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A NEW EXTENSION OF CARLSON'S INEQUALITY

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Abstract. A new extension of Carlson's inequality is obtained by using the Euler-Maclaurin summation formula and an extended Beta function introduced recently in [Appl. Math. Comput. 248 (2014) 631–651]. The integral analogue of this inequality is also presented.

1. Introduction

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers and $f(x)$ denote a Lebesgue measurable nonnegative function on $[0,\infty)$, then Carlson's inequalities are given by

$$\left(\sum_{n=1}^{\infty} a_n\right)^4 \leqslant \pi^2 \left(\sum_{n=1}^{\infty} a_n^2\right)\left(\sum_{n=1}^{\infty} n^2 a_n^2\right) \tag{1}$$

and

$$\left(\int_0^{\infty} f(x)\,\mathrm{d}x\right)^4 \leqslant \pi^2 \left(\int_0^{\infty} f^2(x)\,\mathrm{d}x\right)\left(\int_0^{\infty} x^2 f^2(x)\,\mathrm{d}x\right), \tag{2}$$

where π^2 is the best possible. For more details concerning these inequalities and their variants and extensions, we refer to [2], [3], [4] and [6].

In 2002, Kuang and Debnath [5] proved some extensions of Carlson's inequalities by using a fairly instructive method based on the Euler-Maclaurin summation formula and Beta function. Motivated by this work, we establish in this paper some new extensions of Carlson's inequalities. In order to achieve our aim, we first need the following extension of the Beta function.

The familiar confluent hypergeometric function is defined by (see [9, p. 36, Eq. (3)])

$${}_1F_1(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n}\frac{z^n}{n!} \quad \left(c \notin \mathbb{Z}_0^- := \{0,-1,-2,\cdots\};\ |z| < \infty\right), \tag{3}$$

where $(a)_n$ denotes the familiar Pochhammer symbol defined (for $a \in \mathbb{C}$) by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & \text{if } n = 0 \\ a(a+1)\cdots(a+n-1) & \text{if } n \in \mathbb{N} := \{1,2,\cdots\}. \end{cases}$$

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For $\Re(c) > \Re(a) > 0$, the confluent hypergeometric function (3) has the Euler type integral representation given by (see [9, p. 37, Eq. (6)])

$$ {}_1F_1(a;c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{uz} u^{a-1} (1-u)^{c-a-1} \,\mathrm{d}u. \tag{4} $$

By applying the asymptotic properties of the confluent hypergeometric function, Luo *et al.* [7] introduced a new extension of the Beta function. Here, for the purpose of our paper, we mention its special case as follows:

DEFINITION 1. The extended Beta function $\mathscr{B}_{b;\rho,\lambda}^{(a,c)}(x,y)$ with $b > 0$ is defined by

$$ \mathscr{B}_{b;\rho,\lambda}^{(a,c)}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} {}_1F_1\left(a;c;-\frac{b}{t^{\rho}(1-t)^{\lambda}}\right) \mathrm{d}t, \tag{5} $$

where $\min\{\rho,\lambda\} \geqslant 0$, $\min\{a,c\} > 0$, $x > -\rho a$ and $y > -\lambda a$. When $b = 0$, (5) reduces to the ordinary Beta function $B(x,y)$ $(\min\{x,y\} > 0)$.

The extended Beta function defined by (5) shares many properties of the ordinary Beta function and has been proven to be very useful in giving some new extensions of the generalized hypergeometric functions. By using (5), we establish the following Carlson type inequality.

THEOREM 1. *Let* $S_\alpha = \sum_{n=1}^{\infty} n^{\alpha} a_n^p$, $S_\beta = \sum_{n=1}^{\infty} n^{\beta} a_n^p$, $c \geqslant a > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \beta < p-1 < \alpha$ *and let*

$$ \lambda_\alpha = \frac{p - \alpha q}{p(\alpha - \beta)}, \quad \lambda_\beta = \frac{p - \beta q}{p(\alpha - \beta)}. \tag{6} $$

