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L_p–MIXED INTERSECTION BODIES

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Abstract. In this paper, we prove a dual Bergstrom type inequality for q-dual mixed volumes. Furthermore, we introduce the L_p-mixed intersection bodies for any real number $p \neq 0$, and establish some interesting inequalities for L_p-mixed intersection bodies involving a Minkowski type inequality and three Brunn-Minkowski type inequalities.

1. Introduction

The intersection operator and the class of intersection bodies were introduced by Lutwak [27]. It is well known that the intersection operator and the class of intersection bodies played a critical role in the solution of the famous Busemann-Petty problem, see [12, 32]. During the past 20 years significant advances have been made in our understanding of the intersection operator and the class of intersection bodies (see, e.g., [4, 5, 6, 11, 12, 14, 15], [17]–[23], [29, 31, 32]). As a result of the duality between projection and intersection bodies [27], together with the fact that the projections (onto lower dimensional subspaces) of projection bodies are themselves projection bodies, Lutwak conjectured the 'dual' statement for intersection bodies: when intersection bodies are intersected with lower dimensional subspaces, the results are intersection bodies as well (within the lower dimensional subspaces). This was proved by Fallert, Goodey and Weil [7]. In [25] (see also [26, 28]) Lutwak introduced the mixed projection bodies and derived their fundamental inequalities.

In 2006, Haberl and Ludwig [17] introduced the so-called L_p-intersection bodies. Recently, Zhao and Cheung [34] defined the q-dual mixed volumes of star bodies, which generalized the classical dual mixed volumes, and extended notions of L_p-intersection bodies to L_p-mixed intersection bodies.

In [2] Bergstrom presented an inequality, which later received his name, for symmetric positive definite matrices (see also [1, 8]). It is a natural problem whether there is a version of Bergstrom's inequality in the theory of mixed volumes. This question can be formulated as follows: for which values of i is it true that

$$\frac{W_i(K+L)}{W_{i+1}(K+L)} \geqslant \frac{W_i(K)}{W_{i+1}(K)} + \frac{W_i(L)}{W_{i+1}(L)} \tag{1.1}$$

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for every pair of convex bodies K and L in $\mathbb{R}^n$?

In 2003, Fradelizi, Giannopoulos and Meyer [10] proved that (1.1) holds for every pair of convex bodies K and L in $\mathbb{R}^n$ if and only if $i=n-2$ or $i=n-1$.

In this paper $\mathscr{S}^n$ will denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Under the L_p-radial addition of star bodies (see (2.1) below), we obtain a dual Bergstrom type inequality for q-dual mixed volumes: let $K,L,E\in\mathscr{S}^n$, if $i=n-2$ or $i=n-1$, then for $p\neq 0$,

$$\frac{\tilde{W}^p_{p,i}(K\tilde{+}_pM,E)}{\tilde{W}^p_{p,i+1}(K\tilde{+}_pM,E)}\leqslant\frac{\tilde{W}^p_{p,i}(K,E)}{\tilde{W}^p_{p,i+1}(K,E)}+\frac{\tilde{W}^p_{p,i}(M,E)}{\tilde{W}^p_{p,i+1}(M,E)}, \tag{1.2}$$

where $\tilde{W}_{p,i}(K,E)$ denotes the i-th p-dual relative quermassintegral of K with respect to E.

Next we introduce the notion of L_p-mixed intersection body $\mathrm{I}_p(K_1,\dots,K_{n-1})$ for any $p\neq 0$ (see Definition 4.1), and establish two interesting inequalities for L_p-mixed intersection bodies. One of them is the following Minkowski type inequality: given $K,L\in\mathscr{S}^n$, $0\leqslant i<n$, $0\leqslant j<n-1$, $i,j\in\mathbb{N}$, and any $p\neq 0$, we have

$$\tilde{W}_i^{n-1}(\mathrm{I}_{p,j}(K,L))\leqslant\tilde{W}_i^{n-j-1}(\mathrm{I}_pK)\tilde{W}_i^{j}(\mathrm{I}_pL), \tag{1.3}$$

with equality for $0<j<n-1$ if and only if K is a dilation of L. Here $\mathrm{I}_{p,j}(K,L)$ denotes the j-th L_p-mixed intersection body of K with respect to L and $\mathrm{I}_p(K)$ denotes the L_p-intersection body of K.

The other is a dual Aleksandrov-Fenchel inequality for L_p-mixed intersection bodies: if $K_1,\dots,K_{n-1}\in\mathscr{S}^n$, $1<m\leqslant n-1$, $0\leqslant i<n$, $i\in\mathbb{N}$, then for $p\neq 0$,

$$\begin{aligned}&\tilde{W}_i^{m}(\mathrm{I}_p(K_1,\dots,K_{n-1}))\\&\leqslant\prod_{j=0}^{m-1}\tilde{W}_i(\mathrm{I}_p(\underbrace{K_{n-1-j},\dots,K_{n-1-j}}_{m},K_1,\dots,K_{n-1-m})),\end{aligned} \tag{1.4}$$

with equality if and only if $K_{n-1},\dots,K_{n-m}$ are all dilations of each other.

Particularly, taking $p=1$ in (1.3) and (1.4), they are precisely the Minkowski inequality and the dual Aleksandrov-Fenchel inequality for mixed intersection bodies, respectively.

