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ESSENTIAL NORM AND COMPACTNESS OF THE PRODUCT OF DIFFERENTIATION AND COMPOSITION OPERATORS ON BLOCH TYPE SPACES

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Abstract. In this paper, we give a complete characterization of the compactness of the product of differentiation and composition operators on Bloch type spaces and little Bloch type spaces. Moreover, an estimate of the essential norm for this operator on Bloch type spaces is also given.

1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ denote the unit disk in the complex plane and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let $\alpha \in (0,\infty)$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch type space (or $\alpha-$Bloch space) denoted by $\mathscr{B}^\alpha$, if

$$\|f\|_\alpha = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2)^\alpha < \infty.$$

We say that an $f \in H(\mathbb{D})$ belongs to the little Bloch type space (or little α-Bloch space), denoted by $\mathscr{B}_0^\alpha$, if $\lim_{|z|\to 1} |f'(z)|(1-|z|^2)^\alpha = 0$. The classical Bloch space $\mathscr{B}$ is just $\mathscr{B}^1$. It is well known that $\mathscr{B}^\alpha$ is a Banach space with the norm $\|f\|_{\mathscr{B}^\alpha} = |f(0)| + \|f\|_\alpha$. It is also well known that under the usual integral pairing the dual of $\mathscr{B}_0^\alpha$ is isomorphic to the Bergman space A^1, which is the space of all $f \in H(\mathbb{D})$ such that $\int_{\mathbb{D}} |f(z)|dA(z) < \infty$, where $dA(z)$ is the normalized area measure on $\mathbb{D}$. See [32, 33] for the theory of Bloch type spaces.

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ associated with φ is defined by

$$C_\varphi(f) = f \circ \varphi, \quad f \in H(\mathbb{D}).$$

A basic problem concerning composition operator and its generalizations on various Banach spaces is to relate their operator theoretic properties to the function theoretic properties of the inducing symbols. See [1] and [32] for more results on composition operators on function spaces.

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Let D denote the differentiation operator, *i.e.*, $Df = f'$, $f \in H(\mathbb{D})$. Let m be a nonnegative integer. We denote D^m the m-th differentiation operator, where,

$$(D^m f)(z) = f^{(m)}(z), \qquad f \in H(\mathbb{D}).$$

Now we define the product of m-th differentiation operator and composition operator, denoted by $C_\varphi D^m$, as follows.

$$(C_\varphi D^m f)(z) = f^{(m)}(\varphi(z)), \quad f \in H(\mathbb{D}).$$

The product of composition operator and differentiation operator has been studied, for example, in [3, 4, 6, 7, 9, 10, 11, 18, 20, 21, 22, 25, 27, 28, 31, 34, 35] (see also the related references therein). For some other product-type operators, see, for example, [8, 19, 23] and numerous references therein.

The composition operator C_φ, which induced by any self-map, is automatically bounded on $\mathscr{B}$ by the well-known Schwarz-Pick Lemma. In [24], Tjani proved that $C_\varphi : \mathscr{B} \to \mathscr{B}$ is compact if and only if $\lim_{|a|\to 1^-} \|C_\varphi \sigma_a\|_{\mathscr{B}} = 0$, where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$, $z \in \mathbb{D}$. In [26], Wulan, Zheng and Zhu obtained a very nice characterization for the compactness of the composition operators acting on the Bloch space, *i.e.*, they proved that $C_\varphi : \mathscr{B} \to \mathscr{B}$ is compact if and only if $\lim_{j\to\infty} \|\varphi^j\|_{\mathscr{B}} = 0$. See [1, 12, 13, 14, 15, 16, 24, 26, 29] for more study of the compactness and essential norm of composition operator on Bloch type spaces. Recently, Wu and Wulan extend the above two characterizations to the operator $C_\varphi D^m$ in [25]. Among others, they proved the following result.

