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OPERATOR VERSIONS OF SHANNON TYPE INEQUALITY

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Abstract. In this paper, we present some refinements and precise estimations of parametric extensions of Shannon inequality and its reverse one given by Furuta in Hilbert space operators. We also demonstrate an extension of operator Shannon type inequality.

1. Introduction and preliminaries

Various generalizations of the Shannon inequalities have played an important role in classical information theory. It has been discovered that many of these inequalities have operator generalizations, in which one replaces random variables by Hilbert space operators. The latter are the variables of quantum thermodynamics and quantum information theory. Yanagi et al. proved in [19] some generalized Shannon inequalities. Some other operator inequalities related to Tsallis relative operator entropy were also proved in [10] and [16].

The notion of entropy was introduced in thermodynamics by Clausius in 1850 [3], and some of the main steps towards the consolidation of the concept were taken by Boltzmann and Gibbs. Since then several extensions and reformulations have been developed in various disciplines with motivations and applications in different subjects, such as statistical mechanics, information theory, dynamical systems and ergodic theory, biology, economics, human and social sciences; cf. [15, 13, 14, 18]. There have been investigated the so-called entropy inequalities by some mathematicians, see [1, 8] and references therein.

The generalized relative operator entropy for strictly positive operators A,B and $q \in \mathbb{R}$ defined in [7] by setting

$$S_q(A|B) = A^{1/2}(A^{-1/2}BA^{-1/2})^q(\log A^{-1/2}BA^{-1/2})A^{1/2}.$$

In particular, when $q = 0$, we reach the relative operator entropy defined in [9] as a generalization of the operator entropy as follows:

$$S(A|B) := A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

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Effros [5] considered an operator version of perspective of functions for commuting operators. We introduced in [4] a fully noncommutative generalized perspective of two variables (associated to f and h), by choosing an appropriate ordering. In the notation introduced in [6] we can write:

$$P_{k\Delta h}(A,B) := h(A)^{1/2}k(h(A)^{-1/2}Bh(A)^{-1/2})h(A)^{1/2},$$

where A is a strictly positive matrix and B is a self-adjoint matrix with spectra in the closed interval $\mathbb{J}$ containing 0. We then proved the necessary and sufficient conditions for jointly convexity of a fully noncommutative perspective and generalized perspective function (see also [17]).

Note that the relative operator entropy $S(A|B)$ is perspective of $\log t$ in the sense that $S(A|B) = P_{\log t}(A|B)$ and the generalized relative operator entropy $S_q(A|B)$ is perspective of $t^q \log t$ in the sense that $S_q(A|B) = P_{t^q \log t}(A|B)$.

Throughout the paper the symbol $B(\mathscr{H})$ stands for the C^*-algebra of all bounded linear operators on Hilbert space $\mathscr{H}$ with inner product $\langle \cdot, \cdot \rangle$. A self-adjoint operator A in $B(\mathscr{H})$ is said to be positive, written $A \geqslant 0$, if $\langle Ah, h \rangle \geqslant 0$ for $h \in \mathscr{H}$. If moreover A is invertible, then A is said to be strictly positive, written $A > 0$. For self-adjoint operators A and B in $B(\mathscr{H})$, we write $A \geqslant B$ (resp. $A > B$) if $A - B$ is positive (resp. strictly positive).

A continuous function $f : \mathbb{J} \to \mathbb{R}$ on an interval $\mathbb{J} \subset \mathbb{R}$ is called operator convex if

$$f(\alpha A + (1-\alpha)B) \leqslant \alpha f(A) + (1-\alpha) f(B)$$

for all $\alpha \in [0,1]$ and every self-adjoint operators $A, B \in B(\mathscr{H})$ with spectra $\sigma(A), \sigma(B)$ contained in $\mathbb{J}$. A continuous function $f : \mathbb{J} \to \mathbb{R}$ on an interval $\mathbb{J} \subset \mathbb{R}$ is called operator concave if $-f$ is operator convex.