Finally, under the L_p-radial addition and the log-Minkowski radial addition (see (2.5) below) of star bodies, three Brunn-Minkowski type inequalities for L_p-mixed intersection bodies are obtained: let $K,L,E\in\mathscr{S}^n$; for $0\leqslant i<n$, $0\leqslant j<n-1$, $i,j\in\mathbb{N}$ and $\lambda\in[0,1]$, we have

(I) if $1\geqslant p^2>0$, then

$$\tilde{W}_i(\mathrm{I}_{p,j}((1-\lambda)\cdot K\tilde{+}_p\lambda\cdot L,E))\leqslant(1-\lambda)\tilde{W}_i(\mathrm{I}_{p,j}(K,E))+\lambda\tilde{W}_i(\mathrm{I}_{p,j}(L,E)). \tag{1.5}$$

If $0\leqslant i<n-1$, $0\leqslant j<n-2$, equality for some $\lambda\in(0,1)$ in (1.5) holds if and only if $K=L$;

(II) if $p\neq 0$, then

$$\tilde{W}_i(\mathrm{I}_{p,j}((1-\lambda)\cdot K\tilde{+}_0\lambda\cdot L,E))\leqslant\tilde{W}_i^{1-\lambda}(\mathrm{I}_{p,j}(K,E))\tilde{W}_i^{\lambda}(\mathrm{I}_{p,j}(L,E)), \tag{1.6}$$

with equality for some $\lambda \in (0,1)$ if and only if K is a dilation of L;

(III) if $q \geqslant p^2 \geqslant 1$, then

$$\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}((1-\lambda)\cdot K\tilde{+}\lambda\cdot L,E)) \leqslant (1-\lambda)\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(K,E)) + \lambda\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(L,E)). \tag{1.7}$$

If $0 \leqslant i < n-1$, $0 \leqslant j < n-2$, equality for some $\lambda \in (0,1)$ in (1.7) holds if and only if $K = L$.

Clearly, taking $j = 0$ in (1.5)–(1.7), we get some Brunn-Minkowski type inequalities for L_p-intersection bodies. In the same way, taking $p = 1$ and $q = 1$ in (1.5)–(1.7), we get the corresponding Brunn-Minkowski type inequalities for mixed intersection bodies.

This paper is organized as follows. In Section 2, we collect some basic concepts and various facts that will be used in the proofs of our results. In Sections 3 and 4, we show our main results and their proofs.

2. Notations and preliminaries

Let S^{n-1} and ω_n denote the unit sphere and n-dimensional volume of the unit ball B in $\mathbb{R}^n$, respectively. For a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, its *radial function*, $\rho_K = \rho(K,\cdot) : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$, is defined by

$$\rho(K,x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

If ρ_K is positive and continuous K will be called *a star body* (about the origin). We will denote by $V(K)$ its n-dimensional volume.

Let $K \in \mathscr{S}^n$ and let c be a real number. The *Minkowski scalar multiplication* cK is defined by

$$cK = \{cx : x \in K\}.$$

If c is a positive real number, from the definition of the radial function, it is easy to verify that

$$\rho_{cK}(u) = c\rho_K(u), \quad \rho_K(cu) = c^{-1}\rho_K(u), \quad u \in \mathbb{R}^n.$$

We say that $K, L \in \mathscr{S}^n$ are *dilations of each other* if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

Following [24], given $K, L \in \mathscr{S}^n$ and $\alpha, \beta \geqslant 0$, the *radial linear combination* $\alpha K\tilde{+}\beta L$ is defined by

$$\rho(\alpha K\tilde{+}\beta L,\cdot) = \alpha\rho(K,\cdot) + \beta\rho(L,\cdot). \tag{2.1}$$

Let $K_j (1 \leqslant j \leqslant n) \in \mathscr{S}^n$ be star bodies. The *dual mixed volume* $\tilde{V}(K_1, K_2, \cdots, K_n)$ is defined by (see [24])

$$\tilde{V}(K_1, K_2, \cdots, K_n) = \frac{1}{n}\int_{S^{n-1}} \rho_{K_1}(u)\rho_{K_2}(u)\cdots\rho_{K_n}(u)\mathrm{d}S(u), \tag{2.2}$$

where $\mathrm{d}S(u)$ is the $(n-1)$-dimensional volume element on S^{n-1}. If s,t are non-negative integers whose sum does not exceed n, K, L are star bodies and $\mathfrak{C}$ is the

$(n-s-t)$-tuple of star bodies $(C_1,\cdots,C_{n-s-t})$, $\tilde{V}(K[s],L[t],\mathfrak{C})$ will denote the dual mixed volume $\tilde{V}(K,\cdots,K,L,\cdots,L,C_1,\cdots,C_{n-s-t})$ in which K appears s times and L appears t times. In particular, if $K_1=\cdots=K_{n-i}=K$ and $K_{n-i+1}=\cdots=K_n=L$, then the dual mixed volume $\tilde{V}(K_1,K_2,\cdots,K_n)$ is written as $\tilde{W}_i(K,L)$, and is called the *i-th dual relative quermassintegral* of K with respect to L. Moreover, if $L=B$, then $\tilde{W}_i(K,B)$ is called the *i-th dual quermassintegral of* K and is written as $\tilde{W}_i(K)$.

