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LYAPUNOV–TYPE INEQUALITIES FOR THIRD–ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we obtain new Lyapunov-type inequalities for the third-order linear differential equation $x''' + q(t)x = 0$. Our work provides the sharpest results in the literature and makes corrections to those in a recently published paper [1]. Based on the above, we further establish new Lyapunov-type inequalities for more general third-order linear differential equations. Moreover, by combining these inequalities with the "uniqueness implies existence" theorems by several authors, we establish the uniqueness and hence existence-uniqueness for several classes of boundary value problems for third-order linear equations.

1. Introduction

For the second-order linear differential equation

$$x'' + q(t)x = 0, \tag{1.1}$$

where $q \in C([a,b],\mathbb{R})$, the following result is known as the Lyapunov inequality, see [16, 3].

THEOREM 1.1. *Assume Eq. (1.1) has a solution $x(t)$ satisfying $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$. Then*

$$\int_a^b |q(t)|dt > \frac{4}{b-a}. \tag{1.2}$$

It was first noticed by Wintner [25] and later by several other authors that inequality (1.2) can be improved by replacing $|q(t)|$ by $q_+(t) := \max\{q(t),0\}$, the nonnegative part of $q(t)$, to become

$$\int_a^b q_+(t)dt > \frac{4}{b-a}, \tag{1.3}$$

see Patula [22] for the simplest proof.

The Lyapunov-type inequalities were extended by Hartman [9, Chapter XI] to the more general linear equation

$$(r(t)x')' + q(t)x = 0, \tag{1.4}$$

where $q, r \in C([a,b],\mathbb{R})$ and $r(t) > 0$ for $t \in [a,b]$.

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THEOREM 1.2. *Assume Eq. (1.4) has a solution $x(t)$ satisfying $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$. Then*

$$\int_a^b q_+(t)dt > \frac{4}{\int_a^b r^{-1}(t)dt}. \tag{1.5}$$

The above Lyapunov-type inequalities have been further improved by replacing $\int_a^b q_+(t)dt$ by some integrals of $q(t)$ on parts of or the whole interval $[a,b]$, see Harris and Kong [8] and Brown and Hinton [2] for the details. We note that the number 4 in the above inequalities is the best in the sense that if it is replaced by any larger constant, then the inequalities fail to hold, see [9, p. 345] and [15] for examples.

Lyapunov-type inequalities have been further developed for higher order linear and half-linear differential equations. The reader is referred to Cakmak [4, 5], He and Tang [10], Pachpatte [17, 18], Parhi and Panigrahi [20, 21], Panigrahi [19], Tiryaki, Unal and Cakmak [24], Yang [27], Yang and Lo [26], and Zhang and He [29] for the higher order case; and to Elbert [7], Kisel'ak [14], and Yang [28] for the half-linear case. Also, Pinasco [23] provided an excellent survey on various Lyapunov-type inequalities.

Among the above, Parhi and Panigrahi [20] established Lyapunov-type inequalities for the third-order linear differential equation

$$x''' + q(t)x = 0, \tag{1.6}$$

where $-\infty < a < b < c < \infty$ and $q \in C([a,c],\mathbb{R})$.

THEOREM 1.3. *Assume $x(t)$ is a solution of Eq. (1.6) satisfying $x(a) = x(b) = x(c) = 0$ and $x(t) \neq 0$ for $t \in (a,b) \cup (b,c)$. Then*

$$\int_a^c |q(t)|dt > \frac{4}{(c-a)^2}. \tag{1.7}$$

Recently, Dhar and Kong [6] obtained Lyapunov-type inequalities for third-order half-linear differential equations. Restricting the work in [6] to the linear case (1.6), inequality (1.7) becomes

$$\max_{\xi\in[a,c]} \left\{ \int_a^\xi q_-(s)ds + \int_\xi^c q_+(s)ds \right\} > \frac{4}{(c-a)^2}. \tag{1.8}$$

Clearly, (1.8) improves (1.7) due to the replacement of $|q|$ in the integral of the left-hand side by q_+ and q_-.

In a different approach, by using the Green's function $G(t,s)$ for a corresponding Dirichlét problem defined on $[t_1,t_2] \subset (a,c)$, Aktas, Cakmak, and Tiryaki [1] tried to improve the constant 4 on the right-hand side of (1.7). More specifically, they claim that the following inequality holds:

$$\int_a^c |q(t)|dt \geqslant \frac{16}{(c-a)^2}. \tag{1.9}$$

Unfortunately, there is an error in their proof. In fact, they mistakenly assumed $G(t,s) \geqslant 0$ for all $(t,s) \in [a,c] \times [t_1,t_2]$ which is actually true only when $(t,s) \in [t_1,t_2] \times [t_1,t_2]$. In fact, (2.7) in [1] does not hold unless G is replaced by $|G|$. However, with G replaced by $|G|$, (2.11) in [1] no longer holds. As a result, Lemma 2.1 in [1] fails to hold. Therefore, (1.9) in the above is unjustified and is probably too good to be true.