THEOREM A. *Let φ be an analytic self-map of $\mathbb{D}$ and m a nonnegative integer. Then the following statements are equivalent.*

(i) $C_\varphi D^m : \mathscr{B} \to \mathscr{B}$ *is compact;*

(ii) $\lim_{n\to\infty} \|C_\varphi D^m(z^n)\|_{\mathscr{B}} = 0$;

(iii) $\lim_{|a|\to 1^-} \left\|C_\varphi D^m \sigma_a\right\|_{\mathscr{B}} = 0.$

The condition (ii) in Theorem A has been extended to the case of Bloch type spaces by Liang and Zhou in [11]. They proved that $C_\varphi D^m : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact if and only if $\lim_{n\to\infty} n^{\alpha-1} \|C_\varphi D^m(z^n)\|_{\mathscr{B}^\beta} = 0$. Moreover, they give an estimate for the essential norm of $C_\varphi D^m : \mathscr{B}^\alpha \to \mathscr{B}^\beta$, *i.e.*,

$$\|C_\varphi D^m\|_{e,\mathscr{B}^\alpha \to \mathscr{B}^\beta} \approx \limsup_{n\to\infty} n^{\alpha-1} \|C_\varphi D^m(z^n)\|_{\mathscr{B}^\beta}.$$

Recall that the essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators K mapping X to Y, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K \text{ is compact } \},$$

where X and Y are two Banach spaces and $\|\cdot\|_{X\to Y}$ is the operator norm. The operator T is said to be weakly compact if for every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X, $(L(x_n))_{n\in\mathbb{N}}$ has a weakly convergent subsequence (see [2]).

In [31], Zhou and the author of this paper extended the condition (iii) in Theorem A to Bloch type spaces and obtained the following result.

THEOREM B. *Assume that* $0 < \alpha, \beta < \infty$ *and* m *is a nonnegative integer. Let* φ *be an analytic self-map of* $\mathbb{D}$ *such that* $\varphi \in \mathscr{B}^{\beta}$. *Then the following statements are equivalent.*

(i) $C_{\varphi}D^m : \mathscr{B}^{\alpha} \to \mathscr{B}^{\beta}$ *is compact;*

(ii) $C_{\varphi}D^m : \mathscr{B}_0^{\alpha} \to \mathscr{B}^{\beta}$ *is compact;*

(iii) $\lim_{|a|\to 1^-} \left\| C_{\varphi}D^m \left(\frac{1-|a|^2}{(1-\overline{a}z)^{\alpha}} \right) \right\|_{\mathscr{B}^{\beta}} = 0;$

(iv) $\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)^{\beta}}{(1-|\varphi(z)|^2)^{\alpha+m}} |\varphi'(z)| = 0.$

The purpose of the paper is to give a complete characterization of the compactness for the operator $C_{\varphi}D^m$ between different Bloch type spaces and little Bloch type spaces. In addition. we give another estimate of the essential norm of the operator $C_{\varphi}D^m : \mathscr{B}^{\alpha} \to \mathscr{B}^{\beta}$.

Throughout this paper, we say that $A \lesssim B$ if there exists a constant C such that $A \leqslant CB$. The symbol $A \approx B$ means that $A \lesssim B \lesssim A$.

2. Compactness of $C_{\varphi}D^m : \mathscr{B}^{\alpha}(\mathscr{B}_0^{\alpha}) \to \mathscr{B}^{\beta}(\mathscr{B}_0^{\beta})$

In this section, we characterize the compactness of $C_{\varphi}D^m : \mathscr{B}^{\alpha}(\mathscr{B}_0^{\alpha}) \to \mathscr{B}^{\beta}(\mathscr{B}_0^{\beta})$. For this purpose, we need some Lemmas. From [33], we know that $(\mathscr{B}_0^{\alpha})^* = A^1$ and $(A^1)^* = \mathscr{B}^{\alpha}$. Also, it is well-known that $A^1 \cong l^1$. Since l^1 has the Schur property, we get the following result (see, for example, [30]).