We list some elementary properties of the dual mixed volume:

PROPOSITION 2.1. ([24]) *Let $K,K_i,L_i\in\mathscr{S}^n (1\leqslant i\leqslant n)$. Then*

(1) *$\tilde{V}$ is continuous;*

(2) *$\tilde{V}(K_1,K_2,\cdots,K_n)>0$;*

(3) *$\tilde{V}(\lambda_1K_1,\cdots,\lambda_nK_n)=\lambda_1\cdots\lambda_n\tilde{V}(K_1,\cdots,K_n)$, for $\lambda_1,\cdots,\lambda_n>0$;*

(4) *If $K_i\subset L_i$ for all i, then $\tilde{V}(K_1,\cdots,K_n)\leqslant\tilde{V}(L_1,\cdots,L_n)$ with equality if and only if $K_i=L_i$ for all i;*

(5) *$\tilde{V}(K_1,\cdots,K_i,\cdots,K_j,\cdots,K_n)=\tilde{V}(K_1,\cdots,K_j,\cdots,K_i,\cdots,K_n)$ for all i,j.*

(6) *$\tilde{V}(K,\cdots,K)=V(K)$;*

(7) *$\tilde{V}(K\tilde{+}L,K_2\cdots,K_n)=\tilde{V}(K,K_2\cdots,K_n)+\tilde{V}(L,K_2\cdots,K_n)$.*

The dual Aleksandrov-Fenchel inequality (cf. [24]) states that

$$\tilde{V}^m(K_1,K_2,\cdots,K_n)\leqslant\prod_{i=0}^{m-1}\tilde{V}(K_1,\cdots,K_{n-m},K_{n-i},\cdots,K_{n-i}), \tag{2.3}$$

where $1<m\leqslant n$, and with equality if and only if $K_1,\cdots,K_n$ are dilations of each other (with the origin as the center of dilation).

In the early 1960s, Firey (see e.g., [9], [13] and [30, p. 490]) defined the later so-called Firey L_p-combination (or simply L_p-combination) of convex bodies. Similarly, for $p\neq 0$, $\alpha,\beta\geqslant 0$, the *Minkowski-Firey L_p-radial combination* (or simply the L_p-radial combination) $\alpha\cdot K\tilde{+}_p\beta\cdot L$ of star bodies K,L is defined by

$$\rho(\alpha\cdot K\tilde{+}_p\beta\cdot L,u)=(\alpha\rho^p(K,u)+\beta\rho^p(L,u))^{\frac{1}{p}},\ \ u\in S^{n-1}. \tag{2.4}$$

Clearly, taking $p=1$, $\alpha\cdot K\tilde{+}_p\beta\cdot L$ becomes the classical radial combination.

Recently, the authors introduced the log-Minkowski combination of convex bodies [3]. Similarly, they defined the *log-Minkowski radial combinations* $\alpha\cdot K\tilde{+}_o\beta\cdot L$ of star bodies K,L by

$$\rho(\alpha\cdot K\tilde{+}_o\beta\cdot L,u)=\rho^\alpha(K,u)\rho^\beta(L,u),\ \ u\in S^{n-1}. \tag{2.5}$$

From (2.4) and (2.5) we can easily get that the log-Minkowski radial combination is the limit of the L_p-radial combination as p tends to 0.

3. q-dual mixed volumes

Let $K_1,\dots,K_n \in \mathscr{S}^n$ be star bodies. For a real number $q \neq 0$, Zhao and Cheung defined in [34] the q-dual mixed volume $\tilde{V}_q(K_1,\dots,K_n)$ by

$$\tilde{V}_q(K_1,\dots,K_n) = \omega_n \left(\frac{1}{n\omega_n}\int_{S^{n-1}} \rho^q(K_1,u)\cdots\rho^q(K_n,u)\mathrm{d}S(u)\right)^{\frac{1}{q}}. \tag{3.1}$$

Clearly, $\tilde{V}_1(K_1,\dots,K_n) = \tilde{V}(K_1,\dots,K_n)$, where $\tilde{V}(K_1,\dots,K_n)$ is the classical dual mixed volume which was defined by Lutwak [24]. Particularly, if $K_1 = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then $\tilde{V}_q(K,\dots,K,L,\dots,L)$ is written as $\tilde{W}_{q,i}(K,L)$ and is called the *i-th q-dual relative quermassintegral of K with respect to L*. Moreover, if $L = B$, then $\tilde{W}_{q,i}(K,B)$ is called the *i-th q-dual quermassintegral of K*, and is written as $\tilde{W}_{q,i}(K)$. When $i = 0$, $\tilde{V}_q(\underbrace{K,\dots,K}_{n})$ is written as $\tilde{V}_q(K)$.

For q equal to $-\infty, 0$ or ∞, we define the q-dual mixed volume by

$$\tilde{V}_q(K_1,\dots,K_n) = \lim_{s\to q}\tilde{V}_s(K_1,\dots,K_n).$$

As a direct consequence of Jensen's inequality we have

PROPOSITION 3.1. *Let $K_1,\dots,K_n \in \mathscr{S}^n$, then for $-\infty \leqslant p < q \leqslant \infty$,*

$$\tilde{V}_p(K_1,\dots,K_n) \leqslant \tilde{V}_q(K_1,\dots,K_n),$$

with equality if and only if $\rho(K_1,u)\cdots\rho(K_n,u)$ is constant for all $u \in S^{n-1}$.

For the q-dual mixed volume, Zhao and Cheung [34] got some elementary properties that we collect in the following lemma.