Suppose that $a_n \geqslant 0$, $n = 1, 2, \cdots$, *and* $0 < S_\alpha, S_\beta < \infty$ *so that* $\lambda S_\beta > \rho S_\alpha$ $(\rho,\lambda > 0)$. *Then, we have*

$$ \left(\sum_{n=1}^{\infty} a_n \mathscr{L}(n)\right)^p < 2\mathscr{L}(1) S_\alpha S_\beta \left(\frac{1}{\alpha-\beta} \frac{S_\alpha^{\lambda_\alpha}}{S_\beta^{\lambda_\beta}} \mathscr{B}_{b;\rho,\lambda}^{(a,c)}(\lambda_\beta, -\lambda_\alpha) - \mathscr{C}_1\right)^{\frac{p}{q}}, \tag{7} $$

where $\mathscr{L}(x)$ *is given by*

$$ \mathscr{L}(x) := {}_1F_1\left(a;c;-\frac{b}{S_\beta^{\rho} S_\alpha^{\lambda}} \frac{\left(S_\beta x^{\alpha} + S_\alpha x^{\beta}\right)^{\rho+\lambda}}{x^{\alpha\rho+\beta\lambda}}\right) \tag{8} $$

and

$$ \mathscr{C}_1 := \int_0^1 \frac{\mathscr{L}(x)}{\left(S_\alpha x^{\beta} + S_\beta x^{\alpha}\right)^{\frac{q}{p}}} \mathrm{d}x - {}_1F_1\left(a;c;-\frac{b}{S_\beta^{\rho} S_\alpha^{\lambda}} \left(S_\beta + S_\alpha\right)^{\rho+\lambda}\right) \left(S_\alpha + S_\beta\right)^{-\frac{q}{p}} $$

is finite and $\mathscr{B}_{b;\rho,\lambda}^{(a,c)}(x,y)$ *is the extended Beta function given by* (5).

REMARK 1. By setting $b=0$, the inequality in (7) corresponds to the one given by Kuang and Debnath in [5, p. 396, Theorem 1]. If we specialize the sequence $\{a_n\}_{n=1}^{\infty}$ by putting $a_n = n^{-\frac{1}{p}(\alpha+\beta+1)}$ $(n \geqslant 1)$, then we note that

$$S_\alpha = \sum_{n=1}^{\infty} n^{\alpha} n^{-\alpha-\beta-1} = \sum_{n=1}^{\infty} \frac{1}{n^{\beta+1}} = \zeta(\beta+1)$$

and

$$S_\beta = \sum_{n=1}^{\infty} n^{\beta} n^{-\alpha-\beta-1} = \sum_{n=1}^{\infty} \frac{1}{n^{\alpha+1}} = \zeta(\alpha+1),$$

where $\zeta(s)$ is the Riemann Zeta function (see [8, p. 164]). Upon replacing S_α and S_β with $\zeta(\beta+1)$ and $\zeta(\alpha+1)$, respectively, the inequality (7) can be expressed as

$$\left(\sum_{n=1}^{\infty} \frac{\mathscr{L}(n)}{n^{\frac{1}{p}(\alpha+\beta+1)}}\right)^{p} < 2\mathscr{L}(1)\,\zeta(\alpha+1)\,\zeta(\beta+1) \times \left(\frac{1}{\alpha-\beta}\,\frac{\zeta^{\lambda_\alpha}(\beta+1)}{\zeta^{\lambda_\beta}(\alpha+1)}\,\mathscr{B}_{b;\rho,\lambda}^{(a,c)}(\lambda_\beta,-\lambda_\alpha) - \mathscr{C}_1\right)^{\frac{p}{q}}.$$

The following result is the continuous version of the inequality (7) in which above notations and statements are used.

THEOREM 2. *Let $f(x) \geqslant 0$ and assume that*

$$S_\alpha = \int_{\varepsilon}^{\infty} x^{\alpha} f^{p}(x)\,\mathrm{d}x \quad \text{and} \quad S_\beta = \int_{\varepsilon}^{\infty} x^{\beta} f^{p}(x)\,\mathrm{d}x$$

exist. Moreover, let $\frac{1}{p}+\frac{1}{q}=1$, $c \geqslant a > 0$, $0 < \beta < p-1 < \alpha$ and $\varepsilon \geqslant 1$.