Furuta [7] obtained the following parametric extensions of Shannon inequality:

$$\begin{aligned}\sum_{j=1}^{n} S_{p+1}(A_j|B_j) &\geqslant V_p(1)\log V_p(1) \geqslant \log V_p(1)\\ &\geqslant \sum_{j=1}^{n} S_p(A_j|B_j) \geqslant -\log W_p(1) \geqslant -W_p(1)\log W_p(1)\\ &\geqslant \sum_{j=1}^{n} S_{p-1}(A_j|B_j).\end{aligned}$$

In this paper, we provide a refinement of Furuta's operator extension of Shannon's inequality as follows:

$$\begin{aligned}\sum_{j=1}^{n} S_{p+1}(A_j|B_j) \geqslant \sum_{j=1}^{n} S_{p+q}(A_j|B_j) &\geqslant V_p(1)^q \log V_p(1) \geqslant \log V_p(1)\\ &\geqslant \sum_{j=1}^{n} S_p(A_j|B_j) \geqslant -\log W_p(1) \geqslant -W_p(1)^q \log W_p(1)\\ &\geqslant \sum_{j=1}^{n} S_{p-q}(A_j|B_j) \geqslant \sum_{j=1}^{n} S_{p-1}(A_j|B_j).\end{aligned}$$

2. Refinements and precise estimations of operator Shannon type inequality

Let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathscr{H}$ such that $\sum_{j=1}^{n} A_j \#_p B_j \leqslant I$, where $0 \leqslant p \leqslant 1$ and I means the identity operator on $\mathscr{H}$. Throughout this paper, for the sake of simplified writing, we define

$$U_p := I - \sum_{j=1}^{n} A_j \#_p B_j,$$

$$V_p(t_0) := \sum_{j=1}^{n} A_j \#_{p+1} B_j + t_0 U_p,$$

$$W_p(t_0) := \sum_{j=1}^{n} A_j \#_{p-1} B_j + t_0 U_p$$

for fixed real number $t_0 > 0$, where $A \#_p B = A^{\frac{1}{2}}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^p A^{\frac{1}{2}}$ for $A, B > 0$. Furuta [7, Theorem 2.1] stated the following parametric extensions of Shannon inequality and its reverse one derived from operator concave function $f(t) = \log t$ for $0 \leqslant p \leqslant 1$:

$$\log V_p(t_0) - (\log t_0) U_p \geqslant \sum_{j=1}^{n} S_p(A_j | B_j) \geqslant -\log W_p(t_0) + (\log t_0) U_p. \tag{1}$$

He derived the following parametric extensions of Shannon inequality and its reverse one from operator concave function $f(t) = -t \log t$ for $0 \leqslant p \leqslant 1$:

$$\sum_{j=1}^{n} S_{p+1}(A_j | B_j) \geqslant V_p(t_0) \log V_p(t_0) - (t_0 \log t_0) U_p, \tag{2}$$

$$\sum_{j=1}^{n} S_{p-1}(A_j | B_j) \leqslant -W_p(t_0) \log W_p(t_0) + (t_0 \log t_0) U_p. \tag{3}$$

He then concluded that

$$\begin{aligned}\sum_{j=1}^{n} S_{p+1}(A_j | B_j) &\geqslant V_p(1) \log V_p(1) \geqslant \log V_p(1) \\ &\geqslant \sum_{j=1}^{n} S_p(A_j | B_j) \geqslant -\log W_p(1) \geqslant -W_p(1) \log W_p(1) \\ &\geqslant \sum_{j=1}^{n} S_{p-1}(A_j | B_j)\end{aligned} \tag{4}$$

by letting $t_0 = 1$ in the inequalities (1), (2), and (3) and using the inequality $A \log A \geqslant \log A$ for every $A > 0$.

We present some refinements of parametric extensions of Shannon inequality and its reverse one given by Furuta in Hilbert space operators.

Furuta in [7, Proposition 3.1] provided the following result for an operator concave function. We state the result for an operator convex function. Note that this result is just a trivial and parallel consequence of Jensen's operator inequality [12].

PROPOSITION 1. *If f is a continuous, real function on an interval J, the following conditions are equivalent:*

(i) *f is operator convex;*

(ii) *$f(C^*AC+t_0(I-C^*C) \leqslant C^*f(A)C+f(t_0)(I-C^*C)$ for operator C with $||C|| \leqslant 1$ and self-adjoint operator A with spectra in J and for fixed real number $t_0 \in J$;*

(iii) *$f(\sum_{j=1}^n C_j^*A_jC_j + t_0(I - \sum_{j=1}^n C_j^*C_j) \leqslant \sum_{j=1}^n C_j^* f(A_j)C_j + f(t_0)(I - \sum_{j=1}^n C_j^*C_j)$ for operator C_j with $\sum_{j=1}^n C_j^*C_j \leqslant I$ and self-adjoint operator A_j with spectra in J for $j = 1,...,n$ and for fixed real number $t_0 \in J$.*

We remark that the function $f(t) = t\log t$ is operator convex on $[0,\infty)$. In fact, it is equal to the perspective of operator convex function $k(t) = -\log t$, i.e.,

$$f(A) = A\log A = -A\log A^{-1} = A^{\frac{1}{2}}(-\log A^{-\frac{1}{2}}A^{-\frac{1}{2}})A^{\frac{1}{2}} = P_k(A,1).$$

The result comes from the operator convexity of the function k and [4, Theorem 2.2]. We are going to show that the function $f(t) = t^q\log t$ is operator convex on $[0,\infty)$ for $0 < \delta \leqslant q \leqslant 1$.