LEMMA 3.2. ([34]) *Let $K_1,\dots,K_n \in \mathscr{S}^n$, then for $q \neq 0$,*

(1) *$\tilde{V}_q$ is continuous;*

(2) *$\tilde{V}_q(K_1,K_2,\cdots,K_n) > 0$;*

(3) *$\tilde{V}_q(\lambda_1 K_1,\cdots,\lambda_n K_n) = \lambda_1\cdots\lambda_n \tilde{V}_q(K_1,\cdots,K_n)$, for $\lambda_1,\cdots,\lambda_n > 0$;*

(4) *$\tilde{V}_q(K_1,\cdots,K_i,\cdots,K_j,\cdots,K_n) = \tilde{V}_q(K_1,\cdots,K_j,\cdots,K_i,\cdots,K_n)$ for all $1 \leqslant i,j \leqslant n$.*

For the q-dual mixed volume, in view of (3.1) and Hölder's inequality, a dual Aleksandrov-Fenchel inequality between q-dual mixed volumes is obtained.

LEMMA 3.3. *Let $K_1,\dots,K_n \in \mathscr{S}^n$. If $1 < m \leqslant n$ and $q \neq 0$, then*

$$\tilde{V}_q^{qm}(K_1,\dots,K_n) \leqslant \prod_{i=0}^{m-1} \tilde{V}_q^q(\underbrace{K_{n-i},\dots,K_{n-i}}_{m},K_1,\dots,K_{n-m}), \tag{3.2}$$

with equality if and only if $K_{n-m+1},\dots,K_n$ are all dilations of each other (with the origin as the center of dilation).

Taking $q = 1$ in (3.2), then it becomes the classical dual Aleksandrov-Fenchel inequality between dual mixed volumes, which is due to Lutwak [24].

For the q-dual mixed volume, in view of (3.1) and the definition of the Minkowski-Firey L_p-radial combination, we can get that $\tilde{V}_p^p$ is linear under such an operation. This is the content of the following lemma.

LEMMA 3.4. *Let $K_1,\ldots,K_{n-1},K,L \in \mathscr{S}^n$ and set $\mathscr{M} = (K_1,\ldots,K_{n-1})$. Then for $\alpha > 0, \beta > 0$,*

$$\tilde{V}_p^p(\alpha \cdot K \tilde{+}_p \beta \cdot L, \mathscr{M}) = \alpha \tilde{V}_p^p(K,\mathscr{M}) + \beta \tilde{V}_p^p(L,\mathscr{M}). \tag{3.3}$$

Motivated by [30, Theorem 7.4.3] we establish the following Aleksandrov-Fenchel type inequality for the q-dual mixed volume.

PROPOSITION 3.5. *Let $K,L,M,K_3,\ldots,K_n \in \mathscr{S}^n$ and $p \neq 0$. Set $\mathfrak{C} = (K_3,\cdots,K_n)$, then*

$$\frac{\tilde{V}_p^p(K,K,\mathfrak{C})}{\tilde{V}_p^{2p}(K,M,\mathfrak{C})} - \frac{2\tilde{V}_p^p(K,L,\mathfrak{C})}{\tilde{V}_p^p(K,M,\mathfrak{C})\tilde{V}_p^p(L,M,\mathfrak{C})} + \frac{\tilde{V}_p^p(L,L,\mathfrak{C})}{\tilde{V}_p^{2p}(L,M,\mathfrak{C})} \geqslant 0. \tag{3.4}$$

Proof. Let $K_0,K_1,K_2 \in \mathscr{S}^n$ and $V_{ij} = \tilde{V}_p^p(K_i,K_j,\mathfrak{C})$ for $i,j = 0,1,2$. For $\lambda_1,\lambda_2 \geqslant 0$, Lemmata 3.3 and 3.4 give

$$\begin{aligned}
0 \geqslant & \tilde{V}_p^{2p}(K_1 \tilde{+}_p \lambda_1 \cdot K_0, K_2 \tilde{+}_p \lambda_2 \cdot K_0, \mathfrak{C}) \\
& - \tilde{V}_p^p(K_1 \tilde{+}_p \lambda_1 \cdot K_0, K_1 \tilde{+}_p \lambda_1 \cdot K_0, \mathfrak{C}) \tilde{V}_p^p(K_2 \tilde{+}_p \lambda_2 \cdot K_0, K_2 \tilde{+}_p \lambda_2 \cdot K_0, \mathfrak{C}) \\
= & (V_{12} + \lambda_1 V_{02} + \lambda_2 V_{01} + \lambda_1\lambda_2 V_{00})^2 \\
& - (V_{11} + 2\lambda_1 V_{01} + \lambda_1^2 V_{00})(V_{22} + 2\lambda_2 V_{02} + \lambda_2^2 V_{00}) \\
= & V_{12}^2 - V_{11}V_{22} + 2\lambda_1 V_{12}V_{02} + 2\lambda_2 V_{12}V_{01} - 2\lambda_1 V_{01}V_{22} - 2\lambda_2 V_{11}V_{02} \\
& + \lambda_1^2(V_{02}^2 - V_{00}V_{22}) + \lambda_2^2(V_{01}^2 - V_{00}V_{11}) + 2\lambda_1\lambda_2(V_{12}V_{00} - V_{01}V_{02}).
\end{aligned} \tag{3.5}$$