If $0 < S_\alpha, S_\beta < \infty$ with $\lambda S_\beta > \rho S_\alpha$ $(\rho,\lambda > 0)$, then $f \in L[\varepsilon,\infty)$ and

$$\left(\int_{\varepsilon}^{\infty} f(x)\,\mathscr{L}(x)\,\mathrm{d}x\right)^{p} < 2\mathscr{L}(1)\,S_\alpha S_\beta \left(\mathscr{B}_{b;\rho,\lambda}^{(a,c)}(\lambda_\beta,-\lambda_\alpha) - \mathscr{C}_2\right)^{\frac{p}{q}}, \tag{9}$$

where λ_α and λ_β are defined by (6), $\mathscr{L}(x)$ is given by (8) and

$$\mathscr{C}_2 := \int_{0}^{\varepsilon} \frac{\mathscr{L}(x)}{\left(S_\alpha x^{\beta} + S_\beta x^{\alpha}\right)^{\frac{q}{p}}}\,\mathrm{d}x > 0. \tag{10}$$

REMARK 2. If we put $b=0$ in (9), then the requirement $\varepsilon \geqslant 1$ can be replaced by $\varepsilon > 0$, and the corresponding inequality reduces to the result of Kuang and Debnath [5, p. 397, Theorem 2].

2. Some prerequisite lemmas

Before we prove our main results (Theorems 1 and 2), we in this section prove a few prerequisite results.

LEMMA 1. *Let* $h_\eta(x)$ *be defined by*

$$h_\eta(x) = \left(S_\beta + \frac{S_\alpha}{x^\eta}\right)^\rho \left(S_\alpha + S_\beta x^\eta\right)^\lambda, \quad x \in (0,\infty) \tag{11}$$

with

$$h_\eta(1) = \left(S_\alpha + S_\beta\right)^{\rho+\lambda}, \tag{12}$$

where $\eta,\lambda,\rho > 0$, $0 < S_\alpha, S_\beta < \infty$ *and* $\lambda S_\beta > \rho S_\alpha$. *Then,*

(i) $h_\eta(x)$ *is positive on* $(0,\infty)$*;*

(ii) $h_\eta(x)$ *is strictly increasing on* $[1,\infty)$.

Proof. By differentiating (11) with respect to x, we get

$$\begin{aligned} h'_\eta(x) &= -\rho\eta\left(S_\beta + \frac{S_\alpha}{x^\eta}\right)^{\rho-1}\left(S_\alpha + S_\beta x^\eta\right)^\lambda \frac{S_\alpha}{x^{\eta+1}} \\ &\quad + \lambda\eta\left(S_\beta + \frac{S_\alpha}{x^\eta}\right)^\rho \left(S_\alpha + S_\beta x^\eta\right)^{\lambda-1} S_\beta x^{\eta-1} \\ &= \frac{\eta}{x}\left(S_\beta + \frac{S_\alpha}{x^\eta}\right)^\rho \left(S_\alpha + S_\beta x^\eta\right)^{\lambda-1} \\ &\quad \times \left[-\rho\left(S_\beta + \frac{S_\alpha}{x^\eta}\right)^{-1}\left(S_\alpha + S_\beta x^\eta\right)\frac{S_\alpha}{x^\eta} + \lambda S_\beta x^\eta\right] \\ &= \frac{\eta}{x}\left(S_\beta + \frac{S_\alpha}{x^\eta}\right)^\rho \left(S_\alpha + S_\beta x^\eta\right)^{\lambda-1} H_\eta(x), \end{aligned} \tag{13}$$

where $H_\eta(x) := \lambda S_\beta x^\eta - \rho S_\alpha$. Since $\lambda S_\beta > \rho S_\alpha$, it follows that

$$H_\eta(x) > \lambda S_\beta\left(x^\eta - 1\right) \geqslant 0 \quad x \in [1,\infty).$$

Thus, we have

$$h'_\eta(x) > 0 \quad x \in [1,\infty),$$

and the function $h_\eta(x)$ is strictly increasing on $[1,\infty)$ which proves the positivity of $h_\eta(x)$. □

LEMMA 2. *Let* $l_\eta(x)$ *be defined by*

$$l_\eta(x) := {}_1F_1\left(a;c;-\frac{b}{S_\beta^\rho S_\alpha^\lambda} h_\eta(x)\right), \tag{14}$$

where $c \geqslant a > 0$, $\eta > 0$, $b \geqslant 0$, $0 < S_\alpha, S_\beta < \infty$ *and* $\lambda S_\beta > \rho S_\alpha$ $(\rho,\lambda > 0)$. *Then, we have* *(i)* $l_\eta(x)$ *is positive on* $(0,\infty)$*;*

(ii) $l_\eta(x)$ *is strictly decreasing on* $x\in[1,\infty)$.

