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ALMOST EVERYWHERE CONVERGENCE OF DYADIC TRIANGULAR–FEJÉR MEANS OF TWO–DIMENSIONAL WALSH–FOURIER SERIES

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Abstract. It is proved that the maximal operators of the dyadic triangular-Fejér means of two-dimensional Walsh-Fourier series is of weak type (1,1). Moreover, the dyadic triangular-Fejér means of the function $f \in L_1$ converge almost everywhere to f as $n \to \infty$.

1. Introduction

Lebesgue's [13] theorem is well known for trigonometric Fourier series: the Fejér means $\sigma_n f$ of f converge to f almost everywhere if $f \in L_1(\mathbb{T})$, $\mathbb{T} := [-\pi, \pi)$ (see also Zygmund [26]).

An analogous result for Walsh–Fourier series is due to Fine [1]. Later, Schipp [16] showed that the maximal operator σ^* of the Fejér means of the one-dimensional Walsh–Fourier series is of weak type (1,1), from which the a.e. convergence follows by standard arguments. Schipp's result implies by interpolation also the boundedness of $\sigma^* : L_p(G) \to L_p(G)$, where $1 < p \leqslant \infty$. This fails to hold for $p = 1$, but Fujii [2] proved that σ^* is bounded from the dyadic Hardy space $H_1(G)$ to the space $L_1(G)$ (see also Simon [18]). Fujii's theorem was extended by Weisz [20]. Namely, he proved that σ^* is bounded from the martingale Hardy space $H_p(G)$ to the space $L_p(G)$ for $p > 1/2$. Simon [19] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$. In the endpoint case $p = 1/2$, Weisz [22] proved that σ^* is bounded from the Hardy space $H_{1/2}(G)$ to the space weak-$L_{1/2}(G)$. Goginava proved in [8, 9] that the maximal operator of the Fejér means of the one dimensional Walsh–Fourier series is not bounded from the Hardy space $H_{1/2}(G)$ to the space $L_{1/2}(G)$.

Marcinkiewicz [14] verified for two-dimensional trigonometric Fourier series that the Marcinkiewicz-Fejér means

$$\sigma_n^{\square} f = \frac{1}{n} \sum_{j=0}^{n-1} S_j^{\square}(f)$$

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of a function $f \in L\log L(\mathbb{T}\times\mathbb{T})$ converge a.e. to f as $n\to\infty$, where $S_j^{\square}(f)$ denotes the quadratical partial sums of the Fourier series of f. Later, Zhizhiashvili [24, 25] extended this result to all $f \in L_1(\mathbb{T}\times\mathbb{T})$.

An analogous result for two-dimensional Walsh–Fourier series is due to Weisz [21]. Moreover, he proved that the maximal operator $\sigma_*^{\square} f = \sup\limits_{n\geqslant 1}|\sigma_n^{\square} f|$ is bounded from the dyadic martingale Hardy space $H_p(G\times G)$ to the space $L_p(G\times G)$ for $p>2/3$. The second author [11, 8, 9] proved that the maximal operator $\sigma_*^{\square}$ is bounded from $H_{2/3}(G\times G)$ to weak$-L_{2/3}(G\times G)$ and is not bounded from $H_{2/3}(G\times G)$ to $L_{2/3}(G\times G)$. The first author gave [5] the following necessary and sufficient condition for a.e. convergence of generalized Marcinkiewicz means of two-dimensional integrable functions. Let $\alpha=(\alpha_1,\alpha_2):\mathbb{N}^2\to\mathbb{N}^2$ be a function. Define the following Marcinkiewicz-like means:

$$\sigma_n^{\square,\alpha} f = \frac{1}{n}\sum_{j=0}^{n-1} S^{\square}_{\alpha(|n|,j)}(f).$$

The following properties will play a prominent role in the a.e. convergence behavior of $\sigma_n^{\square,\alpha}$. ($\#B$ denotes the cardinality of set B and $|n|$ is the integer part of $\log_2 n$.) Let $\mathbb{P}=\mathbb{N}\setminus\{0\}$ denote the set of positive integers.

$$\#\{l\in\mathbb{N}:\alpha_j(|n|,l)=\alpha_j(|n|,k), l<n\}\leqslant C \quad (k<n, n\in\mathbb{P},\ j=1,2), \tag{1}$$

$$\max\{\alpha_j(|n|,k):k<n\}\leqslant Cn \quad (n\in\mathbb{P}, j=1,2). \tag{2}$$

Conditions (1) and (2) are necessary and sufficient in the following sense. In [5] the first author proved: Let α satisfy (1) and (2). Then we have the a.e. convergence

$$\sigma_n^{\square,\alpha} f\to f$$

for every $f\in L_1(G\times G)$. It is clear that Condition (1) is necessary, and so does Condition (2) in the following sense (see also [5]). Let $\gamma:\mathbb{N}\to\mathbb{N}$ be any function with property $\gamma(+\infty)=+\infty$. Then there exists a function α satisfying (1) and

$$\max\{\alpha_1(|n|,k):k<n\}\leqslant Cn,\ \max\{\alpha_2(|n|,k):k<n\}\leqslant Cn\gamma(n) \quad (n\in\mathbb{P})$$

and $f\in L_1(G\times G)$ such that $\limsup_{n\in\mathbb{N}}|\sigma_n^{\square,\alpha} f|=+\infty$ almost everywhere.