With a similar argument to [30, Lemma 7.4.1] (or [10, Lemma 2.1]), we can obtain

$$(V_{12}V_{00} - V_{01}V_{02})^2 \leqslant (V_{01}^2 - V_{00}V_{11})(V_{02}^2 - V_{00}V_{22}). \tag{3.6}$$

The proof now concludes by following the same steps as in the proof of [30, Theorem 7.4.3] and taking into account that $V_{01}^2 - V_{00}V_{11} \leqslant 0$ and $V_{02}^2 - V_{00}V_{22} \leqslant 0$. $\square$

From (3.6) we get

$$|V_{12}V_{00} - V_{01}V_{02}| \leqslant (V_{00}V_{11} - V_{01}^2)^{\frac{1}{2}} (V_{00}V_{22} - V_{02}^2)^{\frac{1}{2}}.$$

Thus we have proved the following.

COROLLARY 3.6. *Let $K,L,M \in \mathscr{S}^n$ and $p \neq 0$. Then*

$$\begin{aligned}
& \tilde{V}_p^p(K,M,\mathfrak{C})\tilde{V}_p^p(L,L,\mathfrak{C}) - \tilde{V}_p^p(K,L,\mathfrak{C})\tilde{V}_p^p(M,L,\mathfrak{C}) \\
\leqslant & \left(\tilde{V}_p^p(L,L,\mathfrak{C})\tilde{V}_p^p(K,K,\mathfrak{C}) - \tilde{V}_p^{2p}(K,L,\mathfrak{C})\right)^{\frac{1}{2}} \\
& \times \left(\tilde{V}_p^p(L,L,\mathfrak{C})\tilde{V}_p^p(M,M,\mathfrak{C}) - \tilde{V}_p^{2p}(M,L,\mathfrak{C})\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.7}$$

Based on Corollary 3.6, following the same steps as [10, Proposition 2.1], we can establish the dual Bergstrom type inequality for p-dual mixed volumes.

THEOREM 3.7. *Let K,M,E be star bodies and $p\neq 0$. Then, for $i=n-2$ or $i=n-1$,*

$$\frac{\tilde{W}^p_{p,i}(K\tilde{+}_pM,E)}{\tilde{W}^p_{p,i+1}(K\tilde{+}_pM,E)}\leqslant\frac{\tilde{W}^p_{p,i}(K,E)}{\tilde{W}^p_{p,i+1}(K,E)}+\frac{\tilde{W}^p_{p,i}(M,E)}{\tilde{W}^p_{p,i+1}(M,E)}. \tag{3.8}$$

with equality if K,M,E are all dilations of each other (with the origin as the center of dilation).

4. L_p-mixed intersection bodies

Let $K\in\mathscr{S}^n$. For $p<1$, Haberl and Ludwig [17] introduced the L_p-intersection body I_pK by

$$\rho^p(\mathrm{I}_pK,u)=\int_K|u\cdot x|^{-p}\mathrm{d}x.$$

From [16] one gets

$$v(K\cap u^+)=\lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\int_K|u\cdot x|^{-1+\varepsilon}\mathrm{d}x$$

and

$$\rho(\mathrm{I}K,u)=\lim_{p\to 1_-}\frac{1-p}{2}\rho^p(\mathrm{I}_pK,u),$$

where $u^+=\{x\in\mathbb{R}^n : u\cdot x\geqslant 0\}$.

Let $K_1,\dots,K_{n-1}\in\mathscr{S}^n$. For any $p\neq 0$, we define the L_p-mixed intersection body of $K_1,\dots,K_{n-1}$ as follows.

DEFINITION 4.1. For $p\neq 0$, the L_p-mixed intersection body of $K_1,\dots,K_{n-1}\in\mathscr{S}^n$, $\mathrm{I}_p(K_1,\dots,K_{n-1})$, is defined by

$$\rho^p(\mathrm{I}_p(K_1,\dots,K_{n-1}),u)=\begin{cases}\frac{2}{1-p}\tilde{v}_p(K_1\cap E_u,\dots,K_{n-1}\cap E_u), & p<1;\\ \tilde{v}_p(K_1\cap E_u,\dots,K_{n-1}\cap E_u), & p\geqslant 1.\end{cases}$$

Here E_u denotes the hyperplane passing through the origin which is orthogonal to u, and we write $\tilde{v}_p(K_1\cap E_u,\dots,K_{n-1}\cap E_u)$ to denote the p-dual mixed volume of $K_1\cap E_u,\dots,K_{n-1}\cap E_u$ computed in the $(n-1)$-dimensional space.

In the same way, $\tilde{w}_{p,i}(K\cap E_u,L\cap E_u)$ will denote the i-th p-dual quermassintegral and thus, in particular, $\tilde{w}_{p,i}(K\cap E_u,B\cap E_u)$ will be written for short as $\tilde{w}_{p,i}(K\cap E_u)$ (and consequently $\tilde{w}_{p,0}(K\cap E_u)$ will be denoted as $\tilde{v}_p(K\cap E_u)$).

REMARK 1. When $p<1$, this notation was introduced by Zhao and Cheung [34].