On the other hand, their idea of applying Green's functions for second-order boundary value problems (BVPs) to deal with third-order linear equations are novel and feasible. Motivated by this idea, in Section 2, we will derive new Lyapunov-type inequalities for the third-order linear equation (1.6). In section 3, we will extend the work for Eq. (1.6) to more general third-order linear equations. Finally, in the last section, we will apply the obtained Lyapunov-type inequalities to derive conditions for the nonexistence of nontrivial solutions of third-order homogeneous linear BVPs and the uniqueness of solutions of third-order nonhomogeneous linear BVPs. Furthermore, by employing the "uniqueness implies existence" theorems given by Jackson [11, 12] and by Jackson and Schrader [13], we will establish existence-uniqueness results for several classes of third-order linear BVPs.

2. Main results

In this section, we let $-\infty < a < b < c < \infty$ and consider the third-order linear differential equation

$$x''' + q(t)x = 0, \tag{2.1}$$

where $q \in C([a,c],\mathbb{R})$.

THEOREM 2.1. *Assume Eq. (2.1) has a solution $x(t)$ satisfying*

$$x(a) = x(b) = x(c) = 0 \quad \textit{and} \quad x(t) \neq 0 \textit{ for } t \in (a,b) \cup (b,c). \tag{2.2}$$

Then one of the following holds:

(a) $\displaystyle\int_a^c (t-a)(c-t)q_-(t)dt > 2,$

(b) $\displaystyle\int_a^c (t-a)(c-t)q_+(t)dt > 2,$

(c) $\displaystyle\int_a^b (t-a)(c-t)q_-(t)dt + \int_b^c (t-a)(c-t)q_+(t)dt > 2.$

Proof. By the continuity of $x(t)$ on $[a,c]$, there exist $t_1 \in (a,b)$ and $t_2 \in (b,c)$ such that $|x(t_1)| = \max\{|x(t)| : t \in [a,b]\}$ and $|x(t_2)| = \max\{|x(t)| : t \in [b,c]\}$. As a result,

$$x'(t_1) = x'(t_2) = 0. \tag{2.3}$$

Let

$$G(t,s) = \frac{1}{t_2 - t_1} \begin{cases} (s-t_1)(t_2-t), & t_1 \leqslant s \leqslant t \leqslant t_2; \\ (t-t_1)(t_2-s), & t_1 \leqslant t \leqslant s \leqslant t_2. \end{cases}$$

Then $G(t,s)$ is the Green's function for the BVP

$$-y'' = r(t), \quad y(t_1) = y(t_2) = 0. \tag{2.4}$$

Hence the solution $y(t)$ satisfies

$$y(t) = \int_{t_1}^{t_2} G(t,s)r(s)ds. \tag{2.5}$$

We observe that for the solution $x(t)$ of Eq. (2.1), $y(t) := x'(t)$ satisfies (2.4) with $r(t) = q(t)x(t)$. By (2.5)

$$x'(t) = \int_{t_1}^{t_2} G(t,s)q(s)x(s)ds.$$

It follows that

$$x(t) = \int_b^t \int_{t_1}^{t_2} G(\tau,s)q(s)x(s)dsd\tau = \int_{t_1}^{t_2} \Big(\int_b^t G(\tau,s)d\tau\Big) q(s)x(s)ds. \tag{2.6}$$

Note that for $s \in [t_1,t_2]$

$$\begin{aligned}
\int_{t_1}^{t_2} G(t,s)dt &= \int_{t_1}^{s} G(t,s)dt + \int_{s}^{t_2} G(t,s)dt \\
&= \frac{1}{t_2-t_1}\Big[(t_2-s)\int_{t_1}^{s}(t-t_1)dt + (s-t_1)\int_{s}^{t_2}(t_2-t)dt\Big] \\
&= \frac{1}{t_2-t_1}\Big[\frac{(t_2-s)(s-t_1)^2}{2} + \frac{(t_2-s)^2(s-t_1)}{2}\Big] \\
&= \frac{1}{2}(s-t_1)(t_2-s),
\end{aligned}$$

and hence

$$\int_{t_1}^{t_2} G(t,s)dt \leqslant \frac{1}{2}(s-a)(c-s). \tag{2.7}$$

Without loss of generality, we may assume $x(t)$ satisfies one of following cases:

I. $x(t) > 0$ on $(a,b)\cup(b,c)$ and $x(t_1) \geqslant x(t_2)$;

II. $x(t) > 0$ on $(a,b)\cup(b,c)$ and $x(t_1) < x(t_2)$;

III. $x(t) > 0$ on (a,b), $x(t) < 0$ on (b,c), and $x(t_1) \geqslant -x(t_2)$;

IV. $x(t) > 0$ on (a,b), $x(t) < 0$ on (b,c), and $x(t_1) < -x(t_2)$.

In the sequel, we denote $m = \max\{|x(t_1)|,|x(t_2)|\}$.