LEMMA 1. *Assume that* $0 < \alpha, \beta < \infty$ *and* m *is a nonnegative integer. Let* φ *be an analytic self-map of* $\mathbb{D}$ *such that* $\varphi \in \mathscr{B}_0^{\beta}$. *Then* $C_{\varphi}D^m : \mathscr{B}_0^{\alpha} \to \mathscr{B}^{\beta}$ *(or* $\mathscr{B}_0^{\beta}$*) is weakly compact if and only if* $C_{\varphi}D^m : \mathscr{B}_0^{\alpha} \to \mathscr{B}^{\beta}$ *(or* $\mathscr{B}_0^{\beta}$*) is compact.*

LEMMA 2. [32] *For* $f \in H(\mathbb{D})$, $n \in \mathbb{N}$ *and* $\alpha > 0$. *Then* $f \in \mathscr{B}^{\alpha}$ *if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n-1} |f^{(n)}(z)| < \infty.$$

Moreover,

$$\|f\|_{\mathscr{B}^{\alpha}} \approx \sum_{j=0}^{n-1} |f^{(j)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{\alpha+n-1} |f^{(n)}(z)|.$$

LEMMA 3. [16] *Let* $\beta > 0$. *A closed set* H *in* $\mathscr{B}_0^{\beta}$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1^-} \sup_{f\in H} (1-|z|^2)^{\beta} |f'(z)| = 0.$$

THEOREM 1. *Assume that* $0<\alpha,\beta<\infty$ *and* m *is a nonnegative integer. Let* φ *be an analytic self-map of* $\mathbb{D}$ *such that* $\varphi\in\mathscr{B}_0^\beta$*. Then the following statements are equivalent.*

(1) $C_\varphi D^m:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ *is compact;*
(2) $C_\varphi D^m:\mathscr{B}_0^\alpha\to\mathscr{B}^\beta$ *is compact;*
(3) $C_\varphi D^m:\mathscr{B}^\alpha\to\mathscr{B}_0^\beta$ *is compact;*
(4) $C_\varphi D^m:\mathscr{B}_0^\alpha\to\mathscr{B}_0^\beta$ *is compact;*
(5)

$$\lim_{|a|\to 1^-}\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha}\right)\right\|_{\mathscr{B}^\beta}=0;$$

(6) $C_\varphi D^m\mathscr{B}^\alpha\subset\mathscr{B}_0^\beta$*;*
(7) $C_\varphi D^m:\mathscr{B}_0^\alpha\to\mathscr{B}^\beta$ *is weakly compact;*
(8) $C_\varphi D^m:\mathscr{B}_0^\alpha\to\mathscr{B}_0^\beta$ *is weakly compact;*
(9)

$$\lim_{|\varphi(z)|\to 1^-}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}}|\varphi'(z)|=0.$$

Proof. $(1)\Leftrightarrow(2)\Leftrightarrow(5)\Leftrightarrow(9)$ follows by Theorem B. From Lemma 1, we see that $(8)\Leftrightarrow(4)$ and $(7)\Leftrightarrow(2)$. $(3)\Rightarrow(6)$ and $(4)\Rightarrow(2)$ are obvious. Hence, we only need to show that $(6)\Rightarrow(4)$ and $(2)\Rightarrow(3)$.

$(6)\Rightarrow(4)$. We adopt the method of the proof of Theorem 2 in [5]. Since $C_\varphi D^m:\mathscr{B}_0^\alpha\to\mathscr{B}_0^\beta$ is bounded and $(\mathscr{B}_0^\beta)^*=(\mathscr{B}_0^\alpha)^*=A^1$, then we see that $(C_\varphi D^m)^*:A^1\to A^1$ is bounded. Hence every bounded linear functional T on $\mathscr{B}_0^\beta$ can be identified by a function $g\in A^1$. Therefore for every $f\in\mathscr{B}_0^\alpha$ and $g\in A^1$,

$$\langle C_\varphi D^m(f),g\rangle=\langle f,(C_\varphi D^m)^*(g)\rangle.$$

On the other hand, since $(A^1)^*=\mathscr{B}^\alpha$, we get that $(C_\varphi D^m)^{**}:\mathscr{B}^\alpha\to\mathscr{B}^\beta$ is bounded, where $(C_\varphi D^m)^{**}$ is the second adjoint of $C_\varphi D^m$. Hence every $f\in\mathscr{B}_0^\alpha$ can be viewed as an element of the space $(A^1)^*$ and

$$\langle f,(C_\varphi D^m)^*(g)\rangle=\langle (C_\varphi D^m)^{**}(f),g\rangle.$$