From Definition 4.1, we can get that $\rho^p(\mathrm{I}_p(K_1,\ldots,K_{n-1}),u)$ is continuous with respect to p (see [34, p. 432]). For the L_p-mixed intersection body, $\mathrm{I}_p(K_1,\ldots,K_{n-1})$, if $K_1=\ldots=K_{n-1-i}=K$, $K_{n-i}=\ldots=K_{n-1}=L$, then $\mathrm{I}_p(K_1,\ldots,K_{n-1})$ is written as $\mathrm{I}_{p,i}(K,L)$. If $L=B$, then $\mathrm{I}_{p,i}(K,L)$ is written as $\mathrm{I}_{p,i}(K)$ and is called the *i-th L_p-mixed intersection body* of K. $\mathrm{I}_{p,0}(K)$ is simply written as I_pK, since it is just the L_p-intersection body of K (see [17, 33]). Finally, when $p=1$, $\mathrm{I}(K_1,\ldots,K_{n-1})$ is called the mixed intersection body of $K_1,\ldots,K_{n-1}$.

From (3.1) and Definition 4.1, we easily obtain the following Lemma.

LEMMA 4.2. *Let $K,L\in\mathscr{S}^n$, $0\leqslant i<n$, $0\leqslant j<n-1$, $i,j\in\mathbb{N}$. For $p\neq 0$, $q\neq 0$, then*

$$\tilde{W}_{q,i}(\mathrm{I}_pK)=C(n,p,q)\left(\int_{S^{n-1}}\left(\tilde{v}_p(K\cap E_u)\right)^{\frac{(n-i)q}{p}}\mathrm{d}S(u)\right)^{\frac{1}{q}}, \tag{4.1}$$

$$\tilde{W}_{q,i}(\mathrm{I}_{p,j}K)=C(n,p,q)\left(\int_{S^{n-1}}\left(\tilde{w}_{p,j}(K\cap E_u)\right)^{\frac{(n-i)q}{p}}\mathrm{d}S(u)\right)^{\frac{1}{q}}, \tag{4.2}$$

$$\tilde{W}_{q,i}(\mathrm{I}_{p,j}(K,L))=C(n,p,q)\left(\int_{S^{n-1}}\left(\tilde{w}_{p,j}(K\cap E_u,L\cap E_u)\right)^{\frac{(n-i)q}{p}}\mathrm{d}S(u)\right)^{\frac{1}{q}}, \tag{4.3}$$

where

$$C(n,p,q)=\begin{cases}\omega_n\left(\frac{1}{n\omega_n}\right)^{\frac{1}{q}}\left(\frac{2}{1-p}\right)^{\frac{(n-i)}{p}}, & p<1;\\ \omega_n\left(\frac{1}{n\omega_n}\right)^{\frac{1}{q}}, & p\geqslant 1.\end{cases}$$

The following Minkowski type inequality for L_p-mixed intersection bodies stated in the introduction will be established:

Let $K,L\in\mathscr{S}^n$, $0\leqslant i<n$, $0\leqslant j<n-1$, $i,j\in\mathbb{N}$. Then for $p\neq 0$,

$$\tilde{W}_i^{n-1}(\mathrm{I}_{p,j}(K,L))\leqslant\tilde{W}_i^{n-j-1}(\mathrm{I}_pK)\tilde{W}_i^{j}(\mathrm{I}_pL), \tag{4.4}$$

with equality for $0<j<n-1$ if and only if K and L are dilations of each other.

This is just the case $q=1$ of the following theorem.

THEOREM 4.3. *Let $K,L\in\mathscr{S}^n$, $0\leqslant i<n$, $0\leqslant j<n-1$, $i,j\in\mathbb{N}$. For $p\neq 0$, $q>0$, then*

$$\tilde{W}_{q,i}^{n-1}(\mathrm{I}_{p,j}(K,L))\leqslant\tilde{W}_{q,i}^{n-j-1}(\mathrm{I}_pK)\tilde{W}_{q,i}^{j}(\mathrm{I}_pL), \tag{4.5}$$

with equality for $0<j<n-1$ if and only if K is a dilation of L (with the origin as the center of dilation).

Proof. For $j=0$, it is trivial. For $0<j<n-1$, taking $K_1=\cdots=K_{n-j}=K$, $K_{n-j+1}=\cdots=K_n=L$ and $m=n$ in Lemma 3.3, and in view of taking p for q, we obtain

$$\tilde{W}_{p,j}^{np}(K,L)\leqslant\tilde{V}_p^{(n-j)p}(K)\tilde{V}_p^{jp}(L),$$

with equality if and only if K is a dilation of L. Hence, in the $(n-1)$-dimensional space, we have

$$\tilde{w}_{p,j}^{p(n-1)}(K\cap E_u, L\cap E_u) \leqslant \tilde{v}_p^{(n-j-1)p}(K\cap E_u)\tilde{v}_p^{jp}(L\cap E_u), \tag{4.6}$$

with equality if and only if $K\cap E_u$ and $L\cap E_u$ are dilations of each other for a fixed $u\in S^{n-1}$.