Case I. In this case, $m = x(t_1)$. Then (2.6) with $t = t_1$ shows that

$$m = \int_{t_1}^{t_2} \Big(\int_{t_1}^{b} G(t,s)dt\Big)(-q(s))x(s)ds.$$

Using the facts that $G(t,s) \geqslant 0$ on $[t_1,t_2]\times[t_1,t_2]$, $0 \leqslant x(t) \leqslant m$ and $x(t) \not\equiv m$ on $[t_1,t_2]$, and $-q(t) \leqslant q_-(t)$, we have

$$m < m\int_{t_1}^{t_2}\Big(\int_{t_1}^{t_2} G(t,s)dt\Big)q_-(s)ds.$$

Cancelling m from both sides and using (2.7) we obtain

$$1 < \frac{1}{2}\int_{t_1}^{t_2}(s-a)(c-s)q_-(s)ds \leqslant \frac{1}{2}\int_a^c (s-a)(c-s)q_-(s)ds,$$

i.e., conclusion (a) holds.

Case II. In this case, $m = x(t_2)$. Then (2.6) with $t = t_2$ shows that

$$m = \int_{t_1}^{t_2}\Big(\int_{b}^{t_2} G(t,s)dt\Big)q(s)x(s)ds.$$

Using the facts that $G(t,s) \geqslant 0$ on $[t_1,t_2]\times[t_1,t_2]$, $0 \leqslant x(t) \leqslant m$ and $x(t) \not\equiv m$ on $[t_1,t_2]$, and $q(t) \leqslant q_+(t)$, we have

$$m < m\int_{t_1}^{t_2}\Big(\int_{t_1}^{t_2} G(t,s)dt\Big)q_+(s)ds.$$

Cancelling m from both sides and using (2.7) we obtain

$$1 < \frac{1}{2}\int_{t_1}^{t_2}(s-a)(c-s)q_+(s)ds \leqslant \frac{1}{2}\int_a^c (s-a)(c-s)q_+(s)ds,$$

i.e., conclusion (b) holds.

Case III. In this case, $m = x(t_1)$. Then (2.6) with $t = t_1$ shows that

$$\begin{aligned} m &= \int_{t_1}^{t_2}\Big(\int_{t_1}^{b} G(t,s)dt\Big)(-q(s))x(s)ds \\ &= \int_{t_1}^{b}\Big[\int_{t_1}^{b} G(t,s)(-q(s))x(s)ds + \int_{b}^{t_2} G(t,s)q(s)(-x(s))ds\Big]dt. \end{aligned}$$

Note that $x(t) > 0$ on (a,b) and $x(t) < 0$ on (b,c). Then by a similar argument to Cases I and II, we see that

$$1 < \frac{1}{2}\Big[\int_a^b (s-a)(c-s)q_-(s)ds + \int_b^c (s-a)(c-s)q_+(s)ds\Big],$$

i.e., conclusion (c) holds.

Case IV. The same argument as in Case III also leads to conclusion (c). We omit the detail. □

From the fact that $4\alpha\beta \leqslant (\alpha+\beta)^2$ for any $\alpha,\beta \in \mathbb{R}$, we see that $4(s-a)(c-s) \leqslant (c-a)^2$. Then the following corollary is a direct consequence of Theorem 2.1.

COROLLARY 2.1. *Assume Eq. (2.1) has a solution $x(t)$ satisfying (2.2). Then one of the following holds:*

(a) $\int_a^c q_-(t)dt > \frac{8}{(c-a)^2}$,

(b) $\int_a^c q_+(t)dt > \frac{8}{(c-a)^2}$,

(c) $\int_a^b q_-(t)dt + \int_b^c q_+(t)dt > \frac{8}{(c-a)^2}$.

As a result,

$$\int_a^c |q(t)|dt > \frac{8}{(c-a)^2}.$$

REMARK 2.1. It is clear that the inequalities obtained in this section are sharper than those in (1.7) and (1.8) and hence is the best in the literature. In particular, by Corollary 2.1, the constant 4 on both the right-hand sides is improved to a larger constant 8, the integral of $|q(t)|$ on the left-hand side of (1.7) is replaced by integrals of $q_+(t)$ and $q_-(t)$, and the maximum for $\xi \in [a,c]$ on the left-hand side of (1.8) is replaced by the maximum over three values $\xi = a,b,c$.

3. Generalizations

In this section, we extend the Lyapunov-type inequalities obtained in Section 2 to several more general third-order linear equations. All the results obtained in this section are new in the literature. Here we let $-\infty < a < b < c < \infty$.

We first consider the differential equation

$$(p(t)x'')' + q(t)x = 0, \tag{3.1}$$

where $p,q \in C([a,c],\mathbb{R})$ and $p(t) > 0$ for $t \in [a,c]$. We denote $P = \max\{p(t) : t \in [a,c]\}$.