Weisz [23] studied the triangular partial sums and the Fejér means

$$\sigma_n^{\triangle} f = \frac{1}{n}\sum_{j=0}^{n-1} S_j^{\triangle} f$$

of the two-dimensional trigonometric Fourier series. This summability method is rarely investigated in the literature (see the references in [23]). In [12] Goginava and Weisz proved that the maximal operator $\sigma_{\#}^{\triangle} := \sup\limits_n \left|\sigma_{2^n}^{\triangle} f\right|$ of the Fejér means of the triangular partial sums of the double Walsh–Fourier series is bounded from the dyadic Hardy space $H_p(G\times G)$ to the $L_p(G\times G)$ if $p>1/2$, is bounded from $H_{1/2}(G\times G)$ to the space weak- $L_{1/2}(G\times G)$ and it is not bounded from $H_{1/2}(G\times G)$ to $L_{1/2}(G\times G)$.

For triangular partial sums of two-dimensional Walsh-Fourier series it is well-known [17] the operatos $S_{2^A}^{\triangle}$ are not uniformly bounded on L_p for $1\leqslant p\neq 2$.

In [6] Gát and Goginava proved that the operators $\sigma_n^{\triangle}$ of the triangular-Fejér-means of a two-dimensional Walsh–Fourier series are uniformly bounded from the dyadic Hardy space H_p to L_p for all $4/5<p\leqslant\infty$.

In this paper we prove that the maximal operators of the dyadic triangular-Fejér means of two-dimensional Walsh-Fourier series is of weak type (1,1). Moreover, the dyadic triangular-Fejér means of the function $f\in L_1$ converge almost everywhere to f as $n\to\infty$.

The results for summability of quadratical partial sums of two-dimensional Walsh-Fourier series can be found in [7, 10, 3, 4, 5].

2. Definitions and the notation

Let $\mathbb{P}$ denote the set of positive integers, $\mathbb{N}:=\mathbb{P}\cup\{0\}$. Denote by Z_2 the discrete cyclic group of order 2, that is $Z_2=\{0,1\}$, where the group operation is the modulo 2 addition and every subset is open. A Haar measure on Z_2 is given such that the measure of a singleton is 1/2. Let G be the complete direct product of the countable infinite copies of the compact groups Z_2. The elements of G are of the form $x=(x_0,x_1,\dots,x_k,\dots)$ with $x_k\in\{0,1\}\,(k\in\mathbb{N})$. The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given by

$$I_n(x):=I_n(x_0,\dots,x_{n-1}):=\{y\in G: y=(x_0,\dots,x_{n-1},y_n,y_{n+1},\dots)\},$$

where $I_0(x):=G$ and $x\in G, n\in\mathbb{N}$. These sets are called the dyadic intervals. Let $0=(0:i\in\mathbb{N})\in G$ denote the null element of G, $I_n:=I_n(0)\ (n\in\mathbb{N})$, $\overline{I}_n:=G\backslash I_n$.

For $k\in\mathbb{N}$ and $x\in G$ denote

$$r_k(x):=(-1)^{x_k}\quad (x\in G,\ k\in\mathbb{N})$$

the k-th Rademacher function. If $n\in\mathbb{N}$, then $n=\sum_{i=0}^{\infty}n_i2^i$, where $n_i\in\{0,1\}\ (i\in\mathbb{N})$, i.e. n is expressed in the number system of base 2. Denote $|n|:=\max\{j\in\mathbb{N}: n_j\neq 0\}$, that is, $2^{|n|}\leqslant n<2^{|n|+1}$. Set

$$n_{(s)}:=\sum_{k=0}^{s}n_k2^k, n^{(s)}:=\sum_{k=s}^{\infty}n_k2^k.$$

Define the dyadic addition of natural numbers n and j as

$$n\oplus j:=\sum_{k=0}^{\infty}|n_k-j_k|.$$

Also use the notation $n\wedge j=\min\{n,j\}$. The Walsh–Paley system is defined as the sequence of Walsh–Paley functions:

$$w_n(x):=\prod_{k=0}^{\infty}(r_k(x))^{n_k}=(-1)^{\sum_{k=0}^{|n|}n_kx_k}\quad (x\in G,\ n\in\mathbb{P}).$$

The Walsh–Dirichlet kernel is defined by

$$D_n(x)=\sum_{k=0}^{n-1}w_k(x),\quad D_0(x)=0.$$

Recall that ([17])

$$\begin{aligned} D_{2^n}(x)&=\begin{cases}2^n \text{ if } x\in I_n,\\ 0 \text{ if } x\in \overline{I}_n\end{cases},\\ D_n(x)&=w_n(x)\left(\sum_{r=0}^{i-1}n_r2^r-n_i2^i\right) \text{ for } x\in J_i:=I_i\setminus I_{i+1}. \end{aligned}\tag{3}$$

In this paper we consider the double system $\{w_i(x^1)w_j(x^2):i,j\in\mathbb{N}\}$ on $G\times G$.