From these we obtain

$$\langle C_\varphi D^m(f),g\rangle=\langle (C_\varphi D^m)^{**}(f),g\rangle,$$

for every $g\in A^1$. By a well known consequence of Hann-Banach theorem, for every $f\in\mathscr{B}_0^\alpha$,

$$(C_\varphi D^m)^{**}(f)=C_\varphi D^m(f).$$

Since $\mathscr{B}_0^\alpha$ is w^* dense in $\mathscr{B}^\alpha$, for every $f\in\mathscr{B}^\alpha$,

$$(C_\varphi D^m)^{**}(f)=C_\varphi D^m(f).$$

By Gantmacher's theorem (see [2]), $C_\varphi D^m : \mathscr{B}_0^\alpha \to \mathscr{B}_0^\beta$ is weakly compact if and only if

$$(C_\varphi D^m)^{**}((\mathscr{B}_0^\alpha)^{**}) \subset \mathscr{B}_0^\beta.$$

From $(\mathscr{B}_0^\alpha)^{**} \cong \mathscr{B}^\alpha$ ([33]) and $(C_\varphi D^m)^{**}(f) = C_\varphi D^m(f)$ for every $f \in \mathscr{B}^\alpha$, it follows that $C_\varphi D^m : \mathscr{B}_0^\alpha \to \mathscr{B}_0^\beta$ is weakly compact if and only if $C_\varphi D^m(\mathscr{B}^\alpha) \subset \mathscr{B}_0^\beta$. By Lemma 1 we see that $C_\varphi D^m : \mathscr{B}_0^\alpha \to \mathscr{B}_0^\beta$ is compact.

$(2) \Rightarrow (3)$. From Theorem B, we see that $(2) \Leftrightarrow (9)$. By the assumption that $\varphi \in \mathscr{B}_0^\beta$ and (9), we get

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}} |\varphi'(z)| = 0. \tag{2.1}$$

For any $f \in \mathscr{B}^\alpha$, by Lemma 2 we have

$$\begin{aligned}
&(1-|z|^2)^\beta |(C_\varphi D^m f)'(z)| = (1-|z|^2)^\beta |f^{(m+1)}(\varphi(z))\varphi'(z)| \\
&= |f^{(m+1)}(\varphi(z))|(1-|\varphi(z)|^2)^{\alpha+m} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}} |\varphi'(z)| \\
&\lesssim \|f\|_{\mathscr{B}^\alpha} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+m}} |\varphi'(z)|.
\end{aligned}$$

Taking the supremum in this inequality over all $f \in \mathscr{B}^\alpha$ such that $\|f\|_{\mathscr{B}^\alpha} \leqslant 1$, letting $|z| \to 1^-$ and using (2.1) we obtain that

$$\lim_{|z|\to 1^-} \sup_{\|f\|_{\mathscr{B}^\alpha} \leqslant 1} (1-|z|^2)^\beta |(C_\varphi D^m f)'(z)| = 0.$$

From Lemma 3 it follows that $C_\varphi D^m : \mathscr{B}^\alpha \to \mathscr{B}_0^\beta$ is compact. The proof of the theorem is completed. □

3. An estimate of the essential norm of $C_\varphi D^m$

In this section, we give an estimate for the essential norm of $C_\varphi D^m$. Now we are in a position to state and prove the main result in this section.

THEOREM 2. *Assume that $0 < \alpha, \beta < \infty$ and m is a nonnegative integer. Let φ be an analytic self-map of $\mathbb{D}$ such that $C_\varphi D^m : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is bounded. Then*

$$\|C_\varphi D^m\|_{e,\mathscr{B}_0^\alpha \to \mathscr{B}^\beta} \approx \|C_\varphi D^m\|_{e,\mathscr{B}^\alpha \to \mathscr{B}^\beta} \approx \limsup_{|a|\to 1^-} \left\| C_\varphi D^m \left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha} \right) \right\|_{\mathscr{B}^\beta}.$$