LEMMA 1. *If f is an operator monotone function on $[0,\infty)$ such that $f(0) \leqslant 0$ and $\lim_{t\to\infty} \frac{f(t)}{t} = 0$, then $t^q f(t)$ is operator convex for $0 < \delta \leqslant q \leqslant 1$.*

Proof. The operator monotone function f on $[0,\infty)$ can be represented as

$$f(t) = f(0) + \beta t + \int_0^\infty \frac{\lambda t}{\lambda + t} d\mu(\lambda),$$

where $\beta \geqslant 0$ and μ is a positive measure on $[0,\infty)$; see [2, Chapter V]. Since $\lim_{t\to\infty} \frac{f(t)}{t}$ $= 0$, $\beta = 0$. So by multiplying both sides to t^q we have

$$t^q f(t) = f(0)t^q + \int_0^\infty \frac{\lambda t^{1+q}}{\lambda + t} d\mu(\lambda).$$

The function $f(0)t^q$ is operator convex. Indeed, it is sufficient to prove that the function $\frac{\lambda t^{1+q}}{\lambda+t}$ is operator convex. Define $g(t) = t^q$ and consider two cases:

(i) For $\lambda > 1$ the function $h(t) = \frac{t+1/\lambda}{t} g(t)$ is operator monotone by [2, Corollary V.3.12]. So [12, Theorem 2.4] and [2, Problem V.5.7] show that the function $th(t^{-1})^{-1}$ is operator convex.

(ii) For $0 < \lambda < 1$ the function $h_1(t) = \frac{t+\lambda}{t} g(t)$ is operator monotone by [2, Corollary V.3.12]. So [2, Problem V.5.7] entails that the function $h(t) = h_1(\lambda t^{-1})^{-1}$ is operator monotone. This implies the function $h(\frac{t}{\lambda})$ is also operator monotone. Therefore, the function $\lambda^{2q} th(\frac{t}{\lambda})$ is operator convex by [12, Theorem 2.4]. □

LEMMA 2. *The function $f(t) = t^q \log t$ is operator convex on $[0,\infty)$ for $0 < \delta \leqslant q \leqslant 1$.*

Proof. Let $\varepsilon \in (0,1)$. Then, the function $\log(t+\varepsilon)$ satisfies in the assumptions of Lemma 1 and so $t^q \log(t+\varepsilon)$ is operator convex. Hence, when $\varepsilon \to 0$ we see that the function $f(t) = t^q \log t$ is operator convex for $0 < \delta \leqslant q \leqslant 1$. $\square$

THEOREM 1. *Let $0 \leqslant p \leqslant 1$, $0 < \delta \leqslant q \leqslant 1$ and also let $\{A_1, A_2, \dots, A_n\}$ and $\{B_1, B_2, \dots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathscr{H}$ such that $\sum_{j=1}^n A_j \#_p B_j \leqslant I$. Then,*

$$\sum_{j=1}^n S_{p+q}(A_j|B_j) \geqslant V_p(t_0)^q \log V_p(t_0) - (t_0^q \log t_0) U_p \tag{5}$$

for fixed real number $t_0 > 0$ and

$$\sum_{j=1}^n S_{p-q}(A_j|B_j) \leqslant -W_p(t_0)^q \log W_p(t_0) + (t_0^q \log t_0) U_p \tag{6}$$

for fixed real number $t_0 > 0$.

Proof. We apply Proposition 1 (iii) to state the following inequality for any strictly positive operator $T_j > 0$ for $j = 1, 2, \dots, n$ and for any fixed real number $t_0 > 0$, since by Lemma 2 the function $t^q \log t$ is operator convex for $t > 0$ and $0 < \delta \leqslant q \leqslant 1$:

$$\begin{aligned}\Big(\sum_{j=1}^n C_j^* T_j C_j + t_0(I - \sum_{j=1}^n C_j^* C_j)\Big)^q \log(\sum_{j=1}^n C_j^* T_j C_j + t_0(I - \sum_{j=1}^n C_j^* C_j))&\\ \leqslant \sum_{j=1}^n C_j^* T_j^q (\log T_j) C_j + t_0^q \log t_0 (I - \sum_{j=1}^n C_j^* C_j),&\end{aligned} \tag{7}$$