For $p\neq 0$, $q>0$, from Lemma 4.2, (4.6) and Hölder's inequality for integrals, we have

$$\begin{aligned}
&\tilde{W}_{q,i}^{q}(\mathrm{I}_{p,j}(K,L))\\
\leqslant &C^q(n,p,q)\int_{S^{n-1}}\left(\tilde{v}_p^{\frac{n-j-1}{n-1}}(K\cap E_u)\tilde{v}_p^{\frac{j}{n-1}}(L\cap E_u)\right)^{\frac{(n-i)q}{p}}\mathrm{d}S(u)\\
\leqslant &C^q(n,p,q)\left(\int_{S^{n-1}}\tilde{v}_p^{\frac{(n-i)q}{p}}(K\cap E_u)\mathrm{d}S(u)\right)^{\frac{n-j-1}{n-1}}\left(\int_{S^{n-1}}\tilde{v}_p^{\frac{(n-i)q}{p}}(L\cap E_u)\mathrm{d}S(u)\right)^{\frac{j}{n-1}}\\
= &\left(\tilde{W}_{q,i}(\mathrm{I}_pK)\right)^{\frac{(n-j-1)q}{n-1}}\left(\tilde{W}_{q,i}(\mathrm{I}_pL)\right)^{\frac{jq}{n-1}},
\end{aligned}$$

which together with $q>0$ gives $\tilde{W}_{q,i}^{n-1}(\mathrm{I}_{p,j}(K,L)) \leqslant \tilde{W}_{q,i}^{n-j-1}(\mathrm{I}_pK)\tilde{W}_{q,i}^{j}(\mathrm{I}_pL)$. The equality conditions of both (4.6) and Hölder's inequality for integrals imply that the equality holds in (4.5) if and only if K is a dilation of L. $\square$

The following dual Aleksandrov-Fenchel inequality for L_p-mixed intersection bodies stated in the introduction will be established:

If $K_1,\dots,K_{n-1}\in\mathscr{S}^n$, $1<m\leqslant n-1$, $0\leqslant i<n$, $i\in\mathbb{N}$, then

$$\tilde{W}_i^m(\mathrm{I}_p(K_1,\dots,K_{n-1})) \leqslant \prod_{j=0}^{m-1}\tilde{W}_i(\mathrm{I}_p(\underbrace{K_{n-1-j},\dots,K_{n-1-j}}_{m},K_1,\dots,K_{n-1-m})), \tag{4.7}$$

with equality if and only if $K_{n-m},\dots,K_{n-1}$ are all dilations of each other.

This is just the case $q=1$ of the following theorem.

THEOREM 4.4. *Let* $K_1,\dots,K_{n-1}\in\mathscr{S}^n$, $1<m\leqslant n-1$, $0\leqslant i<n$, $i\in\mathbb{N}$. *For* $p\neq 0$, $q>0$, *we have*

$$\tilde{W}_{q,i}^m(I_p(K_1,\dots,K_{n-1})) \leqslant \prod_{j=0}^{m-1}\tilde{W}_{q,i}(I_p(\underbrace{K_{n-1-j},\dots,K_{n-1-j}}_{m},K_1,\dots,K_{n-1-m})), \tag{4.8}$$

with equality if and only if $K_{n-m},\dots,K_{n-1}$ *are all dilations of each other (with the origin as the center of dilation).*

Proof. It is similar to that of Theorem 4.3. $\square$

Under the L_p-radial combination of star bodies, the following Brunn-Minkowski type inequality for L_p-mixed intersection bodies stated in the introduction will be established: let $K,L,E\in\mathscr{S}^n$, $0\leqslant i<n$, $0\leqslant j<n-1$, $i,j\in\mathbb{N}$. Then for all $\lambda\in[0,1]$

and $1 \geqslant p^2 > 0$,

$$\tilde{W}_i(\mathrm{I}_{p,j}((1-\lambda)\cdot K\tilde{+}_p\lambda\cdot L,E))\leqslant(1-\lambda)\tilde{W}_i(\mathrm{I}_{p,j}(K,E))+\lambda\tilde{W}_i(\mathrm{I}_{p,j}(L,E)). \tag{4.9}$$

When $0 \leqslant i < n-1$, $0 \leqslant j < n-2$, equality holds for some $\lambda \in (0,1)$ in (4.9) if and only if $K = L$.

This is just the case $q = 1$ of the following theorem.

THEOREM 4.5. *Let $K,L,E \in \mathscr{S}^n$, $0 \leqslant i < n$, $0 \leqslant j < n-1$, $i,j \in \mathbb{N}$. Then for all $\lambda \in [0,1]$ and $q \geqslant p^2 > 0$,*

$$\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}((1-\lambda)\cdot K\tilde{+}_p\lambda\cdot L,E))\leqslant(1-\lambda)\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(K,E))+\lambda\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(L,E)). \tag{4.10}$$

Moreover, if $1 \geqslant q \geqslant p^2 > 0$, then

$$\tilde{W}_{q,i}(\mathrm{I}_{p,j}((1-\lambda)\cdot K\tilde{+}_p\lambda\cdot L,E))\leqslant(1-\lambda)\tilde{W}_{q,i}(\mathrm{I}_{p,j}(K,E))+\lambda\tilde{W}_{q,i}(\mathrm{I}_{p,j}(L,E)). \tag{4.11}$$

When $0 \leqslant i < n-1$, $0 \leqslant j < n-2$, equality in (4.10) (respectively (4.11)) for some $\lambda \in (0,1)$ holds if and only if $K = L$.