The rectangular partial sums of the 2-dimensional Walsh–Fourier series are defined as

$$S_{M,N}f(x^1,x^2):=\sum_{i=0}^{M-1}\sum_{j=0}^{N-1}\widehat{f}(i,j)w_i(x^1)w_j(x^2),\tag{4}$$

where the number

$$\widehat{f}(i,j)=\int_{G\times G}f(x^1,x^2)w_i(x^1)w_j(x^2)\,d\mu(x^1,x^2)$$

is said to be the (i,j)th Walsh–Fourier coefficient of the function f. Denote

$$S_M^{\square}f(x^1,x^2):=S_{M,M}f(x^1,x^2).$$

The triangular partial sums of the 2-dimensional Walsh–Fourier series are defined as

$$S_k^{\triangle}f(x^1,x^2):=\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}\widehat{f}(i,j)w_i(x^1)w_j(x^2).$$

Denote

$$D_k^{\square}(x^1,x^2):=D_k(x^1)D_k(x^2)$$

and

$$D_k^{\triangle}(x^1,x^2):=\sum_{i=0}^{k-1}\sum_{j=0}^{k-i-1}w_i(x^1)w_j(x^2).$$

The norm (or the quasinorm) of the space $L_p(G\times G)$ is defined by

$$\|f\|_p:=\left(\int_{G\times G}|f(x^1,x^2)|^p\,d\mu(x^1,x^2)\right)^{1/p}\quad (0<p\leqslant\infty).$$

The space weak-$L_p(G\times G)$ consists of all measurable functions f for which

$$\|f\|_{\text{weak}-L_p(G\times G)} := \sup_{\lambda>0} \lambda\mu\left(|f|>\lambda\right)^{1/p} < +\infty.$$

For $n\in\mathbb{P}$ and a function f the Marcinkiewicz-Fejér means and triangular Fejér means of order n of the 2-dimensional Walsh–Fourier series of a function f is given by

$$\sigma_n^{\square} f(x^1,x^2) = \frac{1}{n}\sum_{j=0}^{n-1} S_j^{\square} f(x^1,x^2)$$

and

$$\sigma_n^{\triangle} f(x^1,x^2) = \frac{1}{n}\sum_{j=0}^{n-1} S_j^{\triangle} f(x^1,x^2),$$

respectively. It is easy to show that

$$\sigma_n^{\square} f(x^1,x^2) = \int_{G\times G} f\left(u^1,u^2\right) K_n^{\square}\left(x^1+u^1,x^2+u^2\right) d\mu\left(u^1,u^2\right)$$

and

$$\sigma_n^{\triangle} f(x^1,x^2) = \int_{G\times G} f\left(u^1,u^2\right) K_n^{\triangle}\left(x^1+u^1,x^2+u^2\right) d\mu\left(u^1,u^2\right), \tag{5}$$

where

$$K_n^{\square}\left(x^1,x^2\right) := \frac{1}{n}\sum_{j=0}^{n-1} D_j^{\square}\left(x^1,x^2\right)$$

and

$$K_n^{\triangle}\left(x^1,x^2\right) := \frac{1}{n}\sum_{j=0}^{n-1} D_j^{\triangle}\left(x^1,x^2\right).$$

It is known that (see [6])

$$K_n^{\triangle}\left(x^1,x^2\right) = \frac{1}{n}\sum_{i=1}^{n-1} D_{n-i}\left(x^1\right) D_i\left(x^2\right). \tag{6}$$

Consequently, from (4) and (6) we can write

$$\sigma_n^{\triangle} f(x^1,x^2) = \frac{1}{n}\sum_{i=1}^{n-1} S_{n-i,i} f(x^1,x^2) = \frac{1}{n}\sum_{i=1}^{n-1} S_{i,n-i} f(x^1,x^2). \tag{7}$$

In ([12]) it is proved that the following is true.

THEOREM GW. (Goginava, Weisz) *Let* $f\in L_1(G\times G)$. *Then*

$$\frac{1}{2^n}\sum_{i=1}^{2^n-1} S_{i,2^n-i} f(x^1,x^2) \to f\left(x^1,x^2\right) \ a.\ e.\ (x^1,x^2)\in G\times G \ as\ n\to\infty.$$

In [5, Corollary 3] it is proved the following theorem.

THEOREM G. (Gát) *Let* $a:\mathbb{N}\to\mathbb{N}$ *be a lacunary sequence. That is,* $a_{n+1}/a_n \geqslant q>1$ *for some* q *and for each* $n\in\mathbb{N}$. *Besides, let* $\alpha_j:\mathbb{N}^2\to\mathbb{N}$ *satisfy the following two conditions (* $j=1,2$ *).*

$$\#\{l\in\mathbb{N}:\alpha_j(n,l)=\alpha_j(n,k), l<a_n\}\leqslant C,\ \max\{\alpha_j(n,k):k<a_n\}\leqslant Ca_n,$$

$(j=1,2)$. *Then for every* $f\in L_1(G\times G)$ *we have*

$$\frac{1}{a_n}\sum_{i=1}^{a_n-1}S_{\alpha_1(n,i),\alpha_2(n,i)}f(x^1,x^2)\to f(x^1,x^2)$$

for a.e. $(x^1,x^2)\in G\times G$ *as* $n\to\infty$.