THEOREM 3.1. *Assume Eq. (3.1) has a solution $x(t)$ satisfying (2.2). Then one of the following holds:*

(a) $\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_-(t)dt > \frac{2}{P}$,

(b) $\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_+(t)dt > \frac{2}{P}$,

(c) $\int_a^b \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_-(t)dt + \int_b^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_+(t)dt > \frac{2}{P}$.

Proof. We first consider the case where $0 < p(t) \leqslant 1$ for $t \in [a,c]$. Let t_1 and t_2 be defined as in the proof of Theorem 2.1 and let

$$G(t,s) = \frac{1}{\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \begin{cases} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_{t}^{t_2} \frac{d\tau}{p(\tau)}, & t_1 \leqslant s \leqslant t \leqslant t_2; \\ \int_{t_1}^{t} \frac{d\tau}{p(\tau)} \int_{s}^{t_2} \frac{d\tau}{p(\tau)}, & t_1 \leqslant t \leqslant s \leqslant t_2. \end{cases} \tag{3.2}$$

Then $G(t,s)$ is the Green's function for the BVP

$$-(p(t)y')' = r(t), \quad y(t_1) = y(t_2) = 0. \tag{3.3}$$

As shown in the proof of Theorem 2.1, (2.6) holds with $G(t,s)$ defined by (3.2), i.e.,

$$x(t) = \int_{t_1}^{t_2} \Big(\int_{b}^{t} G(\tau,s)d\tau \Big) q(s)x(s)ds. \tag{3.4}$$

Note that for $s \in [t_1,t_2]$

$$\begin{aligned} \int_{t_1}^{t_2} G(t,s)dt &= \int_{t_1}^{s} G(t,s)dt + \int_{s}^{t_2} G(t,s)dt \\ &= \frac{1}{\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \Big[\int_{s}^{t_2} \frac{d\tau}{p(\tau)} \int_{t_1}^{s} \int_{t_1}^{t} \frac{d\tau}{p(\tau)} dt + \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_{s}^{t_2} \int_{t}^{t_2} \frac{d\tau}{p(\tau)} dt \Big]. \end{aligned} \tag{3.5}$$

We claim that for $s \in [t_1,t_2]$

$$H(s) := \int_{t_1}^{s} \int_{t_1}^{t} \frac{d\tau}{p(\tau)} dt - \frac{1}{2} \Big(\int_{t_1}^{s} \frac{d\tau}{p(\tau)} \Big)^2 \leqslant 0$$

and hence

$$\int_{t_1}^{s} \int_{t_1}^{t} \frac{d\tau}{p(\tau)} dt \leqslant \frac{1}{2} \Big(\int_{t_1}^{s} \frac{d\tau}{p(\tau)} \Big)^2. \tag{3.6}$$

This is because $H(t_1) = 0$ and

$$H'(s) = \int_{t_1}^{s} \frac{d\tau}{p(\tau)} - \frac{1}{p(s)} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} = \Big(1 - \frac{1}{p(s)}\Big) \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \leqslant 0$$

due to the assumption that $0 < p(t) \leqslant 1$. Similarly, for $s \in [t_1,t_2]$

$$\int_{s}^{t_2} \int_{t}^{t_2} \frac{d\tau}{p(\tau)} dt \leqslant \frac{1}{2} \Big(\int_{s}^{t_2} \frac{d\tau}{p(\tau)} \Big)^2. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5) we see that for $s \in [t_1,t_2]$

$$\begin{aligned} \int_{t_1}^{t_2} G(t,s)dt &\leqslant \frac{1}{2\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \Big[\Big(\int_{t_1}^{s} \frac{d\tau}{p(\tau)} \Big)^2 \int_{s}^{t_2} \frac{d\tau}{p(\tau)} + \Big(\int_{s}^{t_2} \frac{d\tau}{p(\tau)} \Big)^2 \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \Big] \\ &= \frac{1}{2\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_{s}^{t_2} \frac{d\tau}{p(\tau)} \Big[\int_{t_1}^{s} \frac{d\tau}{p(\tau)} + \int_{s}^{t_2} \frac{d\tau}{p(\tau)} \Big] \\ &= \frac{1}{2} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_{s}^{t_2} \frac{d\tau}{p(\tau)} \leqslant \frac{1}{2} \int_{a}^{s} \frac{d\tau}{p(\tau)} \int_{s}^{c} \frac{d\tau}{p(\tau)}. \end{aligned}$$

Then a similar argument to that in the proof of Theorem 2.1 shows that one of the following holds:

(i) $\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_-(t)dt > 2,$

(ii) $\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_+(t)dt > 2,$

(iii) $\int_a^b \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_-(t)dt + \int_b^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) q_+(t)dt > 2.$

