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THE MAXIMUM RELATIVE DIAMETER FOR MULTI–ROTATIONALLY SYMMETRIC PLANAR CONVEX BODIES

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Abstract. In this work we study the maximum relative diameter functional d_M in the class of multi-rotationally symmetric planar convex bodies. A given set C of this class is k-rotationally symmetric for $k \in \{k_1, \dots, k_n\} \subset \mathbb{N}$, and so it is natural to consider the standard k_i-partition P_{k_i} associated to C (which is a minimizing k_i-partition for d_M when $k_i \geqslant 3$) and the corresponding value $d_M(P_{k_i})$. We establish the relation among these values, characterizing the particular sets for which all these values coincide.

1. Introduction

The class of rotationally symmetric planar convex bodies is an interesting family of sets, which constitutes a suitable setting for studying different geometrical problems (for instance, see [5] or [4, Th. 4]). Recall that a planar convex body (and so, consequently compact) is *rotationally symmetric* if it is invariant under the rotation of a certain angle centered at a point (called the center of symmetry of the set).

One of such problems, recently treated in some works (see [6, 2, 3]), is the following: given a k-rotationally symmetric planar convex body C, where $k \in \mathbb{N}$, $k \geqslant 2$ (which indicates that C is invariant for the rotation of angle $2\pi/k$), we can consider a decomposition P of C into k connected subsets $C_1, \dots, C_k$. Then, the *maximum relative diameter* associated to the decomposition P is defined by

$$d_M(P) = \max\{D(C_i) : i = 1, \dots, k\},$$

where $D(C_i)$ denotes the classical Euclidean diameter functional. An interesting question is investigating which decompositions of C give the minimal possible value for the maximum relative diameter. In other words, we search for the subdivisions of C providing the *minimal* largest distance in the corresponding subsets.

At this point, it is convenient to distinguish a specific type of subdivisions called *k-partitions*: they are decompositions given by k simple curves, all of them meeting in an interior point of C, and reaching the boundary of C at different points. The fact that C has a remarkable interior point naturally leads to consider these particular decompositions, originated from an arbitrary interior point, see Figure 1.

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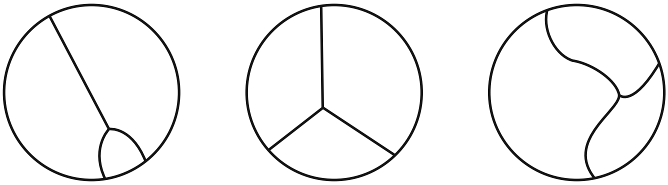

Figure 1: *Three different* 3*-partitions for the circle*

In this setting, the main result in [3] (see also [2]) states that the so-called *standard* k*-partition* is a minimizing k-partition for the maximum relative diameter when $k \geqslant 3$. In view of this result, we can reasonably think about the standard k-partition (precisely described along Section 2) as the optimal k-partition associated to each k-rotationally symmetric planar convex body, provided $k \geqslant 3$.

We shall here focus on *multi-rotationally symmetric* planar convex bodies, that is, the planar convex bodies which are rotationally symmetric under the rotation of *several* angles about the center of symmetry of the set (for instance, a regular hexagon is rotationally invariant for angles π, $2\pi/3$ and $\pi/3$, and a circle is invariant under the rotation of any angle we consider). For a given multi-rotationally symmetric planar convex body C, which is k-rotationally symmetric for $k \in \{k_1, \ldots, k_n\} \subset \mathbb{N}$, $k_i \geqslant 2$, it is natural to consider the standard k_i-partitions associated to C, and the corresponding values $d_M(P_{k_1}), \ldots, d_M(P_{k_n})$ for the maximum relative diameter functional. The aim of this paper is comparing these values, determining the relation among them.

A priori, the monotonicity property of the diameter functional may suggest that the previous values satisfy

$$d_M(P_{k_1}) > \ldots > d_M(P_{k_n}), \tag{1.1}$$

for $k_1 < \ldots < k_n$. However, we shall see that this is not true in general, since some equality sign may appear in (1.1), and even *a chain of equalities* may occur. In fact, in Lemma 3.1 we shall obtain that the general relation is

$$d_M(P_{k_1}) \geqslant \ldots \geqslant d_M(P_{k_n}), \tag{1.2}$$

and our main results will establish when (1.2) is a chain of equalities, under the hypothesis that $k_1 \geqslant 3$. More precisely, our Theorem 3.9 asserts the following:

> *Let* C *be a multi-rotationally symmetric planar convex body for* $\{k_1, \ldots, k_n\}$, *with* $k_1 \geqslant 3$. *Then,* (1.2) *is a chain of equalities if and only if* $k_1 \geqslant 7$.