If $\alpha_1(n,i)=i$ and $\alpha_2(n,i)=a_n-i$, then we have the relation

$$\frac{1}{a_n}\sum_{i=1}^{a_n-1}S_{i,a_n-i}f(x^1,x^2)\to f\left(x^1,x^2\right) \text{ a. e. } (x^1,x^2)\in G\times G \text{ as } n\to\infty.$$

We note that in the case of trigonometric system Weisz [23] proved that for $f\in L_1(\mathbb{T}\times\mathbb{T})$ the almost everywhere convergence $\frac{1}{n}\sum\limits_{i=1}^{n-1}S_{i,n-i}f\to f\ (n\to\infty)$ holds. This issue with respect to the Walsh system is still open.

In this paper we introduce notion of dyadic triangular-Fejér means of two-dimensional Walsh-Fourier series as follows

$$\dot{\sigma}_n^{\triangle}f(x^1,x^2):=\frac{1}{n}\sum_{i=1}^{n-1}S_{i,n\oplus i}f(x^1,x^2).$$

It is easy to show that

$$\dot{\sigma}_n^{\triangle}f(x^1,x^2)=\int_{G\times G}f\left(u^1,u^2\right)\dot{K}_n^{\triangle}\left(x^1+u^1,x^2+u^2\right)d\mu\left(u^1,u^2\right),$$

where

$$\dot{K}_n^{\triangle}\left(x^1,x^2\right):=\frac{1}{n}\sum_{k=1}^{n-1}D_k\left(x^1\right)D_{n\oplus k}\left(x^2\right).$$

The maximal operator of dyadic triangular-Fejér means of the two dimensional Walsh-Fourier series is defined as follows

$$\dot{\sigma}_*^{\triangle}f:=\sup_{n\in\mathbb{P}}\left|\dot{\sigma}_n^{\triangle}f\right|$$

The notation $a\lesssim b$ in the proofs stands for $a<c\cdot b$, where c is an absolute constant.

We study the weak type inequality for the maximal operator of the dyadic triangular-Fejér means of two-dimensional Walsh-Fourier series. In particular, the following is true.

THEOREM 1. *Let* $f \in L_1(G \times G)$. *Then*

$$\left\| \dot{\sigma}_*^{\triangle} f \right\|_{weak-L_1} \lesssim \|f\|_1 .$$

The weak type (1,1) inequality and the usual density argument of Marcinkiewicz and Zygmund [15] and the fact that $\dot{\sigma}_n^{\triangle} P \to P$ everywhere for each two-dimensional Walsh polynomial (which will be proved later) imply

COROLLARY 1. *Let* $f \in L_1(G \times G)$. *Then*

$$\dot{\sigma}_n^{\triangle} f(x^1, x^2) \to f\left(x^1, x^2\right) \text{ a.e. as } n \to \infty.$$

3. Proof of Theorem 1

Proof. Denote by $\overline{I_a \times I_a}$ the complement of the set $I_a \times I_a$ and let $J_k = I_k \setminus I_{k+1}$ ($k \in \mathbb{N}$). First we prove that for any $a \in \mathbb{P}$ the following inequality is true

$$\int\limits_{\overline{I_a \times I_a}} \sup_{n \geqslant 2^{a-1}} \left| \dot{K}_n^{\triangle}\left(x^1, x^2\right) \right| d\mu\left(x^1, x^2\right) \lesssim 1. \tag{8}$$

Indeed, we can write

$$\begin{aligned} &\int\limits_{\overline{I_a \times I_a}} \sup_{n \geqslant 2^{a-1}} \left| \dot{K}_n^{\triangle}\left(x^1, x^2\right) \right| d\mu\left(x^1, x^2\right) \qquad (9)\\ &\lesssim \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int\limits_{J_{t^1} \times J_{t^2}} \sup_{n \geqslant 2^{a-1}} \left| \dot{K}_n^{\triangle}\left(x^1, x^2\right) \right| d\mu\left(x^1, x^2\right) \\ &\quad + \sum_{t^2=0}^{a-1} \sum_{t^1=t^2}^{\infty} \int\limits_{J_{t^1} \times J_{t^2}} \sup_{n \geqslant 2^{a-1}} \left| \dot{K}_n^{\triangle}\left(x^1, x^2\right) \right| d\mu\left(x^1, x^2\right) \\ &:= R_1 + R_2. \end{aligned}$$

For $2^A \leqslant n < 2^{A+1}$ we can write

$$n\dot{K}_n^{\triangle}\left(x^1, x^2\right) = \sum_{s=0}^{A} n_s \sum_{k=0}^{2^s-1} D_{n^{(s+1)}+k}\left(x^1\right) D_{n \oplus \left(n^{(s+1)}+k\right)}\left(x^2\right)$$

and since for $|n|=A$ that is, for $2^A \leqslant n < 2^{A+1}$ it holds $n \geqslant 2^A$, thus we have