Now, we consider the general case with $p(t) > 0$. Dividing both sides of Eq. (3.1) by P we obtain

$$\Big(\frac{p(t)}{P}x''\Big)' + \frac{q(t)}{P}x = 0. \tag{3.8}$$

Note that Eq. (3.8) is in the form of Eq. (3.1) with $p(t)$ and $q(t)$ replaced by $p(t)/P$ and $q(t)/P$, respectively; and $0 < p(t)/P \leqslant 1$. Then the conclusions (a)-(c) follow from (i)-(iii) above with $p(t)$ and $q(t)$ replaced by $p(t)/P$ and $q(t)/P$, respectively. $\square$

Again, from the fact that $4\alpha\beta \leqslant (\alpha+\beta)^2$ for any $\alpha, \beta \in \mathbb{R}$ we see that

$$4\int_a^t \frac{d\tau}{p(\tau)} \int_t^c \frac{d\tau}{p(\tau)} \leqslant \Big(\int_a^t \frac{d\tau}{p(\tau)} + \int_t^c \frac{d\tau}{p(\tau)}\Big)^2 = \Big(\int_a^c \frac{d\tau}{p(\tau)}\Big)^2.$$

Then the following corollary is a direct consequence of Theorem 3.1.

COROLLARY 3.1. *Assume Eq. (3.1) has a solution $x(t)$ satisfying (2.2). Then one of the following holds:*

(a) $\int_a^c q_-(t)dt > \frac{8}{P(\int_a^c p^{-1}(\tau)d\tau)^2},$

(b) $\int_a^c q_+(t)dt > \frac{8}{P(\int_a^c p^{-1}(\tau)d\tau)^2},$

(c) $\int_a^b q_-(t)dt + \int_b^c q_+(t)dt > \frac{8}{P(\int_a^c p^{-1}(\tau)d\tau)^2}.$

As a result,

$$\int_a^c |q(t)|dt > \frac{8}{P(\int_a^c p^{-1}(\tau)d\tau)^2}.$$

The result below is provided by the anonymous referee:

COROLLARY 3.2. *Assume Eq. (3.1) has a solution $x(t)$ satisfying (2.2). Then one of the following holds:*

(a) $\int_a^c q_-(t)dt > \frac{4}{(c-a)\int_a^c p^{-1}(\tau)d\tau},$

(b) $\int_a^c q_+(t)dt > \frac{4}{(c-a)\int_a^c p^{-1}(\tau)d\tau},$

(c) $\int_a^b q_-(t)dt + \int_b^c q_+(t)dt > \frac{4}{(c-a)\int_a^c p^{-1}(\tau)\,d\tau}.$

As a result,

$$\int_a^c |q(t)|dt > \frac{4}{(c-a)\int_a^c p^{-1}(\tau)\,d\tau}.$$

Proof. The proof is a modification of that of Theorem 3.1. We only provide the outline here. For the function $G(t,s)$ given in (3.2) we have

$$G(s,s) = \frac{1}{\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \int_{t_1}^{s} \frac{d\tau}{p(\tau)} \int_{s}^{t_2} \frac{d\tau}{p(\tau)}.$$

Again, using the inequality $4\alpha\beta \leqslant (\alpha+\beta)^2$, we have

$$G(s,s) \leqslant \frac{1}{4\int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}} \Big(\int_{t_1}^{s} \frac{d\tau}{p(\tau)} + \int_{s}^{t_2} \frac{d\tau}{p(\tau)} \Big)^2 = \frac{1}{4} \int_{t_1}^{t_2} \frac{d\tau}{p(\tau)}.$$

Note that $\max_{t_1 \leqslant t \leqslant t_2} G(t,s) = G(s,s)$, we have

$$\int_{t_1}^{t_2} G(t,s)\,dt \leqslant \int_{t_1}^{t_2} G(s,s)\,dt \leqslant \frac{t_2-t_1}{4} \int_{t_1}^{t_2} \frac{d\tau}{p(\tau)} \leqslant \frac{c-a}{4} \int_a^c \frac{d\tau}{p(\tau)}.$$

Then the rest of the proof is essentially in the same way as that of Theorem 2.1 and hence is omitted. □

REMARK 3.1. Corollary 3.2 is a variation of Corollary 3.1. It is not intended to replace Corollary 3.1; however, it provides a better result for the case when $p(t)$ has a large maximum but not a large average. More precisely, the inequalities in Corollary 3.2 are sharper than those in Corollary 3.1 when $P > 2(c-a)/\int_a^c p^{-1}(\tau)\,d\tau$.