This result immediately yields our Theorem 3.10:

> *Let* C *be a multi-rotationally symmetric planar convex body for* $\{k_1, \ldots, k_n\}$, *with* $k_1 \geqslant 3$. *Then,* (1.2) *is a chain of equalities if and only if* k_n *is a product of prime numbers (possibly repeated), all of them greater than or equal to* 7.

We finish this paper by discussing the special case $k_1 = 2$, which needs a particular treatment as explained in Subsection 3.2. In this case, we prove in Lemma 3.13 that

$$d_M(P_{k_1}) > d_M(P_{k_2}) \geqslant \ldots \geqslant d_M(P_{k_n}),$$

which differs from (1.2) in the first inequality, which is always *strict*.

2. Preliminaries

In this work we shall focus on rotationally symmetric planar convex bodies, assuming then the compactness of the sets. Recall that, given $k \in \mathbb{N}$, $k \geqslant 2$, a planar convex body C is said to be k*-rotationally symmetric* if there exists a point $p \in C$ such that C is invariant under the rotation of angle $2\pi/k$ about p. In this setting, p will be referred to as the *center of symmetry* of C. This notion naturally suggests the following definition.

DEFINITION 2.1. Let C be a planar convex body. We will say that C is multi-rotationally symmetric if it is k-rotationally symmetric for more than one value of k.

For instance, the circle is multi-rotationally symmetric since it is k-rotationally symmetric for any $k \in \mathbb{N}$, and the square is also multi-rotationally symmetric since it is 2-rotationally symmetric and 4-rotationally symmetric. Moreover, the regular decagon is k-rotationally symmetric for $k \in \{2, 5, 10\}$, and so it is multi-rotationally symmetric. We stress that any set of this class possesses a rich geometric structure, inherited by the different existing rotational symmetries leaving invariant the set.

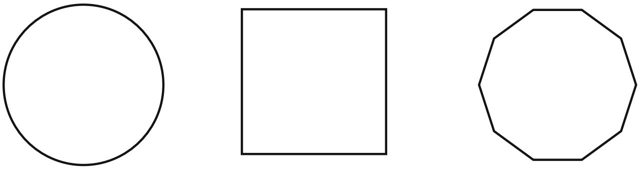

Figure 2: *Some multi-rotationally symmetric planar convex bodies: the circle, the square and the regular decagon*

REMARK 2.2. Any regular polygon E_m of m edges is k-rotationally symmetric for any divisor k of m. Hence, if m is not a prime number, E_m is multi-rotationally symmetric.

REMARK 2.3. Examples of multi-rotationally symmetric planar convex bodies can be constructed by the following procedure: for a given $k \in \mathbb{N}$, $k \geqslant 2$, consider the circular sector of angle $2\pi/k$, modify the curved piece of the boundary and apply successively $k-1$ times the rotation of angle $2\pi/k$, in such a way that the resulting set C is convex, see Figure 3. In that case, if k is not a prime number, then C is multi-rotationally symmetric.

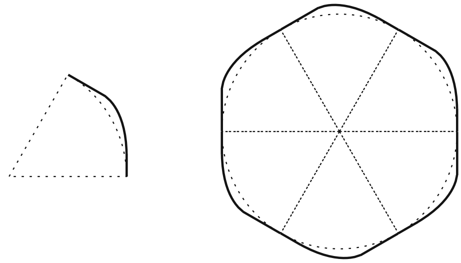

Figure 3: *A modified circular sector of angle* $2\pi/6$*, and the resulting multi-rotationally symmetric planar convex body*

The following definition concerns a remarkable natural number associated to a given multi-rotationally symmetric planar convex body (different from a circle). We shall see in Lemma 2.6 that this number determines the rotational properties of the set.

DEFINITION 2.4. Let C be a multi-rotationally symmetric planar convex body different from a circle. The largest natural number k for which C is k-rotationally symmetric will be called the maximal degree of C, and will be denoted by k_C.

REMARK 2.5. Any circle $\mathscr{C}$ can be seen as a *degenerate* multi-rotationally symmetric set, since it is k-rotationally symmetric for any $k \in \mathbb{N}$, $k \geqslant 2$, and so its associated maximal degree $k_{\mathscr{C}}$ could be set as $+\infty$. In fact, the circles are the only sets with this property.

