sum_{r=0}^{n} \frac{p_r}{(1+\gamma)^r} \sum_{k=0}^{r} \binom{r}{k} \gamma^k S_k f(x) - f(x)\right|$$
$$= O_x\left(\frac{(n+1)^{\beta}}{\sum_{\nu=0}^{n} p_{\nu}} \sum_{k=0}^{n} p_k \left[w_x^p f\left(\frac{\pi}{n+1}\right)_{\beta} + \overline{w}_x^1 f\left(\frac{\pi}{k+1}\right)_{\beta}\right]\right),$$

and

$$\left|\frac{1}{\sum_{\nu=0}^{n} p_{\nu}} \sum_{r=0}^{n} \frac{p_{n-r}}{(1+\gamma)^r} \sum_{k=0}^{r} \binom{r}{k} \gamma^k S_k f(x) - f(x)\right|$$
$$= O_x\left(\frac{(n+1)^{\beta}}{\sum_{\nu=0}^{n} p_{\nu}} \sum_{k=0}^{n} p_{n-k} \left[w_x^p f\left(\frac{\pi}{n+1}\right)_{\beta} + \overline{w}_x^1 f\left(\frac{\pi}{k+1}\right)_{\beta}\right]\right),$$

with $p_{\nu} > 0$, $p_{\nu} \leqslant p_{\nu+1}$ *and* $\gamma > 0$, *for almost all considered* x.

REMARK 1. In special case $p_r = p_{n-r} = 1$ we have estimates

$$\left| \frac{1}{n+1} \sum_{r=0}^{n} \frac{1}{(1+\gamma)^r} \sum_{k=0}^{r} \binom{r}{k} \gamma^k S_k f(x) - f(x) \right|$$
$$= O_x \left((n+1)^{\beta-1} \sum_{k=0}^{n} \left[w_x^p f\left(\frac{\pi}{n+1}\right)_\beta + \overline{w}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \right] \right).$$

REMARK 2. If we take $\beta = 0$ in the above considerations then we have to estimate the quantities $|I_2|$ and $\|I_2\|_{L^p}$ using the Hölder inequality analogously as in estimate of $|I_1|$. Thus we obtain in the all above estimates $(n+1)^{1/p}$ instead of $(n+1)^\beta$.

3. Auxiliary results

We begin this section by some notations following A. Zygmund [8, Section 5 of Chapter II].

It is clear that

$$S_k f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_k(t)\, dt$$

and

$$T_{n,A,B} f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} D_k(t)\, dt,$$

where

$$D_k(t) = \frac{1}{2} + \sum_{\nu=1}^{k} \cos \nu t = \frac{\sin \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}}.$$

Hence

$$T_{n,A,B} f(x) - f(x) = \frac{1}{\pi} \int_{0}^{\pi} \varphi_x(t) \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} D_k(t)\, dt.$$

Now, we formulate some estimates for the Dirichlet kernel.

LEMMA 1. (see [8]) *If* $0 < |t| \leqslant \pi/2$, *then*

$$|D_k(t)| \leqslant \frac{\pi}{2|t|}$$

and for any real t *we have*

$$|D_k(t)| \leqslant k+1.$$

LEMMA 2. *Let* $(b_{r,k})_{k=0}^{r}$ *be such that the condition (1) holds for* $0 \leqslant \mu \leqslant \nu$. *If* $(a_{n,k})_{k=0}^{n} \in MRBVS$, *then*

$$\left| \sum_{r=0}^{n} \sum_{k=0}^{r} a_{n,r} b_{r,k} D_k(t) \right| \ll \tau A_{n,\tau}$$

and if $\left(a_{n,k}\right)_{k=0}^{n} \in MHBVS$, *then*

$$\left|\sum_{r=0}^{n}\sum_{k=0}^{r} a_{n,r}b_{r,k}D_k\left(t\right)\right| \ll \tau\overline{A}_{n,n-\tau}$$

with $\tau = [\pi/t]$ *and* $t \in \left[\frac{\pi}{n+1}, \pi\right]$, *for* $n = 0,1,2,\ldots$.