We next consider the equation

$$x''' + f(t)x'' + h(t)x = 0, \tag{3.9}$$

where $f,h \in C([a,c],\mathbb{R})$. We denote

$$p(t) = \exp\Big(\int_a^t f(s)ds\Big) \text{ and } P = \exp\Big(\max\Big\{\int_a^t f(s)ds : t \in [a,c]\Big\}\Big).$$

THEOREM 3.2. *Assume Eq. (3.9) has a solution $x(t)$ satisfying (2.2). Then one of the following holds:*

(a) $\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) p(t)h_-(t)dt > \frac{2}{P},$

(b) $\int_a^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big) p(t)h_+(t)dt > \frac{2}{P},$

$$(c) \int_a^b \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big)p(t)h_-(t)dt$$
$$+\int_b^c \Big(\int_a^t \frac{d\tau}{p(\tau)}\Big)\Big(\int_t^c \frac{d\tau}{p(\tau)}\Big)p(t)h_+(t)dt > \frac{2}{P}.$$

Proof. Multiplying both sides of (3.9) by the integral factor $p(t)$ and letting $q(t) = h(t)\exp(\int_a^t f(s)ds)$, we obtain Eq. (3.1). Note that $p(t) > 0$ for $t \in [a,c]$. Then the result follows from Theorem 3.1 immediately. □

The following corollary is an immediate consequence of Theorem 3.2.

COROLLARY 3.3. *Assume Eq. (3.9) has a solution $x(t)$ satisfying (2.2). Then one of the following holds:*

$$(a) \int_a^c p(t)h_-(t)dt > \frac{8}{P\big(\int_a^c p^{-1}(\tau)d\tau\big)^2},$$

$$(b) \int_a^c p(t)h_+(t)dt > \frac{8}{P\big(\int_a^c p^{-1}(\tau)d\tau\big)^2},$$

$$(c) \int_a^b p(t)h_-(t)dt + \int_b^c p(t)h_+(t)dt > \frac{8}{P\big(\int_a^c p^{-1}(\tau)d\tau\big)^2}.$$

As a result,

$$\int_a^c |h(t)|dt > \frac{8}{P\big(\int_a^c p^{-1}(\tau)d\tau\big)^2}.$$

Finally, we consider the general third-order linear equation

$$x''' + f(t)x'' + g(t)x' + h(t)x = 0, \tag{3.10}$$

where $f,g,h \in C([a,c],\mathbb{R})$. We denote $p(t) = \exp(\int_a^t f(s)ds)$.

THEOREM 3.3. *Assume Eq. (3.10) has a solution $x(t)$ satisfying (2.2). Then*

$$\int_a^c \Big(|g(t)| + (c-a)|h(t)|\Big)p(t)dt > \frac{4}{\int_a^c p^{-1}(t)dt}. \tag{3.11}$$

Proof. Multiplying both sides of Eq. (3.10) by the integral factor $p(t)$ we obtain the equation

$$-(p(t)x'')' = r(t), \tag{3.12}$$

where $r(t) = p(t)(g(t)x'(t) + h(t)x(t))$. Let t_1 and t_2 be defined as in the proof of Theorem 2.1. Let $G(t,s)$ be given by (3.2). By (3.12),

$$\begin{aligned} x'(t) &= \int_{t_1}^{t_2} G(t,s)r(s)ds \\ &= \int_{t_1}^{t_2} G(t,s)p(s)(g(s)x'(s) + h(s)x(s))ds \\ &= \int_{t_1}^{t_2} G(t,s)p(s)\Big(g(s)x'(s) + h(s)\int_b^s x'(\tau)d\tau\Big)ds. \end{aligned}$$

Taking absolute value on both sides and considering that $G(t,s) \geqslant 0$ on $[t_1,t_2]\times[t_1,t_2]$ and $p(t) > 0$, we have

$$|x'(t)| \leqslant \int_{t_1}^{t_2} G(t,s)p(s)\Big(|g(s)||x'(s)| + |h(s)|\Big|\int_b^s x'(\tau)d\tau\Big|\Big)ds. \tag{3.13}$$

Note $G(t,s) \leqslant G(s,s)$ and $4\alpha\beta \leqslant (\alpha+\beta)^2$. Then

$$G(t,s) \leqslant \frac{1}{\int_{t_1}^{t_2}\frac{d\tau}{p(\tau)}}\int_{t_1}^{s}\frac{d\tau}{p(\tau)}\int_{s}^{t_2}\frac{d\tau}{p(\tau)} \leqslant \frac{1}{4}\int_{t_1}^{t_2}\frac{d\tau}{p(\tau)} \leqslant \frac{1}{4}\int_a^c\frac{d\tau}{p(\tau)}. \tag{3.14}$$

Let $m_1 = \max\{|x'(t)| : t\in[t_1,t_2]\}$. Taking the maximum of $|x'(t)|$ on (3.13) and using the fact that $0 \leqslant |x'(t)| \leqslant m_1$ and $|x'(t)| \not\equiv m_1$ on $[t_1,t_2]$, we have

$$\begin{aligned} m_1 &= m_1\int_{t_1}^{t_2} G(s,s)p(s)\Big(|g(s)| + |s-b||h(s)|\Big)ds \\ &\leqslant m_1\int_a^c G(s,s)p(s)\Big(|g(s)| + (c-a)|h(s)|\Big). \end{aligned}$$

By substituting (3.14) in the above and cancelling m_1 from both sides, we obtain (3.11). □

If $f(t) \equiv 0$, then Eq. (3.10) becomes

$$x''' + g(t)x' + h(t)x = 0; \tag{3.15}$$

and if $h(t) \equiv 0$, then (3.10) becomes

$$x''' + f(t)x'' + g(t)x' = 0. \tag{3.16}$$

We observe that $p(t) \equiv 1$ for the former case. As direct consequences of Theorem 3.3, we have the following results.