Proof. By using the integral representation (4), we can express $l_\eta(x)$ as

$$l_\eta(x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 e^{-\frac{b}{S_\beta^\rho S_\alpha^\lambda}h_\eta(x)}t^{a-1}(1-t)^{c-a-1}\,\mathrm{d}t \quad (c>a>0). \tag{15}$$

We note that the integrand of (15) is positive on $(0,1)$ and the factor $\Gamma(c)/\Gamma(a)\Gamma(c-a)$ is also positive for $c>a>0$. Hence, $l_\eta(x)>0$ for all $x\in(0,\infty)$.

Since e^{-x} is a strictly decreasing function on $\mathbb{R}$ and $h_\eta(x)$ has been proved in Lemma 1 to be a strictly increasing function on $[1,\infty)$, we have then

$$e^{-\frac{b}{S_\beta^\rho S_\alpha^\lambda}h_\eta(x_1)}>e^{-\frac{b}{S_\beta^\rho S_\alpha^\lambda}h_\eta(x_2)},\quad 1\leqslant x_1<x_2<\infty.$$

Thus, we obtain

$$l_\eta(x_1)>l_\eta(x_2),\quad 1\leqslant x_1<x_2<\infty,$$

which means that the function $l_\eta(x)$ is strictly decreasing on $x\in[1,\infty)$.

For $a=c$, the confluent hypergeometric functions in (14) reduces to the exponential function and the results follow immediately with similar arguments as above. □

LEMMA 3. *Let* $0<S_\alpha,S_\beta<\infty$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\beta<p-1<\alpha$, *then*

$$\int_0^\infty \frac{l_{\alpha-\beta}(x)}{\left(S_\alpha x^\beta+S_\beta x^\alpha\right)^{\frac{q}{p}}}\mathrm{d}x=\frac{1}{\alpha-\beta}\frac{S_\alpha^{\lambda_\alpha}}{S_\beta^{\lambda_\beta}}\mathscr{B}_{b;\rho,\lambda}^{(a,c)}(\lambda_\beta,-\lambda_\alpha), \tag{16}$$

where λ_α *and* λ_β *are given by* (6)*, and* $\mathscr{B}_{b;\rho,\lambda}^{(a,c)}(x,y)$ *is the extended Beta function defined by* (5)*.*

Proof. In view of Lemmas 1 and 2, we need to prove that

$$\int_0^\infty \frac{1}{\left(S_\alpha x^\beta+S_\beta x^\alpha\right)^{\frac{q}{p}}}\,{}_1F_1\left(a,c;-b\frac{\left(S_\beta x^\alpha+S_\alpha x^\beta\right)^{\rho+\lambda}}{S_\beta^\rho S_\alpha^\lambda x^{\alpha\rho+\beta\lambda}}\right)\mathrm{d}x$$
$$=\frac{1}{\alpha-\beta}\frac{S_\alpha^{\lambda_\alpha}}{S_\beta^{\lambda_\beta}}\mathscr{B}_{b;\rho,\lambda}^{(a,c)}(\lambda_\beta,-\lambda_\alpha). \tag{17}$$

Let I denote the left-hand side of (17), then

$$I=\frac{1}{S_{\beta}^{\frac{q}{p}}}\int_{0}^{\infty}\frac{x^{-\frac{q\beta}{p}}}{\left(S_{\alpha}/S_{\beta}+x^{\alpha-\beta}\right)^{\frac{q}{p}}}{}_{1}F_{1}\left(a,c;-b\left(\frac{S_{\alpha}}{S_{\beta}x^{\alpha-\beta}}\right)^{\rho}\left(S_{\beta}/S_{\alpha}x^{\alpha-\beta}+1\right)^{\rho+\lambda}\right)\mathrm{d}x$$

$$=\frac{1}{(\alpha-\beta)S_{\beta}^{\frac{q}{p}}}\int_{0}^{\infty}\frac{u^{-\frac{1}{\alpha-\beta}\frac{q\beta}{p}+\frac{1}{\alpha-\beta}-1}}{\left(S_{\alpha}/S_{\beta}+u\right)^{\frac{q}{p}}}{}_{1}F_{1}\left(a;c;-b\left(\frac{S_{\alpha}}{S_{\beta}u}\right)^{\rho}\left(S_{\beta}/S_{\alpha}u+1\right)^{\rho+\lambda}\right)\mathrm{d}u$$