Proof. The proof is analogous to the proof of [5, Lemma 11]. □

4. Proofs of the results

4.1. Proof of Theorem 1

We start with the obvious relations

$$\begin{aligned} T_{n,A,B}f(x) - f(x) &= \frac{1}{\pi}\int_0^{\frac{\pi}{n+1}} \varphi_x\left(t\right)\sum_{r=0}^{n}\sum_{k=0}^{r} a_{n,r}b_{r,k}D_k\left(t\right)dt \\ &\quad + \frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi} \varphi_x\left(t\right)\sum_{r=0}^{n}\sum_{k=0}^{r} a_{n,r}b_{r,k}D_k\left(t\right) \\ &= I_1 + I_2 \end{aligned}$$

and

$$\left|T_{n,A}f(x) - f(x)\right| \leqslant |I_1| + |I_2|.$$

By the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ and Lemma 1, for $\beta < 1-\frac{1}{p}$, we have

$$\begin{aligned} |I_1| &\ll (n+1)\int_0^{\frac{\pi}{n+1}} |\varphi_x(t)|\,dt \leqslant (n+1)\int_0^{\frac{\pi}{n+1}} |\varphi_x(t)|\sin^{\beta}\frac{t}{2}\sin^{-\beta}\frac{t}{2}dt \\ &\leqslant \left\{(n+1)\int_0^{\frac{\pi}{n+1}}\left[|\varphi_x(t)|\sin^{\beta}\frac{t}{2}\right]^p dt\right\}^{1/p}\left\{(n+1)\int_0^{\frac{\pi}{n+1}}\sin^{-\beta q}\frac{t}{2}dt\right\}^{1/q} \\ &\ll w_x^p f\left(\frac{\pi}{n+1}\right)_{\beta}\left\{(n+1)\int_0^{\frac{\pi}{n+1}} t^{-\beta q}dt\right\}^{1/q} \ll (n+1)^{\beta}\, w_x^p f\left(\frac{\pi}{n+1}\right)_{\beta}. \end{aligned}$$

Using Lemma 2 we obtain

$$\begin{aligned} |I_2| &\ll \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t}\sum_{k=0}^{\tau} a_{n,k}dt = \sum_{m=1}^{n}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{|\varphi_x(t)|}{t}\sum_{k=0}^{\tau} a_{n,k}dt \\ &\leqslant \sum_{m=1}^{n}\sum_{k=0}^{m+1} a_{n,k}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{|\varphi_x(t)|}{t}dt \\ &= \sum_{m=1}^{n}\sum_{k=1}^{m} a_{n,k}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{|\varphi_x(t)|}{t}dt + \sum_{m=1}^{n}\left(a_{n,0}+a_{n,m+1}\right)\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{|\varphi_x(t)|}{t}dt \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{k=1}^{n} a_{n,k}\sum_{m=k}^{n} + a_{n,0}\sum_{m=1}^{n} + \sum_{m=1}^{n} a_{n,m+1}\right)\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}} \frac{|\varphi_x(t)|}{t}dt \\
&\ll \left(\sum_{k=1}^{n} a_{n,k}\sum_{m=k}^{n} + a_{n,0}\sum_{m=1}^{n}\right)\left\{\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left[\frac{|\varphi_x(t)|\sin^{\beta}\frac{t}{2}}{t^{1+\beta}}\right]dt\right\} \\
&\quad + \sum_{m=1}^{n} a_{n,m+1}\left\{\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left[|\varphi_x(t)|\sin^{\beta}\frac{t}{2}\right]dt\right\}\left[\frac{1}{\frac{\pi}{m+1}\sin^{\beta}\frac{\pi/2}{m+1}}\right] \\
&\ll \left(\sum_{k=1}^{n} a_{n,k}\sum_{m=k}^{n} + a_{n,0}\sum_{m=1}^{n}\right)\left\{\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left[\frac{|\varphi_x(t)|\sin^{\beta}\frac{t}{2}}{t^{1+\beta}}\right]dt\right\} \\
&\quad + \sum_{m=1}^{n} a_{n,m+1}(m+1)^{\beta}\left\{\frac{m+1}{\pi}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\left[|\varphi_x(t)|\sin^{\beta}\frac{t}{2}\right]dt\right\} \\
&\ll \sum_{k=0}^{n} a_{n,k}\left\{\int_{\frac{\pi}{n+1}}^{\frac{\pi}{k+1}} t^{-1-\beta}\frac{d}{dt}\left[\int_0^t |\varphi_x(u)|\sin^{\beta}\frac{u}{2}du\right]dt\right\} \\
&\quad + (n+1)^{\beta}\sum_{m=1}^{n} a_{n,m+1}w_x^1 f\left(\frac{\pi}{m+1}\right)_{\beta} \\
&\ll \sum_{k=0}^{n} a_{n,k}\left\{\left[\frac{1}{t^{1+\beta}}\int_0^t |\varphi_x(u)|\sin^{\beta}\frac{u}{2}du\right]_{\frac{\pi}{n+1}}^{\frac{\pi}{k+1}}\right. \\
&\quad \left. + (1+\beta)\int_{\frac{\pi}{n+1}}^{\frac{\pi}{k+1}}\left[\frac{1}{t^{2+\beta}}\int_0^t |\varphi_x(u)|\sin^{\beta}\frac{u}{2}du\right]dt\right\} \\
&\quad + (n+1)^{\beta}\sum_{m=1}^{n} a_{n,m+1}w_x^1 f\left(\frac{\pi}{m+1}\right)_{\beta} \\
&\ll \sum_{k=0}^{n} a_{n,k}\left\{\left[(k+1)^{1+\beta}\int_0^{\frac{\pi}{k+1}} |\varphi_x(u)|\sin^{\beta}\frac{u}{2}du\right]\right. \\
&\quad \left. + (1+\beta)\int_{\frac{\pi}{n+1}}^{\frac{\pi}{k+1}}\left[\frac{1}{t^{1+\beta}}w_x^1 f(t)_{\beta}\right]dt\right\} \\
&\quad + (n+1)^{\beta}\sum_{m=1}^{n} a_{n,m+1}w_x^1 f\left(\frac{\pi}{m+1}\right)_{\beta} \\
&\ll \sum_{k=0}^{n} a_{n,k}\left\{\left[(k+1)^{\beta}w_x^1 f\left(\frac{\pi}{k+1}\right)_{\beta}\right]\right. \\
&\quad \left. + (1+\beta)\int_{\frac{\pi}{n+1}}^{\frac{\pi}{k+1}}\left[\frac{1}{t^{1+\beta}}w_x^1 f(t)_{\beta}\right]dt\right\} \\
&\quad + (n+1)^{\beta}\sum_{m=1}^{n} a_{n,m+1}w_x^1 f\left(\frac{\pi}{m+1}\right)_{\beta}
\end{aligned}
$$