where $\sum_{j=1}^n C_j^* C_j \leqslant I$. Note that $\sum_{j=1}^n C_j^* C_j = \sum_{j=1}^n A_j \#_p B_j$ and hence $\sum_{j=1}^n A_j \#_p B_j \leqslant I$. Set $T_j = (A_j^{-1/2} B_j A_j^{-1/2})^\alpha$ for a real number α and $C_j = (A_j^{-1/2} B_j A_j^{-1/2})^{\frac{p}{2}} A_j^{1/2}$ in (7). Then, (7) entails that

$$\begin{aligned}&\Big(\sum_{j=1}^n A_j^{1/2}(A_j^{-1/2} B_j A_j^{-1/2})^{p+\alpha} A_j^{1/2} + t_0(I - \sum_{j=1}^n A_j \#_p B_j)\Big)^q\\ &\quad \times \log(\sum_{j=1}^n A_j^{1/2}(A_j^{-1/2} B_j A_j^{-1/2})^{p+\alpha} A_j^{1/2} + t_0(I - \sum_{j=1}^n A_j \#_p B_j))\\ &\leqslant \sum_{j=1}^n A_j^{1/2}(A_j^{-1/2} B_j A_j^{-1/2})^{p+\alpha q}\Big(\log(A_j^{-1/2} B_j A_j^{-1/2})^\alpha\Big) A_j^{1/2}\\ &\quad + t_0^q \log t_0 (I - \sum_{j=1}^n A_j \#_p B_j).\end{aligned}$$

Hence,

$$\Big(\sum_{j=1}^{n} A_j\#_{p+\alpha}B_j + t_0(I-\sum_{j=1}^{n} A_j\#_p B_j)\Big)^q \log(\sum_{j=1}^{n} A_j\#_{p+\alpha}B_j + t_0(I-\sum_{j=1}^{n} A_j\#_p B_j))$$
$$\leqslant \alpha\sum_{j=1}^{n} S_{p+\alpha q}(A_j|B_j) + t_0^q \log t_0(I-\sum_{j=1}^{n} A_j\#_p B_j). \tag{8}$$

Setting $\alpha = 1$ and $\alpha = -1$ in (8) respectively and using our notation, we have

$$V_p(t_0)^q \log V_p(t_0) \leqslant \sum_{j=1}^{n} S_{p+q}(A_j|B_j) + (t_0^q \log t_0)U_p$$

for fixed real number $t_0 > 0$ and

$$-W_p(t_0)^q \log W_p(t_0) \geqslant \sum_{j=1}^{n} S_{p-q}(A_j|B_j) - (t_0^q \log t_0)U_p.$$

for fixed real number $t_0 > 0$. □

Applying Theorem 1 we give the following result:

COROLLARY 1. *Let $0 \leqslant p \leqslant 1$, $0 < \delta \leqslant q \leqslant 1$ and also let $\{A_1, A_2, \ldots, A_n\}$ and $\{B_1, B_2, \ldots, B_n\}$ be two sequences of strictly positive operators on a Hilbert space $\mathscr{H}$ such that $\sum_{j=1}^{n} A_j\#_p B_j \leqslant I$. Then,*

$$\begin{aligned}\sum_{j=1}^{n} S_{p+1}(A_j|B_j) &\geqslant \sum_{j=1}^{n} S_{p+q}(A_j|B_j)\\ &\geqslant V_p(t_0)^q \log V_p(t_0) - (t_0^q \log t_0)U_p\\ &\geqslant \log V_p(t_0) - (t_0 \log t_0)U_p, \qquad (9)\\ \sum_{j=1}^{n} S_{p-1}(A_j|B_j) &\leqslant \sum_{j=1}^{n} S_{p-q}(A_j|B_j)\\ &\leqslant -W_p(t_0)^q \log W_p(t_0) + (t_0^q \log t_0)U_p\\ &\leqslant -\log W_p(t_0) + (t_0 \log t_0)U_p \qquad (10)\end{aligned}$$

for fixed real number $t_0 > 0$.

Proof. The function $\Gamma_A(x) = A^x \log A$ is an increasing function of x for every $A > 0$. On the other hand, $S_q(A|B) = A^{1/2}\Gamma_{A^{-1/2}BA^{-1/2}}(q)A^{1/2}$ and so the generalized relative operator entropy is an increasing function of q for every $A, B > 0$. The results now follow from Theorem 1 and the fact that the function $\Gamma_A(x)$ is increasing. □

REMARK 1. We note that the inequalities (9) and (10) recover the inequalities (2) and (3) proved by Furuta in [7, Theorem 2.2], if we put $q = 1$.