$$
\begin{aligned}
R_1 &\lesssim \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}\times J_{t^2}} \sup_{A\geqslant a-1} \sup_{n:|n|=A} \frac{1}{2^A} \sum_{s=0}^{A} \left| \sum_{k=0}^{2^s-1} D_{n^{(s+1)}+k}\left(x^1\right) \right. \\
&\quad \left. \times D_{n\oplus\left(n^{(s+1)}+k\right)}\left(x^2\right) d\mu\left(x^1,x^2\right) \right| d\mu\left(x^1,x^2\right) \\
&\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}\times J_{t^2}} \sup_{n:|n|=A} \frac{1}{2^A} \sum_{s=0}^{A} \left| \sum_{k=0}^{2^s-1} D_{n^{(s+1)}+k}\left(x^1\right) \right. \\
&\quad \left. \times D_{n\oplus\left(n^{(s+1)}+k\right)}\left(x^2\right) \right| d\mu\left(x^1,x^2\right) \\
&\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}\times J_{t^2}} \sup_{n:|n|=A} \frac{1}{2^A} \sum_{s=0}^{t^1-1} \left| \sum_{k=0}^{2^s-1} D_{n^{(s+1)}+k}\left(x^1\right) \right. \\
&\quad \left. \times D_{n\oplus\left(n^{(s+1)}+k\right)}\left(x^2\right) d\mu\left(x^1,x^2\right) \right| d\mu\left(x^1,x^2\right) \\
&\quad + \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}\times J_{t^2}} \sup_{n:|n|=A} \frac{1}{2^A} \sum_{s=t^1}^{t^2\wedge A} \left| \sum_{k=0}^{2^s-1} D_{n^{(s+1)}+k}\left(x^1\right) \right. \\
&\quad \left. \times D_{n\oplus\left(n^{(s+1)}+k\right)}\left(x^2\right) \right| d\mu\left(x^1,x^2\right) \\
&\quad + \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1}\times J_{t^2}} \sup_{n:|n|=A} \frac{1}{2^A} \sum_{s=t^2+1}^{A} \left| \sum_{k=0}^{2^s-1} D_{n^{(s+1)}+k}\left(x^1\right) \right. \\
&\quad \left. \times D_{n\oplus\left(n^{(s+1)}+k\right)}\left(x^2\right) \right| d\mu\left(x^1,x^2\right) \\
&= R_{11}+R_{12}+R_{13}.
\end{aligned} \tag{10}
$$

Recall that s goes from 0 to A and therefore one can find in R_{12} that s goes to $t^2\wedge A$. Remark that for $k<2^s$ we have

$$
n\oplus\left(n^{(s+1)}+k\right)=n_{(s)}\oplus k.
$$

Since

$$
\left|D_{n_{(s)}\oplus k}\left(x^2\right)\right| \leqslant n_{(s)}\oplus k \lesssim 2^s
$$

and

$$
\left|D_{n^{(s+1)}\oplus k}\left(x^1\right)\right| \lesssim 2^{t^1}, x^1\in J_{t^1}
$$

for R_{11} we can write

$$R_{11} \lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1} \times J_{t^2}} \sup_{n:|n|=A} \frac{1}{2^A} \sum_{s=0}^{t^1-1} \left| \sum_{k=0}^{2^s-1} D_{n^{(s+1)}+k}\left(x^1\right) \right. \tag{11}$$

$$\left. \times D_{n_{(s)} \oplus k}\left(x^2\right) \right| d\mu\left(x^1, x^2\right)$$

$$\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \int_{J_{t^1} \times J_{t^2}} \frac{1}{2^A} \sum_{s=0}^{t^1-1} \sum_{k=0}^{2^s-1} 2^{t^1+s}$$

$$\lesssim \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \frac{2^{3t^1}}{2^{t^1+t^2+a}} \lesssim 1.$$

From (3) we have ($x^2 \in J_{t^2}$)

$$D_{n_{(s)} \oplus k}\left(x^2\right) = w_{n_{(s)} \oplus k}\left(x^2\right) \left(\sum_{i=0}^{t^2-1} (n \oplus k)_i 2^i - (n \oplus k)_{t^2} \wedge 2^{t^2} \right)$$

$$= w_{n_{(s)} \oplus k}\left(x^2\right) \left(\sum_{i=0}^{t^2-1} |n_i - k_i| 2^i - |n_{t^2} - k_{t^2}| 2^{t^2} \right).$$

On the other hand, for $x^1 \in J_{t^1}$ and $s \geqslant t^1$ we have

$$D_{n^{(s+1)} \oplus k}\left(x^1\right) = w_{n^{(s+1)} \oplus k}\left(x^1\right) \left(\sum_{j=0}^{t^1-1} k_j 2^j - k_{t^1} 2^{t^1} \right)$$

$$= w_{n^{(s+1)}}\left(x^1\right) D_k\left(x^1\right), k < 2^s.$$

Consequently, for R_{13} we obtain

$$R_{13} \lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{A-1} \int_{J_{t^1} \times J_{t^2}} \sup_{n:|n|=A} \frac{1}{2^A} \sum_{s=t^2+1}^{A} \left| \sum_{k=0}^{2^s-1} D_k\left(x^1\right) w_k\left(x^2\right) \right.$$

$$\left. \times \left(\sum_{i=0}^{t^2-1} |n_i - k_i| 2^i - |n_{t^2} - k_{t^2}| 2^{t^2} \right) \right| d\mu\left(x^1, x^2\right)$$

$$\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{A-1} \sum_{i=0}^{t^2} \int_{J_{t^1} \times J_{t^2}} \sup_{n:|n|=A} \frac{1}{2^A} \sum_{s=t^2+1}^{A} \left| \sum_{k=0}^{2^s-1} D_k\left(x^1\right) w_k\left(x^2\right) \right.$$

$$\left. \times |n_i - k_i| 2^i \right| d\mu\left(x^1, x^2\right).$$

Set

$$D := \sum_{k=0}^{2^s-1} D_k\left(x^1\right) w_k\left(x^2\right) |n_i - k_i| 2^i$$

$$= \sum_{k_0=0}^{1} \cdots \sum_{k_{s-1}=0}^{1} D_k\left(x^1\right) w_k\left(x^2\right) |n_i - k_i| 2^i.$$