LEMMA 2.6. *Let C be a multi-rotationally symmetric planar convex body, with maximal degree $k_C \in \mathbb{N}$. Then C is k-rotationally symmetric for $k \in \mathbb{N}$ if and only if k is a divisor of k_C.*

Proof. It is clear that if $k \in \mathbb{N}$ is a divisor of k_C, then C is k-rotationally symmetric. Assume now that C is k-rotationally symmetric, but k is not a divisor of k_C. Let $d = gcd(k, k_C)$. Then $k = d\,m_1$ and $k_C = d\,m_2$, for certain $m_1, m_2 \in \mathbb{N}$. By Bezout's identity (see [1, Th. 1.7]), we can find $a, b \in \mathbb{Z} - \{0\}$ solving the diophantine equation $k\,a + k_C\,b = d$, which gives

$$\varphi_{b\frac{2\pi}{k}} \circ \varphi_{a\frac{2\pi}{k_C}} = \varphi_{\frac{2\pi d}{k k_C}} = \varphi_{\frac{2\pi}{m_1 k_C}},$$

where φ_α denotes the rotation of angle α about the center of symmetry of C. Such an equality implies that C is $(m_1\,k_C)$-rotationally symmetric, which is contradictory since $m_1\,k_C > k_C$, and k_C is the maximal degree of C. $\square$

We now define the *minimal degree* of a multi-rotationally symmetric planar convex body, which will play an important role in Section 3.

DEFINITION 2.7. Let C be a multi-rotationally symmetric planar convex body. The smallest natural number k for which C is k-rotationally symmetric will be called the minimal degree of C, and will be denoted by χ_C.

REMARK 2.8. We point out that for any multi-rotationally symmetric planar convex body C, the minimal degree χ_C is a prime number, which will be equal to 2 if the maximal degree of C is *even*, in view of Lemma 2.6.

REMARK 2.9. Given a planar convex body C, we have that C is k-rotationally symmetric if and only if C is invariant under the action of the cyclic group generated by the rotation $\varphi_{2\pi/k}$ of angle $2\pi/k$. In addition, if C is multi-rotationally symmetric, taking into account the previous definitions, the maximal degree k_C and the minimal degree χ_C will determine the cyclic groups of rotations with largest and smallest orders leaving invariant C, respectively. This algebraic setting allows to obtain alternative proofs for some results herein (for instance, Lemma 2.6 can be derived using these equivalent formulations).

The following definition describes the decompositions we shall consider for multi-rotationally symmetric planar convex bodies. Since this kind of sets have a special interior point (which is the center of symmetry), it is natural to work with a particular type of divisions called *k-partitions*, where $k \in \mathbb{N}$, see [3].

DEFINITION 2.10. Let C be a k-rotationally symmetric planar convex body, where $k \in \mathbb{N}$, $k \geqslant 2$. A k-partition of C is a decomposition of C into k connected subsets, given by k simple curves starting at an interior point of C and meeting the boundary of C at different points.

REMARK 2.11. We stress that, in the previous definition, the interior point of a k-partition does not coincide, in general, with the center of symmetry of the set, and moreover, the corresponding subsets need not enclose equal areas.

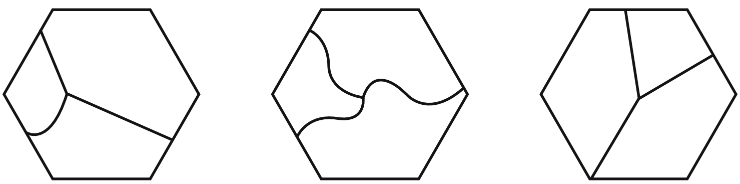

Figure 4: *Three different 3-partitions for the regular hexagon*

We now recall the definition of the maximum relative diameter functional, which is given by means of the classical diameter functional.

DEFINITION 2.12. Let C be a k-rotationally symmetric planar convex body, and let P be a k-partition of C into subsets $C_1, \ldots, C_k$. The maximum relative diameter associated to P is given by

$$d_M(P,C) = \max\{D(C_i) : i = 1, \ldots, k\},$$

where $D(C_i) = \max\{d(x,y) : x, y \in C_i\}$ denotes the Euclidean diameter of C_i.

REMARK 2.13. We remark that the existence of the maximum relative diameter $d_M(P,C)$ associated to a k-partition P of C is assured due to the classical Weierstrass theorem. If no confusion may arise, we shall simply denote it by $d_M(P)$.