COROLLARY 3.4. *(a) Assume Eq. (3.15) has a solution $x(t)$ satisfying (2.2). Then*

$$\int_a^c \Big(|g(t)| + (c-a)|h(t)|\Big)dt > \frac{4}{c-a}.$$

(b) Assume Eq. (3.16) has a solution $x(t)$ satisfying (2.2). Then

$$\int_a^c p(t)|g(t)|dt > \frac{4}{\int_a^c p^{-1}(t)dt}.$$

REMARK 3.2. We note that Eq. (3.10) also includes Eqs. (2.1) and (3.9) as special cases. However, Theorem 3.3 does not cover the results in Theorems 2.1 and 3.2. This is due to the complexity caused by the appearance of the term $g(t)x'$, and hence the result has to be derived from an inequality for x' rather than from one for x as in the proofs of Theorems 2.1 and 3.2. Moreover, since the zeros of x' may have accumulation points in $[a,c]$, we cannot locate any specific subintervals of $[a,c]$ where x' has a fixed sign. Therefore, the Lyapunov-type inequalities given in Theorem 3.3 and Corollary 3.4 can only involve $|g|$ and $|h|$ instead of $g_\pm$ and $h_\pm$.

4. Applications to boundary value problems

In the last section, we apply the results on the Lyapunov-type inequalities obtained in Sections 2 and 3 to study the nonexistence, uniqueness, and existence-uniqueness for solutions of certain third-order BVPs.

We first consider the BVP consisting of Eq. (2.1) and the following boundary condition (BC)

$$x(a) = x(b) = x(c) = 0, \tag{4.1}$$

where $-\infty < a < b < c < \infty$ and $q \in C([a,c],\mathbb{R})$.

The result below is on the nonexistence of nontrivial solutions of the three point BVP (2.1), (4.1).

THEOREM 4.1. *Assume all the following three conditions are satisfied:*

(a) $\displaystyle\int_a^c q_-(t)dt \leqslant \frac{8}{(c-a)^2}$,

(b) $\displaystyle\int_a^c q_+(t)dt \leqslant \frac{8}{(c-a)^2}$,

(c) $\displaystyle\int_a^b q_-(t)dt + \int_b^c q_+(t)dt \leqslant \frac{8}{(c-a)^2}$.

Then BVP (2.1), (4.1) has only the trivial solution.

Proof. Assume the contrary, i.e., BVP (2.1), (4.1) has a nontrivial solution $x(t)$. Then there exist $a_1 \in [a,b)$ and $c_1 \in (b,c]$ such that $x(a_1) = x(c_1) = 0$ and $x(t) \neq 0$ for $t \in (a_1,b) \cup (b,c_1)$. It follows that one of the conclusions (a)-(c) in Corollary 2.1 holds with a and c replaced by a_1 and c_1, respectively. Without loss of generality, we assume (a) holds. Hence,

$$\int_a^c q_-(t)dt \geqslant \int_{a_1}^{c_1} q_-(t)dt > \frac{8}{(c_1-a_1)^2} \geqslant \frac{8}{(c-a)^2}$$

which contradicts condition (a) in the theorem. □

Next we consider the third-order nonhomogeneous linear BVPs consisting of the equation

$$x''' + q(t)x = w(t) \text{ on } (A,B), \tag{4.2}$$

where $-\infty < A < B < \infty$ and $q, w \in C((A,B),\mathbb{R})$; and one of the following BCs

$$x(t_1) = k_1,\ x(t_2) = k_2,\ x'(t_2) = k_3, \tag{4.3}$$

$$x(t_1) = k_1,\ x'(t_1) = k_2,\ x(t_2) = k_3, \tag{4.4}$$

and

$$x(t_1) = k_1,\ x(t_2) = k_2,\ x(t_3) = k_3, \tag{4.5}$$

where

$$A < t_1 < t_2 < t_3 < B \text{ and } k_1, k_2, k_3 \in \mathbb{R}. \tag{4.6}$$

Here we present a criterion for the above BVPs to have a unique solution.