Let $i \leqslant t^1$ and $x^1_{t^1+l} = 1$ for some $l = 1,2,...,t^2 - t^1 - 1$. Then it is easy to show that

$$D = \sum_{k_0=0}^{1} \cdots \sum_{k_{s-1}=0}^{1} w_{k\left(t^1\right)}\left(x^1\right)\left(\sum_{j=0}^{t^1-1} k_j 2^j - k_{t^1} 2^{t^1}\right) w_{k\left(t^2\right)}\left(x^2\right)|n_i - k_i| 2^i$$
$$= \sum_{k_{t^1+l}=0}^{1} (-1)^{k_{t^1+l}} \Phi = 0,$$

where $\left(x^1, x^2\right) \in J_{t^1} \times J_{t^2}$ and the function Φ does not depend on k_{t^1+l}. Let $x^1_{t^2+l} \neq x^2_{t^2+l}$ for some $l = 1,2,...,s - t^2 - 1$. Then for $\left(x^1, x^2\right) \in J_{t^1} \times J_{t^2}$ analogously we can write

$$D = \sum_{k_{t^2+l}=0}^{1} (-1)^{k_{t^2+l}} \Phi_1 = 0,$$

where the function Φ_1 does not depend on k_{t^2+l}.

Consequently, $D \neq 0$ imply that $\left(i \leqslant t^1\right)$

$$\begin{aligned} x^1_{t^1+1} &= \cdots = x^1_{t^2-1} = 0; \\ x^1_{t^2+1} &= x^1_{t^2+1}, ..., x^1_{s-1} = x^2_{s-1}. \end{aligned} \tag{12}$$

Let $t^1 < i < t^2$. Then analogously, we can prove that $D \neq 0$ imply that

$$\begin{aligned} x^1_{t^1} &= \cdots = x^1_{i-1} = x^1_{i+1} = \cdots = x^1_{t^2-1} = 0; \\ x^1_{t^2+1} &= x^1_{t^2+1}, ..., x^1_{s-1} = x^2_{s-1}. \end{aligned} \tag{13}$$

Since

$$\left|\sum_{k=0}^{2^s-1} D_k\left(x^1\right) w_k\left(x^2\right)|n_i - k_i| 2^i\right| \lesssim 2^{s+t^1+i} \qquad \left(x^1 \in J_{t^1}\right)$$

from (12) and (13) we conclude that

$$\begin{aligned} R_{13} &\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{A-1} \sum_{i=0}^{t^2} \sum_{n_i=0}^{1} \sum_{s=t^2+1}^{A} \frac{2^{s+t^1+i}}{2^A} \frac{2^{s-t^2}}{2^{2s}} \\ &\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{A-1} \sum_{s=t^2+1}^{A} 2^{t^1-A} \\ &\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{A-1} 2^{t^1-A}\left(A - t^2\right) \\ &\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} 2^{t^1-A}\left(A - t^1\right)^2 \\ &\lesssim \sum_{A=a-1}^{\infty} \frac{(A-a)^2}{2^{A-a}} \lesssim 1. \end{aligned} \tag{14}$$

Let $t^1 \leqslant s < t^2$. This gives

$$D_{n_{(s)} \oplus k}\left(x^2\right) = n_{(s)} \oplus k = \sum_{j=0}^{s} \left|n_j - k_j\right| 2^j, x^2 \in J_{t^2}$$

and

$$D_{n^{(s+1)}+k}\left(x^1\right) = w_{n^{(s+1)}}\left(x^1\right) D_k\left(x^1\right), x^1 \in J_{t^1}.$$

Consequently,

$$\begin{aligned}
&\left|\sum_{k=0}^{2^s-1} D_{n^{(s+1)}+k}\left(x^1\right) D_{n_{(s)}\oplus k}\left(x^2\right)\right| \\
&= \left|\sum_{k=0}^{2^s-1} D_k\left(x^1\right) D_{n_{(s)}\oplus k}\left(x^2\right)\right| \\
&= \left|\sum_{k=0}^{2^s-1} \left(\sum_{i=0}^{t^1-1} k_i 2^i - k_{t^1} 2^{t^1}\right) w_{k^{\left(t^1\right)}}\left(x^1\right) \sum_{j=0}^{s} \left|n_j - k_j\right| 2^j\right| \\
&= \left|\sum_{j=0}^{s} \sum_{k_0=0}^{1} \cdots \sum_{k_{s-1}=0}^{1} \left(\sum_{i=0}^{t^1-1} k_i 2^i - k_{t^1} 2^{t^1}\right) w_{k^{\left(t^1\right)}}\left(x^1\right) \left|n_j - k_j\right| 2^j\right|.
\end{aligned}$$