Given a k-rotationally symmetric planar convex body C, an interesting question is the study of the minimizing k-partitions for the maximum relative diameter d_M. That is, among all the k-partitions of C, we search for the ones providing the minimal possible value for d_M. A complete characterization of a particular minimizing k-partition has been recently obtained when $k \geqslant 3$ [3, Th. 4.5]. We shall describe the construction of this remarkable minimizing k-partition, called *standard k-partition*, which can be considered as the optimal and most appropriate k-partition for the maximum relative diameter functional when $k \geqslant 3$.

Let C be a k-rotationally symmetric planar convex body, with $k \geqslant 2$, being p the center of symmetry of C. Let $x_1, \ldots, x_k$ be points in ∂C at minimal distance to p, symmetrically placed along ∂C. By considering the line segments $\overline{px_i}$ (joining p with each point x_i) we will obtain a k-partition of C into k connected congruent subsets, see Figure 5. This k-partition is called the *standard k-partition* associated to C, and will be denoted by $P_k(C)$, or simply P_k. The points $x_1, \ldots, x_k$ are called the *endpoints* of P_k.

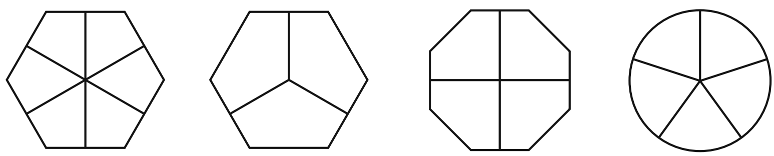

Figure 5: *Standard 6-partition and 3-partition for the regular hexagon, standard 4-partition for the regular octogon, and standard 5-partition for the circle*

The following lemma allows to compute easily the maximum relative diameter associated to any standard k-partition, when $k \geqslant 3$.

LEMMA 2.14. [3, Lemma 3.2] *Let C be a k-rotationally symmetric planar convex body, with $k \geqslant 3$, and let P_k be its associated standard k-partition. Then,*

$$d_M(P_k, C) = \max\{R, 2r\sin(\pi/k)\},$$

where R and r are the circumradius and the inradius of C, respectively.

We finish this section with the following result, which will be used later.

LEMMA 2.15. *Let C be a k-rotationally symmetric planar convex body, being p its center of symmetry, with $k \geqslant 2$. Let $x \in \partial C$ be an endpoint of the standard k-partition associated to C. Let s be the line orthogonal to the segment $\overline{px}$ passing through x. Then s is a supporting line of C.*

Proof. Let s^+, s^- be the (open) halfplanes determined by s, with $p \in s^-$. Assume that there exists $q \in \partial C$ such that $q \in s^+$. As x is an endpoint of the standard k-partition of C, we have that $B \subset C$, where B is the ball centered at p with radius $r = d(p,x)$. Notice that s is then the tangent line to ∂B at x.

Let B' be the convex hull of $B \cup \{q\}$. Then $B' \subset C$, due to the convexity of C, and so $\mathrm{int}(B') \subset \mathrm{int}(C)$. On the other hand, the construction of B' gives that $x \in \mathrm{int}(B')$ (since $q \in s^+$), which implies that $x \in \mathrm{int}(C)$, a contradiction. □

3. Main results

In this section we shall prove the main results of the paper. First of all, we shall state precisely our problem. Let C be a multi-rotationally symmetric planar convex body, with associated maximal degree $k_C \in \mathbb{N}$. Unless explicitly indicated, R and r will denote the circumradius and the inradius of C, respectively. In view of Lemma 2.6, C will be k-rotationally symmetric for any divisor k of k_C. Let $\{k_1 = \chi_C, \dots, k_n = k_C\}$ be the set of divisors of k_C, with $k_1 < \dots < k_n$.

In this setting, we shall consider the standard k_i-partition P_{k_i} associated to C, $i = 1, \dots, n$, which is the *optimal* k_i-partition for the maximum relative diameter when $k_i \geqslant 3$ [3, Th. 4.5], and the corresponding value $d_M(P_{k_i})$. In this work we investigate the relation among the values $d_M(P_{k_1}), \dots, d_M(P_{k_n})$.

At first glance, it could seem that

$$d_M(P_{k_1}) > \dots > d_M(P_{k_n}), \tag{3.1}$$

with strict inequalities, since when $k_i < k_j$, each subset of C given by P_{k_j} is strictly contained, up to a proper rotation (if necessary), in a subset provided by P_{k_i}, see Figure 6.