$$\left(\text{by setting } x=u^{\frac{1}{\alpha-\beta}}\right)$$

$$=\frac{1}{(\alpha-\beta)}\frac{S_{\alpha}^{\lambda_{\beta}-\frac{q}{p}}}{S_{\beta}^{\lambda_{\beta}}}\int_{0}^{\infty}\frac{v^{\lambda_{\beta}-1}}{(1+v)^{\frac{q}{p}}}{}_{1}F_{1}\left(a;c;-b\frac{(v+1)^{\rho+\lambda}}{v^{\rho}}\right)\mathrm{d}v\quad\left(\text{by setting } u=\frac{S_{\alpha}}{S_{\beta}}v\right)$$

$$=\frac{1}{(\alpha-\beta)}\frac{S_{\alpha}^{\lambda_{\alpha}}}{S_{\beta}^{\lambda_{\beta}}}\int_{0}^{\infty}\frac{v^{\lambda_{\beta}-1}}{(1+v)^{\frac{q}{p}}}{}_{1}F_{1}\left(a;c;-b\frac{(v+1)^{\rho+\lambda}}{v^{\rho}}\right)\mathrm{d}v. \tag{18}$$

Finally, by setting $v=\frac{t}{1-t}$ in (18), we get

$$I=\frac{1}{(\alpha-\beta)}\frac{S_{\alpha}^{\lambda_{\alpha}}}{S_{\beta}^{\lambda_{\beta}}}\int_{0}^{1}t^{\lambda_{\beta}-1}(1-t)^{\frac{q}{p}-\lambda_{\beta}-1}{}_{1}F_{1}\left(a;c;-\frac{b}{t^{\rho}(1-t)^{\lambda}}\right)\mathrm{d}t$$

$$=\frac{1}{(\alpha-\beta)}\frac{S_{\alpha}^{\lambda_{\alpha}}}{S_{\beta}^{\lambda_{\beta}}}\int_{0}^{1}t^{\lambda_{\beta}-1}(1-t)^{-\lambda_{\alpha}-1}{}_{1}F_{1}\left(a;c;-\frac{b}{t^{\rho}(1-t)^{\lambda}}\right)\mathrm{d}t$$

$$=\frac{1}{(\alpha-\beta)}\frac{S_{\alpha}^{\lambda_{\alpha}}}{S_{\beta}^{\lambda_{\beta}}}\mathscr{B}_{b;\rho,\lambda}^{(a,c)}\left(\lambda_{\beta},-\lambda_{\alpha}\right). \tag{19}$$

This completes the proof. □

We also need the following Euler-Maclaurin summation formula.

LEMMA 4. ([1]) *If f is positive and strictly decreasing on $[1,\infty)$, then there is a positive constant $C(f)<f(1)$ and a sequence $\{E_f(n)\}$, with $0<E_f(n)<f(n)$, such that*

$$\sum_{k=1}^{n}f(k)=\int_{1}^{n}f(x)\,\mathrm{d}x+C(f)+E_f(n),\,n=2,3,\cdots. \tag{20}$$

Note that if $f(n)\to 0$ as $n\to+\infty$, then $E_f(n)\to 0$.

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. Let g be defined by

$$g(x):=\frac{l_{\alpha-\beta}(x)}{\left(S_{\alpha}x^{\beta}+S_{\beta}x^{\alpha}\right)^{\frac{q}{p}}}, \tag{21}$$

where $l_{\alpha-\beta}(x)$ is defined by (14). In view of Lemma 2, we easily observe that $g(x)$ is positive on $(0,\infty)$ and that $g(x)$ is strictly decreasing on $[1,\infty)$ and also $\lim_{x\to\infty} g(x) = 0$.

By applying the Euler-Maclaurin summation formula (20), we have

$$\sum_{n=1}^{\infty} g(n) < \int_1^{\infty} g(x)\,\mathrm{d}x + g(1) = \int_0^{\infty} g(x)\,\mathrm{d}x - \mathscr{C}_1, \tag{22}$$

where

$$\mathscr{C}_1 = \int_0^1 g(x)\,\mathrm{d}x - g(1) = \int_0^1 g(x)\,\mathrm{d}x - {}_1F_1\left(a;c;-\frac{b}{S_\beta^\rho S_\alpha^\lambda}\left(S_\beta+S_\alpha\right)^{\rho+\lambda}\right)\left(S_\alpha+S_\beta\right)^{-\frac{q}{p}}.$$

Using (16) and (21), we have

$$\int_0^{\infty} g(x)\,\mathrm{d}x = \frac{1}{(\alpha-\beta)}\frac{S_\alpha^{\lambda_\alpha}}{S_\beta^{\lambda_\beta}}\mathscr{B}_{b;\rho,\lambda}^{(a,c)}\left(\lambda_\beta,-\lambda_\alpha\right). \tag{23}$$