Analogously, as above we can prove that the inner sum can be different from zero only in the case when

$$x^1_{t^1+1} = \cdots = x^1_{s-1} = 0 \qquad \left(x^1 \in J_{t^1}\right),$$

where index j is excluded from the list of $t^1+1,\ldots,s-1$. That is, the measure of the subset of $J_{t^1} \times J_{t^2}$, where this sum can be different from zero is bounded by $c2^{-s-t^2}$. Taking account $x^1 \in J_{t^1}$ and $n_{(s)} \oplus k \lesssim 2^s$ also give

$$\left|\sum_{k=0}^{2^s-1} D_{n^{(s+1)}+k}\left(x^1\right) D_{n_{(s)}\oplus k}\left(x^2\right)\right| \lesssim 2^{2s+t^1}.$$

Consequently, for R_{12} we can write (recall that s is not greater than A as one can see the formula for $n\dot{K}_n^{\triangle}$ after the definition of R_1.)

$$\begin{aligned}
R_{12} &\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{\infty} \sum_{s=t^1}^{t^2 \wedge A} \frac{1}{2^A} \frac{2^{2s+t^1}}{2^{s+t^2}} \qquad (15)\\
&\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{A} \sum_{s=t^1}^{t^2} \frac{2^{s+t^1}}{2^{A+t^2}} + \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=A+1}^{\infty} \sum_{s=t^1}^{A} \frac{2^{s+t^1}}{2^{A+t^2}} \\
&\lesssim \sum_{A=a}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=t^1}^{A} 2^{t^1-A} + \sum_{A=a}^{\infty} \sum_{t^1=0}^{a-1} \sum_{t^2=A+1}^{\infty} 2^{t^1-t^2} \\
&\lesssim \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \frac{A-t^1}{2^{A-t^1}} + \sum_{A=a-1}^{\infty} \sum_{t^1=0}^{a-1} \frac{1}{2^{A-t^1}} \lesssim 1.
\end{aligned}$$

Combining (12)–(15) we conclude that

$$R_1 \lesssim 1. \tag{16}$$

Analogously, we can prove that

$$R_2 \lesssim 1. \tag{17}$$

From (9), (16) and (17) we obtain the proof of (8).

In the sequal we prove that the maximal operator $\dot{\sigma}_*^{\triangle}$ is quasi-local. This reads as follows:

Let $f \in L_1(G\times G)$, $\operatorname{supp} f \subset I_a(u^1)\times I_a(u^2)$ and

$$\int\limits_{I_a(u^1)\times I_a(u^2)} f(x^1,x^2)\,d\mu(x^1,x^2)=0$$

for some $u=(u^1,u^2)\in G$. Then

$$\int\limits_{(G\times G)\setminus(I_a(u^1)\times I_a(u^2))} \dot{\sigma}_*^{\triangle} f(x^1,x^2)\,d\mu(x^1,x^2) \lesssim \|f\|_1 .$$

Indeed, by the shift invariancy of the Haar measure it can be supposed that $u^1=u^2=0$. If $n<2^{a-1}$ then

$$\begin{aligned}\dot{\sigma}_n^{\triangle} f(x^1,x^2) &= \int\limits_{G\times G} f(u^1,u^2)\,\dot{K}_n^{\triangle}(x^1+u^1,x^2+u^2)\,d\mu(u^1,u^2)\\ &= \int\limits_{I_a\times I_a} f(u^1,u^2)\,\dot{K}_n^{\triangle}(x^1+u^1,x^2+u^2)\,d\mu(u^1,u^2)\\ &= \dot{K}_n^{\triangle}(x^1,x^2)\int\limits_{I_a\times I_a} f(u^1,u^2)\,d\mu(u^1,u^2)=0.\end{aligned}$$

Consequently, $n\geqslant 2^{a-1}$ can be supposed. Then by $\overline{I_a\times I_a}=(\overline{I_a}\times G)\cup(G\times\overline{I_a})$ and from (8) we have

$$\int\limits_{\overline{I_a}\times G}\sup_{n\geqslant 2^{a-1}}\left|\dot{K}_n^{\triangle}(x^1,x^2)\right|d\mu(x)\lesssim 1.$$

Since for any fixed $(u^1,u^1)\in I_a\times I_a$ we have that (x^1+u^1,x^2+u^2) ranges over $\overline{I_a}\times G$ as (x^1,x^2) ranges over $\overline{I_a}\times G$ ($x^1\in\overline{I_a}, u^1\in I_a$ implies $x^1+u^1\in\overline{I_a}$), then we also have

$$\int\limits_{\overline{I_a}\times G}\sup_{n\geqslant 2^{a-1}}\left|\dot{K}_n^{\triangle}(x^1+u^1,x^2+u^2)\right|d\mu(x)\lesssim 1.$$

Similarly,

$$\int\limits_{G\times\overline{I_a}}\sup_{n\geqslant 2^{a-1}}\left|\dot{K}_n^{\triangle}(x^1+u^1,x^2+u^2)\right|d\mu(x)\lesssim 1.$$