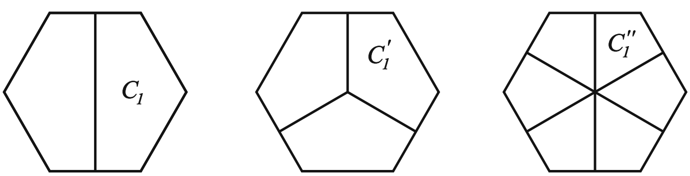

Figure 6: *For the regular hexagon, the subsets C_1, C_1' and C_1'', determined by the standard 2-partition, 3-partition and 6-partition, are related by strict inclusions*

However, we shall see that the above strict inequalities in (3.1) do not hold in general, and it is even possible *a chain of equalities* in (3.1) in some particular situations. The case of equalities in (3.1) is especially interesting, since it implies that the considered set can be divided in different natural ways (into different numbers of connected subsets), yielding the *same* value for the maximum relative diameter functional. In other words, the number k of subsets given by an associated standard k-partition will not have influence on this functional in such a case.

The following result establishes the general chain of inequalities satisfied in this setting.

LEMMA 3.1. *Let C be a multi-rotationally symmetric planar convex body, with maximal degree k_C and minimal degree χ_C. Let $\{k_1 = \chi_C, \dots, k_n = k_C\}$ be the set of divisors of k_C, with $k_1 < \dots < k_n$. Then,*

$$d_M(P_{k_1}) \geqslant \dots \geqslant d_M(P_{k_n}), \tag{3.2}$$

where P_k denotes the standard k-partition associated to C.

Proof. If $k_i < k_j$, then any subset of C given by P_{k_j} is contained in a subset given by P_{k_i} (up to a proper rotation), and so the statement is clear due to the monotonicity of the diameter functional. □

REMARK 3.2. There are examples where (3.2) holds *with strict inequalities*. For instance, consider a regular hexagon E_6, which is k-rotationally symmetric for $k \in \{2,3,6\}$, with fixed circumradius R. By Lemma 2.14, we have $d_M(P_3,E_6) = 3/2R$ and $d_M(P_6,E_6) = R$, and straightforward computations give $d_M(P_2,E_6) = \sqrt{13}/2R$, obtaining

$$d_M(P_2,E_6) > d_M(P_3,E_6) > d_M(P_6,E_6).$$

An identical behavior occurs for E_9 and E_{15}. On the other hand, (3.2) may combine strict inequalities and equalities, as for the regular dodecagon E_{12}: it is k-rotationally symmetric for $k \in \{2,3,4,6,12\}$, and straightforward computations give

$$d_M(P_2,E_{12}) > d_M(P_3,E_{12}) > d_M(P_4,E_{12}) > d_M(P_6,E_{12}) = d_M(P_{12},E_{12}).$$

The same happens for the regular polygons E_{20} and E_{45}.

In order to determine which are the multi-rotationally symmetric planar convex bodies providing *a chain of equalities* in (3.2), we shall distinguish two cases, depending on the minimal degree of our set (the case $k = 2$ requires some special considerations, as explained in Subsection 3.2).

3.1. Minimal degree greater than 2

If the minimal degree of our set is greater than 2, we can obtain the following characterization result.

LEMMA 3.3. *Let C be a multi-rotationally symmetric planar convex body, with minimal degree $\chi_C \geqslant 3$. Then, we have a chain of equalities in (3.2) if and only if $d_M(P_{\chi_C}) = R$.*

Proof. Let k_C be the maximal degree of C, and let $\{k_1 = \chi_C, \dots, k_n = k_C\}$ be the set of divisors of k_C, with $k_1 < \dots < k_n$. Assume that $d_M(P_{\chi_C}) = R$. Then, (3.2) turns

$$R = d_M(P_{k_1}) \geqslant d_M(P_{k_2}) \geqslant \dots \geqslant d_M(P_{k_n}) = \max\{R, 2r\sin(\pi/k_n)\} \geqslant R,$$

by using Lemma 2.14. Thus $d_M(P_{k_i})$ equals R, for any $i \in \{1, \dots, n\}$.

Assume now that we have a chain of equalities in (3.2). Due to Lemma 2.14, we have that $d_M(P_{k_i}) = \max\{R, 2r\sin(\pi/k_i)\}$, $i = 1, \dots, n$. Taking into account that

$\sin(\pi/k_1) > \sin(\pi/k_2) > \ldots > \sin(\pi/k_n)$, the only admissible possibility in this case is $d_M(P_{k_i}) = R$, for any $i \in \{1,\ldots,n\}$. □

The previous Lemma 3.3 allows to find out the values of the minimal degree for which all equalities hold in (3.2). The following result shows that the above condition is satisfied when the minimal degree is greater than or equal to 6.