Proof. The lower estimate of the essential norm. Let $a \in \mathbb{D}$. We define

$$f_a(z) = \frac{1-|a|^2}{(1-\overline{a}z)^\alpha}, \quad z \in \mathbb{D}.$$

It is easy to check that $f_a \in \mathscr{B}_0^\alpha$, $\|f_a\|_{\mathscr{B}^\alpha} \leqslant 1+\alpha 2^{\alpha+1}$ for all $a \in \mathbb{D}$ and f_a goes to zero uniformly on compact subsets of $\mathbb{D}$ as $|a| \to 1^-$. Moreover, f_a converges to zero weakly in $\mathscr{B}_0^\alpha$ as $|a| \to 1^-$. Thus, by using the argument in [17, Theorem 3.4], we have that for any compact operator $K : \mathscr{B}_0^\alpha \to \mathscr{B}^\beta$

$$\lim_{|a|\to 1^-} \|Kf_a\|_{\mathscr{B}^\beta} = 0.$$

Hence

$$\begin{aligned}\|C_\varphi D^m - K\|_{\mathscr{B}_0^\alpha \to \mathscr{B}^\beta} &\gtrsim \|(C_\varphi D^m - K)f_a\|_{\mathscr{B}^\beta} \\ &\gtrsim \|C_\varphi D^m f_a\|_{\mathscr{B}^\beta} - \|Kf_a\|_{\mathscr{B}^\beta}.\end{aligned}$$

Taking $\limsup_{|a|\to 1^-}$ to the last inequality on both sides, we obtain

$$\|C_\varphi D^m - K\|_{\mathscr{B}_0^\alpha \to \mathscr{B}^\beta} \gtrsim \limsup_{|a|\to 1} \left\| C_\varphi D^m \left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha} \right) \right\|_{\mathscr{B}^\beta}.$$

Therefore, by the definition of the essential norm, we get

$$\begin{aligned}\|C_\varphi D^m\|_{e,\mathscr{B}^\alpha \to \mathscr{B}^\beta} &\geqslant \|C_\varphi D^m\|_{e,\mathscr{B}_0^\alpha \to \mathscr{B}^\beta} = \inf_K \|C_\varphi D^m - K\|_{\mathscr{B}_0^\alpha \to \mathscr{B}^\beta} \\ &\gtrsim \limsup_{|a|\to 1^-} \left\| C_\varphi D^m \left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha} \right) \right\|_{\mathscr{B}^\beta}.\end{aligned}$$

The upper estimate of the essential norm. We will show that there exists a constant $C_{\alpha,m} > 0$, depending only on α and m, such that

$$\|C_\varphi D^m\|_{e,\mathscr{B}^\alpha \to \mathscr{B}^\beta} \leqslant C_{\alpha,m} \limsup_{|a|\to 1^-} \left\| C_\varphi D^m \left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha} \right) \right\|_{\mathscr{B}^\beta}.$$

For $r \in [0,1)$, set $K_r : H(\mathbb{D}) \to H(\mathbb{D})$ by

$$(K_r f)(z) = f_r(z) = f(rz), \quad f \in H(\mathbb{D}).$$

It is clear that $f_r \to f$ uniformly on compact subsets of $\mathbb{D}$ as $r \to 1^-$. Moreover, the operator K_r is compact on $\mathscr{B}^\alpha$ and $\|K_r\|_{\mathscr{B}^\alpha \to \mathscr{B}^\alpha} \leqslant 1$. Let $\{r_n\} \subset (0,1)$ be a sequence such that $r_n \to 1^-$ as $n \to \infty$. Then for all positive integer n, the operator $C_\varphi D^m K_{r_n} : \mathscr{B}^\alpha \to \mathscr{B}^\beta$ is compact. By the definition of the essential norm we have

$$\|C_\varphi D^m\|_{e,\mathscr{B}^\alpha \to \mathscr{B}^\beta} \leqslant \limsup_{n\to\infty} \|C_\varphi D^m - C_\varphi D^m K_{r_n}\|_{\mathscr{B}^\alpha \to \mathscr{B}^\beta}. \tag{3.1}$$

Thus, we only need to show that

$$\begin{aligned}&\limsup_{n\to\infty} \|C_\varphi D^m - C_\varphi D^m K_{r_n}\|_{\mathscr{B}^\alpha \to \mathscr{B}^\beta} \\ &\leqslant C_{\alpha,m} \limsup_{|a|\to 1^-} \left\| C_\varphi D^m \left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha} \right) \right\|_{\mathscr{B}^\beta}.\end{aligned} \tag{3.2}$$