THEOREM 4.2. *Assume*

$$\max_{\xi\in[A,B]} \left\{ \int_A^{\xi} q_-(t)dt + \int_{\xi}^{B} q_+(t)dt \right\} \leqslant \frac{8}{(B-A)^2}. \tag{4.7}$$

Then each of the BVPs (4.2), (4.3); (4.2), (4.4); and (4.2), (4.5) has a unique solution on (A,B) for any t_1,t_2,t_3 and k_1,k_2,k_3 satisfying (4.6).

To prove Theorem 4.2, we first introduce the following results by Jackson and Schrader [13] and Jackson [12] on BVPs associated with the general third-order equation

$$x''' = F(t,x,x',x''), \tag{4.8}$$

where

(i) $F \in C((A,B)\times\mathbb{R}^3,\mathbb{R})$, and

(ii) any initial value problem associated with Eq. (4.8) has a unique solution which exists on the whole interval.

LEMMA 4.1. *Assume (i) and (ii) are satisfied and BVP (4.8), (4.5) has at most one solution on (A,B). Then each of the BVPs (4.8), (4.3); (4.8), (4.4); and (4.8), (4.5) has a unique solution on (A,B).*

Under assumptions (i) and (ii), Jackson and Schrader [13, Theorem 2, 3] showed that if the solution of BVP (4.8), (4.5), when exists, is unique on (A,B), then each of the BVPs (4.8), (4.3); (4.8), (4.4); and (4.8), (4.5) has at least one solution on (A,B). Furthermore, Jackson [12, Theorem 1, 2] proved that the uniqueness of the solution of BVP (4.8), (4.5) guarantees the uniqueness of the solution of BVPs (4.8), (4.3); (4.8), (4.4); and (4.8), (4.5).

Using Lemma 4.1, we prove Theorem 4.2 below.

Proof of Theorem 4.2. Note that assumptions (i) and (ii) are satisfied by Eq. (4.2) since it is linear. First we show that BVP (4.2), (4.5) has at most one solution for any t_1,t_2,t_3 and k_1,k_2,k_3 satisfying (4.6). For otherwise, it has two solutions $x_1(t)$

and $x_2(t)$ in (A,B). Define $x(t) = x_1(t) - x_2(t)$. Then $x(t)$ is a solution of the BVP consisting of Eq. (2.1) and the BC

$$x(t_1) = 0,\ x(t_2) = 0,\ x(t_3) = 0. \tag{4.9}$$

From (4.7) we have for any $\xi \in [t_1, t_3]$

$$\max_{\xi \in [t_1,t_3]} \left\{ \int_{t_1}^{\xi} q_-(t)dt + \int_{\xi}^{t_3} q_+(t)dt \right\} \leqslant \frac{8}{(t_3 - t_1)^2}.$$

In particular, for $\xi = t_3, t_1, t_2$, respectively, we see that conditions (a)-(c) in Theorem 4.1 are satisfied with $a = t_1, b = t_2, c = t_3$. Then by Theorem 4.1, $x(t) \equiv 0$, i.e., $x_1(t) \equiv x_2(t)$. This shows the uniqueness of the solution of BVP (4.2), (4.5). By Lemma 3.1, each of the BVPs (4.2), (4.3); (4.2), (4.4); and (4.2), (4.5) has a unique solution on (A,B). $\square$

Now we consider the BVPs consisting of one of the equations

$$x''' + f(t)x'' + h(t)x = r(t) \tag{4.10}$$

and

$$x''' + f(t)x'' + g(t)x' + h(t)x = r(t), \tag{4.11}$$

where $f, g, h, r \in C((A,B), \mathbb{R})$; and one of the BCs (4.3), (4.4), (4.5). In the following, we state criteria for these BVPs to have a unique solution. As before, we denote

$$p(t) = \exp\Big(\int_A^t f(s)ds\Big) \text{ and } P = \exp\Big(\max\Big\{\int_A^t f(s)ds : t \in [a,c]\Big\}\Big).$$

THEOREM 4.3. *Assume*

$$\max_{\xi \in [A,B]} \left\{ \int_A^{\xi} p(t)h_-(t)dt + \int_{\xi}^{B} p(t)h_+(t)dt \right\} \leqslant \frac{8}{P(\int_A^B p^{-1}(\tau)d\tau)^2}. \tag{4.12}$$

Then each of the BVPs (4.10), (4.3); (4.10), (4.4); and (4.10), (4.5) has a unique solution on (A,B) for any t_1,t_2,t_3 and k_1,k_2,k_3 satisfying (4.6).

THEOREM 4.4. *Assume*

$$\int_A^B \Big(|g(t)| + (B-A)|h(t)|\Big)p(t)dt \leqslant \frac{4}{\big(\int_A^B p^{-1}(\tau)d\tau\big)^2}.$$

Then each of the BVPs (4.11), (4.3); (4.11), (4.4); and (4.11), (4.5) has a unique solution on (A,B) for any t_1,t_2,t_3 and k_1,k_2,k_3 satisfying (4.6).

The proofs of Theorems 4.3 and 4.4 are similar to that of Theorem 4.2. We omit the details.

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