$$\ll \sum_{k=0}^{n} a_{n,k}\left\{\int_{k+1}^{n+1}\left[\frac{1}{t^{1-\beta}}w_x^1 f\left(\frac{\pi}{t}\right)_\beta\right]dt\right\}$$
$$+(n+1)^\beta \sum_{m=1}^{n} a_{n,m+1} w_x^1 f\left(\frac{\pi}{m+1}\right)_\beta$$

Since $\overline{w}_x^p f(\delta)_\beta$ is nondecreasing majorant of $w_x^p f(\delta)_\beta$ we have with $\beta > 0$

$$\ll \sum_{k=0}^{n} a_{n,k}\overline{w}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta \int_{k+1}^{n+1}\frac{1}{t^{1-\beta}}dt$$
$$+(n+1)^\beta \sum_{m=1}^{n} a_{n,m+1}\overline{w}_x^1 f\left(\frac{\pi}{m+1}\right)_\beta$$
$$\ll (n+1)^\beta \sum_{k=0}^{n} a_{n,k}\overline{w}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta .$$

Collecting these estimates we obtain the desired result. □

4.2. Proof of Theorem 2

Let as usual

$$T_{n,A,B}f(x)-f(x)=I_1+I_2$$

and

$$\left|T_{n,A,B}f(x)-f(x)\right| \leqslant |I_1|+|I_2| .$$

The term $|I_1|$ we can estimate by the same way as in the proof of Theorem 1. Therefore

$$|I_1| \ll (n+1)^\beta\, w_x^p f\left(\frac{\pi}{n+1}\right)_\beta .$$

Analogously to the above, by the Hölder inequality $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ and Lemma 2

$$|I_2| \ll \int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|}{t}\sum_{k=n-\tau}^{n} a_{n,k}dt = \sum_{m=1}^{n}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{|\varphi_x(t)|}{t}\sum_{k=0}^{\tau} a_{n,n-k}dt$$
$$\leqslant \sum_{m=1}^{n}\sum_{k=0}^{m+1} a_{n,n-k}\int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\frac{|\varphi_x(t)|}{t}dt$$
$$\ll (n+1)^\beta \sum_{k=0}^{n} a_{n,n-k}\overline{w}_x^1 f\left(\frac{\pi}{k+1}\right)_\beta .$$

Collecting these estimates we obtain the desired result. □

4.3. Proofs of Theorems 3-4

The proofs are similar to these above and follows from the evident inequality

$$\left\|w_\cdot^p f(\delta)_\beta\right\|_{L^p} \ll \omega_\beta f(\delta)_{L^p} ,$$

because modulus of continuity $\omega_\beta f(\delta)_{L^p}$ is nondecreasing function of δ. □

4.4. Proof of corollary 1

The proof is analogous to the proof of [5, Corollary 6]. □

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