LEMMA 3.4. *Let C be a multi-rotationally symmetric planar convex body. If the minimal degree χ_C of C is greater than or equal to 6, then $d_M(P_{\chi_C}) = R$.*

Proof. Since $\chi_C \geqslant 6$, it is clear that $\sin(\pi/\chi_C) \leqslant \sin(\pi/6) = 1/2$, and so we have $2r\sin(\pi/\chi_C) \leqslant r \leqslant R$. Then, $d_M(P_{\chi_C}) = \max\{R, 2r\sin(\pi/\chi_C)\} = R$, as stated. □

REMARK 3.5. The proof of Lemma 3.4 also shows that $d_M(P_k) = R$ for any k-rotationally symmetric planar convex body with $k \geqslant 6$. This implies that equality signs will appear in (3.2) when $k_j \geqslant 6$.

We point out that, in view of Remark 2.8, the previous Lemma 3.4 applies in fact to $\chi_C \geqslant 7$, since 6 is not a prime number. On the other hand, when $\chi_C \leqslant 5$, we only have to analyze the cases $\chi_C = 3$ and $\chi_C = 5$, since 4 is not prime either. The following results show that we do not have a chain of equalities in (3.2) in any of these two cases.

LEMMA 3.6. *Let C be a multi-rotationally symmetric planar convex body, with minimal degree χ_C equal to 3. Then $d_M(P_{\chi_C}) \neq R$.*

Proof. Recall that, by Lemma 2.14, $d_M(P_{\chi_C}) = \max\{R, 2r\sin(\pi/\chi_C)\}$. Suppose that $d_M(P_{\chi_C}) = R$. Then $R \geqslant 2r\sin(\pi/\chi_C)$ and so $R/r \geqslant \sqrt{3}$.

Fix two consecutive endpoints v_1, v_2 of P_{k_C}, where k_C is the maximal degree of C, and let $x_R \in \partial C$ be a point with $d(p, x_R) = R$, which can be assumed to lie in the piece of ∂C delimited by v_1 and v_2. Let α_1 be the angle determined by the segments $\overline{pv_1}$ and $\overline{px_R}$, and α_2 the angle determined by $\overline{px_R}$ and $\overline{pv_2}$. Since $\alpha_1 + \alpha_2 = 2\pi/k_C$ due to the existing rotational symmetry, we can assume without loss of generality that $\alpha_1 \leqslant \pi/k_C$.

Let us now consider the triangle with vertices p, v_1, x_R, with associated angles α_1, β, γ, which will add up to π radians, see Figure 7.

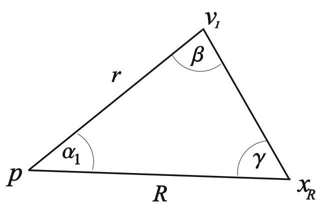

Figure 7: *Triangle with vertices p, v_1, x_R*

By using the sine's theorem we get

$$\sqrt{3} \leqslant \frac{R}{r} = \frac{\sin(\beta)}{\sin(\gamma)} \leqslant \frac{1}{\sin(\gamma)},$$

which gives $\sin(\gamma) \leqslant 1/\sqrt{3}$, and so $\gamma \leqslant \arcsin(1/\sqrt{3})$. Moreover, from Lemma 2.15 we have that $\beta \leqslant \pi/2$. Then

$$\pi = \alpha_1 + \beta + \gamma \leqslant \pi/k_C + \pi/2 + \arcsin(1/\sqrt{3}),$$

yielding

$$k_C \leqslant \frac{\pi}{\pi/2 - \arcsin(1/\sqrt{3})} < 4,$$

which is not possible, since there are no multiples of $\chi_C = 3$ satisfying that condition. Then $d_M(P_{\chi_C}) \neq R$, which finishes the proof. □

LEMMA 3.7. *Let C be a multi-rotationally symmetric planar convex body, with minimal degree χ_C equal to 5. Then $d_M(P_{\chi_C}) \neq R$.*

Proof. The proof is analogous to the one from Lemma 3.6, taking into account that now, by assuming $d_M(P_{\chi_C}) = R$, we shall get

$$R/r \geqslant 2\sin(\pi/5) = \sqrt{\frac{5-\sqrt{5}}{2}},$$

which finally yields $k_C < 6$, a contradiction. □

REMARK 3.8. The previous Lemmata 3.6 and 3.7 only hold for multi-rotationally symmetric sets. For an equilateral triangle T, which is *only* 3-rotationally symmetric, it can be checked that $d_M(P_3, T) = R$ (the same happens for a regular pentagon).

The following Theorem 3.9 summarizes the previous lemmata, characterizing when (3.2) is a chain of equalities provided the minimal degree is greater than or equal to 3.