For any $f \in \mathscr{B}^\alpha$ such that $\|f\|_{\mathscr{B}^\alpha} \leqslant 1$, we consider

$$\begin{aligned}&\|(C_\varphi D^m - C_\varphi D^m K_{r_n})f\|_{\mathscr{B}^\beta}\\ = &|f^{(m)}(\varphi(0)) - r_n^m f^{(m)}(r_n\varphi(0))| + \|(f-f_{r_n})^{(m)} \circ \varphi\|_\beta.\end{aligned}$$

It is clear that

$$\lim_{n\to\infty} |f^{(m)}(\varphi(0)) - r_n^m f^{(m)}(r_n\varphi(0))| = 0.$$

Now we estimate

$$\begin{aligned}&\limsup_{n\to\infty} \|(f-f_{r_n})^{(m)} \circ \varphi\|_\beta\\ = &\limsup_{n\to\infty} \sup_{|\varphi(z)|\leqslant r_N} (1-|z|^2)^\beta |(f-f_{r_n})^{(m+1)}(\varphi(z))||\varphi'(z)|\\ &+\limsup_{n\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\beta |(f-f_{r_n})^{(m+1)}(\varphi(z))||\varphi'(z)|\\ = &Q_1 + Q_2,\end{aligned} \tag{3.3}$$

where $N \in \mathbb{N}$ is large enough such that $r_n \geqslant \frac{1}{2}$ for all $n \geqslant N$,

$$Q_1 := \limsup_{n\to\infty} \sup_{|\varphi(z)|\leqslant r_N} (1-|z|^2)^\beta |(f-f_{r_n})^{(m+1)}(\varphi(z))||\varphi'(z)|$$

and

$$Q_2 := \limsup_{n\to\infty} \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\beta |(f-f_{r_n})^{(m+1)}(\varphi(z))||\varphi'(z)|.$$

Since $\varphi \in \mathscr{B}^\beta$ and $r_n^{m+1} f_{r_n}^{(m+1)} \to f^{(m+1)}$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, we have

$$\begin{aligned}Q_1 &\leqslant \|\varphi\|_{\mathscr{B}^\beta} \limsup_{n\to\infty} \sup_{|w|\leqslant r_N} |f^{(m+1)}(w) - r_n^{m+1} f^{(m+1)}(r_n w)|\\ &= 0.\end{aligned} \tag{3.4}$$

Next we consider Q_2. We have $Q_2 \leqslant \limsup_{n\to\infty}(S_1+S_2)$, where

$$S_1 := \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\beta |f^{(m+1)}(\varphi(z))||\varphi'(z)|$$

and

$$S_2 := \sup_{|\varphi(z)|> r_N} (1-|z|^2)^\beta r_n^{m+1} |f^{(m+1)}(r_n\varphi(z))||\varphi'(z)|.$$

First we estimate S_1. Using the fact that $\|f\|_{\mathscr{B}^\alpha}\leqslant 1$, we have

$$\begin{aligned}
S_1 = & \sup_{|\varphi(z)|>r_N}(1-|z|^2)^\beta|f^{(m+1)}(\varphi(z))||\varphi'(z)|\frac{(1-|\varphi(z)|^2)^{\alpha+m}}{\alpha(\alpha+1)\cdots(\alpha+m)|\varphi(z)|^{m+1}}\\
&\times\frac{\alpha(\alpha+1)\cdots(\alpha+m)|\varphi(z)|^{m+1}}{(1-|\varphi(z)|^2)^{\alpha+m}}\\
< & \frac{1}{\alpha(\alpha+1)\cdots(\alpha+m)r_N^{m+1}}\|f\|_{\mathscr{B}^\alpha}\sup_{|\varphi(z)|>r_N}(1-|z|^2)^\beta|\varphi'(z)|\\
&\times\frac{\alpha(\alpha+1)\cdots(\alpha+m)|\varphi(z)|^{m+1}}{(1-|\varphi(z)|^2)^{\alpha+m}}\\
\leqslant & \frac{2^{m+1}}{\alpha(\alpha+1)\cdots(\alpha+m)}\sup_{|\varphi(z)|>r_N}\sup_{|a|>r_N}(1-|z|^2)^\beta|\varphi'(z)|\\
&\times\frac{\alpha(\alpha+1)\cdots(\alpha+m)|\varphi(z)|^{m+1}}{(1-|\varphi(z)|^2)^{\alpha+m}}\\
\leqslant & \frac{2^{m+1}}{\alpha(\alpha+1)\cdots(\alpha+m)}\sup_{|a|>r_N}\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha}\right)\right\|_\beta\\
\leqslant & \frac{2^{m+1}}{\alpha(\alpha+1)\cdots(\alpha+m)}\sup_{|a|>r_N}\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha}\right)\right\|_{\mathscr{B}^\beta}.
\end{aligned}$$