This gives

$$\begin{aligned}
&\int\limits_{\overline{I_a\times I_a}} \dot{\sigma}_*^{\triangle} f\left(x^1,x^2\right) d\mu\left(x^1,x^2\right) \\
&\lesssim \int\limits_{I_a\times I_a} \left|f\left(u^1,u^2\right)\right| \Big(\int\limits_{\overline{I_a\times I_a}} \sup_{n\geqslant 2^{a-1}} \left|\dot{K}_n^{\triangle}\left(x^1+u^1,x^2+u^2\right)\right| d\mu\left(x\right)\Big) d\mu\left(u\right) \\
&\lesssim \|f\|_1 .
\end{aligned}$$

Now, we prove that the maximal operator $\dot{\sigma}_*^{\triangle}$ is of type (∞,∞). Let $2^{A-1}\leqslant n<2^A$. Since

$$|D_{k\oplus n}|\lesssim 2^A, \quad k=0,1,...,n-1$$

from (8) we obtain

$$\begin{aligned}
&\int\limits_{G\times G} \left|\dot{K}_n^{\triangle}\left(x^1,x^2\right)\right| d\mu\left(x^1,x^2\right) \\
&= \int\limits_{I_A\times I_A} \left|\dot{K}_n^{\triangle}\left(x^1,x^2\right)\right| d\mu\left(x^1,x^2\right) + \int\limits_{\overline{I_A\times I_A}} \left|\dot{K}_n^{\triangle}\left(x^1,x^2\right)\right| d\mu\left(x^1,x^2\right) \\
&\lesssim 1.
\end{aligned}$$

Hence, the maximal operator $\dot{\sigma}_*^{\triangle}$ is of type (∞,∞).

Theorem 1 is proved. □

Finally, we prove Corollary 1.

Proof. The only thing left to proof is the a.e. relation $\dot{\sigma}_n^{\triangle}P\to P$ for each two-dimensional Walsh polynomial. We prove a bit more. That is, we prove everywhere convergence. Let P be a two-dimensional Walsh polynomial with

$$P(x^1,x^2)=\sum_{l_1=0}^{2^\tau-1}\sum_{l_2=0}^{2^\tau-1} c_{l_1,l_2}w_{l_1}(x^1)w_{l_2}(x^2).$$

This gives that for any $n_1,n_2\geqslant 2^\tau$ we have $S_nP=S_{n_1,n_2}P=P$. Set

$$T_n^1:=\{j\in\mathbb{N}:j<2^\tau\}, \quad T_n^2:=\{j\in\mathbb{N}:n\oplus j<2^\tau\}.$$

If we suppose that $n\geqslant 2^\tau$, then one can find imeditely that $T^1\cap T^2=\emptyset$. Next, we prove the relation

$$\left|S_{j,n\oplus j}P(y)-P(y)\right|\leqslant 2^{\tau+1}\|P\|_\infty$$

for every $y\in G\times G$ and $j\in T_n:=T_n^1\cup T_n^2$. Since T_n^1 and T_n^2 are disjoint, then either $j\in T_n^1\setminus T_n^2$ or $j\in T_n^2\setminus T_n^1$. Since the two situations can be intestigated in the same way we can suppose the first one. That is, $j < 2^\tau$ and $n \oplus j \geqslant 2^\tau$. This follows

$$\begin{aligned} S_{j,n\oplus j}P(y) &= \int_G D_j(y^1+x^1) \int_G \sum_{l_1=0}^{2^\tau-1}\sum_{l_2=0}^{2^\tau-1} c_{l_1,l_2} w_{l_1}(x^1) w_{l_2}(x^2) \\ &\quad \times D_{n\oplus j}(y^2+x^2)d\mu(x^2)d\mu(x^1) \\ &= \int_G D_j(y^1+x^1) \sum_{l_1=0}^{2^\tau-1}\sum_{l_2=0}^{2^\tau-1} c_{l_1,l_2} w_{l_1}(x^1) w_{l_2}(y^2) d\mu(x^1) \\ &= \int_G D_j(y^1+x^1) P(x^1,y^2) d\mu(x^1). \end{aligned}$$

Since $\int_G D_j(y^1+x^1)\mu(x^1) = 1$, then we have

$$|S_{j,n\oplus j}P(y) - P(y)| = \left| \int_G D_j(y^1+x^1)\left(P(x^1,y^2) - P(y^1,y^2)\right) d\mu(x^1) \right|.$$

This gives the relation for $\left|S_{j,n\oplus j}P(y) - P(y)\right|$. Since for $j \notin T_n$ we have $S_{j,n\oplus j}P(y) = P(y)$, then

$$\begin{aligned} \left| \frac{1}{n}\sum_{j=0}^{n-1} S_{j,n\oplus i}P(y) - P(y) \right| &\leqslant \frac{1}{n}\sum_{j=0}^{n-1} \left|S_{j,n\oplus j}P(y) - P(y)\right| \\ &\leqslant \frac{1}{n}\sum_{j\in T_n} \left|S_{j,n\oplus j}P(y) - P(y)\right| + \frac{1}{n}\sum_{j\notin T_n} \left|S_{j,n\oplus j}P(y) - P(y)\right| \\ &= \frac{1}{n}\sum_{j\in T_n} \left|S_{j,n\oplus j}P(y) - P(y)\right| \leqslant \frac{\#T_n 2^{\tau+1}\|P\|_\infty}{n} \to 0, \end{aligned}$$

since $\#T_n = \#T_n^1 + \#T_n^2 \leqslant 2^{\tau+1}$. □

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