THEOREM 3.9. *Let C be a multi-rotationally symmetric planar convex body with minimal degree $\chi_C \geqslant 3$. Then (3.2) is a chain of equalities if and only if $\chi_C \geqslant 7$.*

The characterization from Theorem 3.9 is stated in terms of the minimal degree. Regarding the maximal degree, we immediately obtain the following Theorem 3.10.

THEOREM 3.10. *Let C be a multi-rotationally symmetric planar convex body, with maximal degree k_C and minimal degree $\chi_C \geqslant 3$. Then, (3.2) is a chain of equalities if and only if k_C is a product of prime numbers (possibly repeated), all of them greater than or equal to 7.*

EXAMPLE 3.11. For instance, if a multi-rotationally symmetric planar convex body has minimal degree equal to 7, we will have a chain of equalities in (3.2), due to Theorem 3.9, and the admissible values for its maximal degree will be $7 \cdot 7 = 49$, $7 \cdot 11 = 77$, $7 \cdot 13 = 91$, and so on, in view of Theorem 3.10. Notice that, in particular, Theorem 3.10 implies that (3.2) will be a chain of equalities only for *relatively large* values of the associated maximal degree.

We finish this Subsection 3.1 by studying the following related question. Let C be a multi-rotationally symmetric planar convex body, with maximal degree k_C and minimal degree $\chi_C \geqslant 3$. Denote by $\{k_1 = \chi_C, \ldots, k_n = k_C\}$ the set of divisors of k_C. In this setting, we can search for the minimal value in $\{d_M(P_{k_1}), \ldots, d_M(P_{k_n})\}$, where P_{k_i} is the standard k_i-partition associated to C, which represents the *global minimal value* for the maximum relative diameter, taking into account [3, Th. 4.5]. Additionally, this will give, in some sense, a comparison among the *optimal* k_i-partitions of C in terms of our functional. From Lemma 3.1, it clearly follows that

$$\min\{d_M(P_{k_1}), \ldots, d_M(P_{k_n})\} = d_M(P_{k_n}),$$

but we can consider a further question: is there any other standard k_i-partition, apart from P_{k_n}, attaining also that minimum value? The following lemma answers this question.

LEMMA 3.12. *Let C be a multi-rotationally symmetric planar convex body, with maximal degree k_C and minimal degree $\chi_C \geqslant 3$. Let $\{k_1 = \chi_C, \ldots, k_n = k_C\}$ be the set of divisors of k_C, with $k_1 < \ldots < k_n$, and denote by P_{k_i} the standard k_i-partition of C, $i = 1, \ldots, n$. Then, the minimal value in*

$$\{d_M(P_{k_1}), \ldots, d_M(P_{k_n})\}$$

is uniquely attained by $d_M(P_{k_n})$ if and only if $d_M(P_{k_{n-1}}) \neq R$.

Proof. Taking into account Lemma 3.1, the considered minimal value is uniquely attained by $d_M(P_{k_n})$ if and only if $d_M(P_{k_{n-1}}) > d_M(P_{k_n})$. Recall that $d_M(P_k) = \max\{R, 2r\sin(\pi/k)\}$, for any $k \in \{k_1, \ldots, k_n\}$, due to Lemma 2.14.

Assume firstly that $d_M(P_{k_{n-1}}) \neq R$. Then $d_M(P_{k_{n-1}}) = 2r\sin(\pi/k_{n-1}) > R$. Since $2r\sin(\pi/k_{n-1}) > 2r\sin(\pi/k_n)$, both inequalities yield that

$$d_M(P_{k_{n-1}}) = 2r\sin(\pi/k_{n-1}) > \max\{R, 2r\sin(\pi/k_n)\} = d_M(P_{k_n}),$$

as desired.

Assume now that $d_M(P_{k_{n-1}}) = R$. Then $R \geqslant 2r\sin(\pi/k_{n-1}) > 2r\sin(\pi/k_n)$, and so $d_M(P_{k_n}) = \max\{R, 2r\sin(\pi/k_n)\} = R$, which implies that the referred uniqueness does not hold. □

3.2. Minimal degree equal to 2

If we consider a multi-rotationally symmetric planar convex body C with minimal degree equal to 2, the situation is different from the one corresponding to Subsection 3.1. The reason is that C is, in particular, 2-rotationally symmetric, and so the *optimal* 2*-partition* for the maximum relative diameter is not completely characterized in this case. It is proved in [6] that a minimizing 2-partition (into two subsets of equal areas) will consist of a line segment passing through the center of symmetry of the set, but a more precise description is not known. In fact, the corresponding standard 2-partition is not minimizing in general, see Figure 8.