Taking limit as $N\to\infty$ we obtain

$$\limsup_{n\to\infty}S_1\leqslant\frac{2^{m+1}}{\alpha(\alpha+1)\cdots(\alpha+m)}\limsup_{|a|\to1^-}\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha}\right)\right\|_{\mathscr{B}^\beta}.$$

Similarly, we have

$$\limsup_{n\to\infty}S_2\leqslant\frac{2^{m+1}}{\alpha(\alpha+1)\cdots(\alpha+m)}\limsup_{|a|\to1^-}\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha}\right)\right\|_{\mathscr{B}^\beta},$$

i.e., we get that

$$Q_2\leqslant\frac{2^{m+2}}{\alpha(\alpha+1)\cdots(\alpha+m)}\limsup_{|a|\to1^-}\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha}\right)\right\|_{\mathscr{B}^\beta}. \tag{3.5}$$

Hence, by (4), (5) and (6) we get

$$\begin{aligned}
&\limsup_{n\to\infty}\|C_\varphi D^m-C_\varphi D^mK_{r_n}\|_{\mathscr{B}^\alpha\to\mathscr{B}^\beta}\\
&=\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant1}\|(C_\varphi D^m-C_\varphi D^mK_{r_n})f\|_{\mathscr{B}^\beta}\\
&=\limsup_{n\to\infty}\sup_{\|f\|_{\mathscr{B}^\alpha}\leqslant1}\|(f-f_{r_n})^{(m)}\circ\varphi\|_\beta\\
&\leqslant\frac{2^{m+2}}{\alpha(\alpha+1)\cdots(\alpha+m)}\limsup_{|a|\to1^-}\left\|C_\varphi D^m\left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha}\right)\right\|_{\mathscr{B}^\beta}.
\end{aligned}$$

Therefore, by (2) and the last inequality, we obtain

$$\|C_\varphi D^m\|_{e,\mathscr{B}^\alpha\to\mathscr{B}^\beta} \leqslant \frac{2^{m+2}}{\alpha(\alpha+1)\cdots(\alpha+m)} \limsup_{|a|\to 1^-} \left\| C_\varphi D^m \left(\frac{1-|a|^2}{(1-\overline{a}z)^\alpha} \right) \right\|_{\mathscr{B}^\beta}.$$

This completes the proof of the theorem. □

In [29], Zhao proved that the essential norm of $C_\varphi : \mathscr{B} \to \mathscr{B}$ is

$$\|C_\varphi\|_{e,\mathscr{B}\to\mathscr{B}} = \frac{e}{2} \limsup_{n\to\infty} \|\varphi^n\|_{\mathscr{B}}.$$

From the proof of Theorem 2, by taking $\alpha = 1$ and $m = 0$, we obtain a new estimate for the essential norm of $C_\varphi : \mathscr{B} \to \mathscr{B}$ as follows.

COROLLARY 1. *Let φ be an analytic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{e,\mathscr{B}\to\mathscr{B}} \approx \limsup_{|a|\to 1^-} \left\| \frac{1-|a|^2}{1-\overline{a}\varphi(z)} \right\|_{\mathscr{B}} = \limsup_{|a|\to 1^-} \|C_\varphi \sigma_a\|_{\mathscr{B}}.$$

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