Now, by using the Hölder inequality, we find that

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n l_{\alpha-\beta}(n) &= \sum_{n=1}^{\infty} a_n \left(S_\beta n^\alpha + S_\alpha n^\beta\right)^{\frac{1}{p}} l_{\alpha-\beta}^{\frac{1}{p}}(n)\left(S_\beta n^\alpha + S_\alpha n^\beta\right)^{-\frac{1}{p}} l_{\alpha-\beta}^{\frac{1}{q}}(n) \\
&\leqslant \left[\sum_{n=1}^{\infty} a_n^p \left(S_\beta n^\alpha + S_\alpha n^\beta\right) l_{\alpha-\beta}(n)\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty}\left(S_\beta n^\alpha + S_\alpha n^\beta\right)^{-\frac{q}{p}} l_{\alpha-\beta}(n)\right] \\
&\leqslant l_{\alpha-\beta}^{\frac{1}{p}}(1)\left[S_\beta \sum_{n=1}^{\infty} n^\alpha a_n^p + S_\alpha \sum_{n=1}^{\infty} n^\beta a_n^p\right]^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} g(n)\right)^{\frac{1}{q}} \\
&\leqslant l_{\alpha-\beta}^{\frac{1}{p}}(1)\left(2S_\alpha S_\beta\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} g(n)\right)^{\frac{1}{q}}.
\end{aligned} \tag{24}$$

Hence from (22), (23) and (24), we obtain that

$$\begin{aligned}
\left(\sum_{n=1}^{\infty} a_n l_{\alpha-\beta}(n)\right)^p &\leqslant 2 l_{\alpha-\beta}(1) S_\alpha S_\beta \left(\sum_{n=1}^{\infty} g(n)\right)^{\frac{p}{q}} \\
&< 2 l_{\alpha-\beta}(1) S_\alpha S_\beta \left(\frac{1}{\alpha-\beta}\frac{S_\alpha^{\lambda_\alpha}}{S_\beta^{\lambda_\beta}}\mathscr{B}_{b;\rho,\lambda}^{(a,c)}\left(\lambda_\beta,-\lambda_\alpha\right) - \mathscr{C}_1\right)^{\frac{p}{q}},
\end{aligned} \tag{25}$$

which upon using the notation $\mathscr{L}(x) \equiv l_{\alpha-\beta}(x)$ gives the desired result (7). □

Proof of Theorem 2. Making use of the Hölder's inequality, we have

$$\begin{aligned}
&\int_{\varepsilon}^{\infty} f(x)\, l_{\alpha-\beta}(x)\,\mathrm{d}x \\
&= \int_{\varepsilon}^{\infty} f(x)^{\frac{1}{p}} \left(S_\beta x^\alpha + S_\alpha x^\beta\right)^{\frac{1}{p}} l_{\alpha-\beta}^{\frac{1}{p}}(x) \left(S_\beta x^\alpha + S_\alpha x^\beta\right)^{-\frac{1}{p}} l_{\alpha-\beta}^{\frac{1}{q}}(x)\,\mathrm{d}x \\
&\leqslant \left[\int_{\varepsilon}^{\infty} f^p(x) \left(S_\beta x^\alpha + S_\alpha x^\beta\right) l_{\alpha-\beta}(x)\,\mathrm{d}x\right]^{\frac{1}{p}} \left[\int_{\varepsilon}^{\infty} \left(S_\beta x^\alpha + S_\alpha x^\beta\right)^{-\frac{q}{p}} l_{\alpha-\beta}(x)\,\mathrm{d}x\right]^{\frac{1}{q}} \\
&\leqslant l_{\alpha-\beta}^{\frac{1}{p}}(\varepsilon) \left(2S_\alpha S_\beta\right)^{\frac{1}{p}} \left[\mathscr{B}_{b;\rho,\lambda}^{(a,c)}\left(\lambda_\beta, -\lambda_\alpha\right) - \int_0^{\varepsilon} \frac{l_{\alpha-\beta}(x)}{\left(S_\beta x^\alpha + S_\alpha x^\beta\right)^{\frac{q}{p}}}\mathrm{d}x\right]^{\frac{1}{q}}. \qquad (26)
\end{aligned}$$

The desired result (9) follows now upon using (10) and the notation that $\mathscr{L}(x) \equiv l_{\alpha-\beta}(x)$. □

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