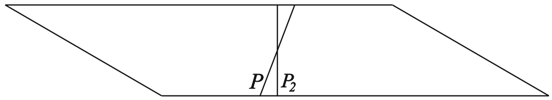

Figure 8: *An example where the standard 2-partition P_2 is not minimizing for d_M, since $d_M(P_2) > d_M(P)$, where P is obtained by a slight rotation of P_2*

Therefore, in this case we cannot make the discussion on the *best values* for the maximum relative diameter functional, as in Subsection 3.1. At least, we can study our problem partially, investigating the relation among the values

$$\{d_M(P_{k_1}),\dots,d_M(P_{k_n})\},$$

where $\{k_1 = 2,\dots,k_n\}$ is the set of divisors of the associated maximal degree of C, taking into account that $d_M(P_{k_1})$ is not optimal in general, and cannot be computed using Lemma 2.14. The existing relation is given by the following result.

LEMMA 3.13. *Let C be a multi-rotationally symmetric planar convex body, with maximal degree $k_C \in \mathbb{N}$ and minimal degree $\chi_C = 2$. Let $\{k_1 = \chi_C, k_2,\dots,k_n = k_C\}$ be the set of divisors of k_C, with $k_1 < k_2 < \dots < k_n$. Then*

$$d_M(P_2) > d_M(P_{k_2}) \geqslant \dots \geqslant d_M(P_{k_n}),$$

where P_k is the standard k-partition associated to C.

Proof. It suffices to prove that $d_M(P_2) > d_M(P_{k_2})$, in view of Lemma 3.1. Let v_1, v_2 be the endpoints of the standard 2-partition P_2. Recall that, by Lemma 2.14, $d_M(P_{k_2}) = \max\{R, 2r\sin(\pi/k_2)\}$. We shall distinguish two cases:

If $d_M(P_{k_2}) = R$, call $x_R \in \partial C$ such that $d(p,x_R) = R$, which can be assumed to be different from an endpoint of P_2 (otherwise $R = r$, being C a circle, for which the maximal degree is not formally defined). Let α_1 be the angle at p determined by the line segments $\overline{pv_1}$ and $\overline{px_R}$, and α_2 the angle determined by $\overline{px_R}$ and $\overline{pv_2}$. Clearly $\alpha_1 + \alpha_2 = \pi$, and so we can assume that $\alpha_1 \geqslant \pi/2$. Then

$$d(v_1,x_R)^2 = d(p,x_R)^2 + d(p,v_1)^2 - 2\,d(p,x_R)\,d(p,v_1)\cos(\alpha_1) > d(p,x_R)^2,$$

yielding

$$d_M(P_2) \geqslant d(v_1,x_R) > d(p,x_R) = d_M(P_{k_2}),$$

as desired.

On the other hand, if $d_M(P_{k_2}) = 2r\sin(\pi/k_2)$, since $k_2 > 2$, we have that

$$d_M(P_{k_2}) = 2r\sin(\pi/k_2) < 2r\sin(\pi/2) = 2r = d(v_1,v_2) \leqslant d_M(P_2),$$

which proves the statement. $\square$

REMARK 3.14. As a consequence of Lemma 3.13, we have that for any multi-rotationally symmetric planar convex body C with minimal degree equal to 2, the corresponding values

$$\{d_M(P_{k_1}),\ldots,d_M(P_{k_n})\},$$

where $\{k_1=2,\ldots,k_n\}$ is the set of divisors of the associated maximal degree of C, will not *all* coincide in any case. In fact, it follows from the proof of Lemma 3.13 that $d_M(P_2)>R$, and so the previous Lemma 3.3 cannot be applied in this situation.

REMARK 3.15. For a circle $\mathscr{C}$, whose minimal degree is equal to 2, straightforward computations using Lemma 2.14 give that

$$d_M(P_2)>d_M(P_3)>d_M(P_4)>d_M(P_5)>d_M(P_6)=d_M(P_k),$$

for any $k\in\mathbb{N}$, $k\geqslant 7$.

REMARK 3.16. For a given set C of our family, with minimal degree equal to 2, we cannot discuss which is the *global minimum value* for the maximum relative diameter in this setting, as in Subsection 3.1, since we cannot estimate

$$\min\{d_M(P,C): P \text{ is a 2-partition of } C\}.$$

Recall that such a minimum is not necessarily provided by the standard 2-partition, as the example from Figure 8 shows.

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