Utilizing inequalities (9) and (10) for the special case $t_0 = 1$, we get the following result. This is indeed a generalization and refinement of inequality (4) and its reverse one proved by Furuta in Hilbert space operators [7, Corollary 2.3], that is, we could obtain more precise estimation (11) than (4) thanks to operator convexity of the function $f(t) = t^q \log t$ for $0 < \delta \leqslant q \leqslant 1$.

COROLLARY 2. *Let* $0 \leqslant p \leqslant 1, 0 < \delta \leqslant q \leqslant 1$ *and also let* $\{A_1, A_2, \ldots, A_n\}$ *and* $\{B_1, B_2, \ldots, B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* $\mathscr{H}$ *such that* $\sum_{j=1}^{n} A_j \#_p B_j \leqslant I$. *Then,*

$$\begin{aligned}\sum_{j=1}^{n} S_{p+1}(A_j|B_j) &\geqslant \sum_{j=1}^{n} S_{p+q}(A_j|B_j) \geqslant V_p(1)^q \log V_p(1) \geqslant \log V_p(1) \\ &\geqslant \sum_{j=1}^{n} S_p(A_j|B_j) \geqslant -\log W_p(1) \geqslant -W_p(1)^q \log W_p(1) \\ &\geqslant \sum_{j=1}^{n} S_{p-q}(A_j|B_j) \geqslant \sum_{j=1}^{n} S_{p-1}(A_j|B_j). \end{aligned} \tag{11}$$

Proof. Since the generalized relative operator entropy $S_x(A,B)$ is an increasing function of x for every $A, B > 0$, the first inequality and eighth one in (11) hold. Let $t_0 = 1$ in (5) and (6) to obtain the second and seventh inequalities in (11). Since the function $\Gamma_A(x)$, defined in the proof of Corollary 1, is increasing on x for every $A > 0$, the third and sixth inequalities in (11) are true. The inequality (1) entails the fourth and fifth inequalities in (11) by putting $t_0 = 1$. $\square$

The original Shannon inequality and its reverse one state that

$$0 \geqslant \sum_{j=1}^{n} a_j \log \frac{b_j}{a_j} \geqslant -\log \sum_{j=1}^{n} \frac{a_j^2}{b_j}$$

for two probability vectors $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ with $\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} b_j = 1$ and $a_j, b_j > 0$. Furuta in [7, Corollary 2.4] gave an operator version of Shannon inequality and its reverse one. We now give a refinement of the operator version of Shannon inequality and its reverse one.

COROLLARY 3. *Let* $\{A_1, A_2, \ldots, A_n\}$ *and* $\{B_1, B_2, \ldots, B_n\}$ *be two sequences of strictly positive operators on a Hilbert space* $\mathscr{H}$ *such that* $\sum_{j=1}^{n} A_j \leqslant I$ *and* $\sum_{j=1}^{n} B_j \leqslant I$ *and let* $0 < \delta \leqslant q \leqslant 1$. *Then,*

$$\begin{aligned}\sum_{j=1}^{n} S_2(A_j|B_j) &\geqslant \sum_{j=1}^{n} S_{q+1}(A_j|B_j) \\ &\geqslant \Big(\sum_{j=1}^{n} B_j A_j^{-1} B_j\Big)^q \log\Big(\sum_{j=1}^{n} B_j A_j^{-1} B_j\Big) \geqslant \log\Big(\sum_{j=1}^{n} B_j A_j^{-1} B_j\Big)\end{aligned}$$

$$\begin{aligned}
&\geqslant \sum_{j=1}^{n} S_1(A_j|B_j) \geqslant \sum_{j=1}^{n} S_q(A_j|B_j) \geqslant 0 \geqslant \sum_{j=1}^{n} S_{1-q}(A_j|B_j) \geqslant \sum_{j=1}^{n} S(A_j|B_j) \\
&\geqslant -\log\Big(\sum_{j=1}^{n} B_jA_j^{-1}B_j\Big) \geqslant -\Big(\sum_{j=1}^{n} B_jA_j^{-1}B_j\Big)^q \log\Big(\sum_{j=1}^{n} B_jA_j^{-1}B_j\Big) \\
&\geqslant \sum_{j=1}^{n} S_{-q}(A_j|B_j) \geqslant \sum_{j=1}^{n} S_{-1}(A_j|B_j). \qquad (12)
\end{aligned}$$

Proof. Set $p = 0$ and $p = 1$ in Corollary 2. Note that $U_0 = U_1 = 0$, $V_1(1) = W_0(1) = \sum_{j=1}^{n} B_jA_j^{-1}B_j$, and $V_0(1) = W_1(1) = I$. Combining the resulting inequalities, we have (12). □

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