Proof. Set $M_{\lambda,p} = (1-\lambda)\cdot K\tilde{+}_p\lambda\cdot L$ and $u \in S^{n-1}$. For any $v \in S^{n-1}\cap E_u$, from (2.4) one can get

$$\begin{aligned}\rho^p(M_{\lambda,p}\cap E_u,v) =&\rho^p(M_{\lambda,p},v) = (1-\lambda)\rho^p(K,v)+\lambda\rho^p(L,v)\\ =&(1-\lambda)\rho^p(K\cap E_u,v)+\lambda\rho^p(L\cap E_u,v).\end{aligned}$$

By Jensen's inequality, one gets

$$\begin{aligned}&\rho^{p(n-1-j)}(M_{\lambda,p}\cap E_u,v)\\ \leqslant&(1-\lambda)\rho^{p(n-1-j)}(K\cap E_u,v)+\lambda\rho^{p(n-1-j)}(L\cap E_u,v),\end{aligned} \tag{4.12}$$

where equality for some $\lambda \in (0,1)$ and all $v \in S^{n-1}\cap E_u$, since $0 \leqslant j < n-2$, holds if and only if $K\cap E_u = L\cap E_u$.

It follows from (3.1) that

$$\begin{aligned}&\tilde{w}^p_{p,j}(M_{\lambda,p}\cap E_u,E\cap E_u)\\ =&\frac{\omega^p_{n-1}}{(n-1)\omega_{n-1}}\int_{S^{n-1}\cap E_u}\rho^{p(n-1-j)}(M_{\lambda,p}\cap E_u,v)\rho^{pj}(E\cap E_u,v)\mathrm{d}\mu(v),\end{aligned} \tag{4.13}$$

where $\mu(v)$ denotes the $(n-2)$-dimensional Lebesgue measure.

Combining (4.12) with (4.13) we get

$$\begin{aligned}&\tilde{w}^p_{p,j}(M_{\lambda,p}\cap E_u,E\cap E_u)\\ \leqslant&(1-\lambda)\tilde{w}^p_{p,j}(K\cap E_u,E\cap E_u)+\lambda\tilde{w}^p_{p,j}(L\cap E_u,E\cap E_u),\end{aligned} \tag{4.14}$$

where equality for some $\lambda \in (0,1)$ holds if and only if $K\cap E_u = L\cap E_u$.

By Lemma 4.2 one can get

$$\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(M_{\lambda,p},E)) = C^q(n,p,q)\int_{S^{n-1}}[\tilde{w}^p_{p,j}(M_{\lambda,p}\cap E_u, E\cap E_u)]^{\frac{(n-i)q}{p^2}}\,\mathrm{d}S(u). \tag{4.15}$$

From (4.14) and (4.15) we have

$$\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(M_{\lambda,p},E)) \leqslant C^q(n,p,q)$$
$$\times\int_{S^{n-1}}[(1-\lambda)\tilde{w}^p_{p,j}(K\cap E_u,E\cap E_u)+\lambda\tilde{w}^p_{p,j}(L\cap E_u,E\cap E_u)]^{\frac{(n-i)q}{p^2}}\,\mathrm{d}S(u).$$

Since $q \geqslant p^2 > 0$, $\frac{(n-i)q}{p^2} \geqslant n-i \geqslant 1$, applying Jensen's inequality, one can obtain

$$\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(M_{\lambda,p},E)) \leqslant (1-\lambda)\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(K,E)) + \lambda\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(L,E)).$$

If $0 \leqslant i < n-1$, $0 \leqslant j < n-2$ and $\lambda \in (0,1)$, the equality condition in (4.14) implies that the equality in (4.10) holds if and only if $K = L$.

If $1 \geqslant q \geqslant p^2 > 0$, (4.11) is a direct consequence of (4.10). □

For the log-Minkowski radial combination, the following Brunn-Minkowski type inequality for L_p-mixed intersection bodies stated in the introduction will be established: let $0 \leqslant i < n$, $0 \leqslant j < n-1$, $i,j \in \mathbb{N}$. Then for all $\lambda \in [0,1]$ and $p \neq 0$

$$\tilde{W}_i(\mathrm{I}_{p,j}((1-\lambda)\cdot K \tilde{+}_0 \lambda\cdot L, E)) \leqslant \tilde{W}_i^{1-\lambda}(\mathrm{I}_{p,j}(K,E))\tilde{W}_i^{\lambda}(\mathrm{I}_{p,j}(L,E)), \tag{4.16}$$

with equality for some $\lambda \in (0,1)$ if and only if K is a dilation of L.

This is just the case $q = 1$ of the following theorem.

THEOREM 4.6. *Let $K,L,E \in \mathscr{S}^n$, $0 \leqslant i < n$, $0 \leqslant j < n-1$, $i,j \in \mathbb{N}$. For all $\lambda \in [0,1]$ and $p \neq 0$, $q > 0$,*

$$\tilde{W}_{q,i}(\mathrm{I}_{p,j}((1-\lambda)\cdot K \tilde{+}_0 \lambda\cdot L, E)) \leqslant \tilde{W}_{q,i}^{1-\lambda}(\mathrm{I}_{p,j}(K,E))\tilde{W}_{q,i}^{\lambda}(\mathrm{I}_{p,j}(L,E)), \tag{4.17}$$

with equality for some $\lambda \in (0,1)$ if and only if K is a dilation of L (with the origin as the center of dilation).

Proof. Setting $M_\lambda = (1-\lambda)\cdot K \tilde{+}_0 \lambda\cdot L$ and $u \in S^{n-1}$, it is immediate to see that, for any $v \in S^{n-1}\cap E_u$,

$$\rho(M_\lambda\cap E_u, v) = \rho^{1-\lambda}(K\cap E_u, v)\rho^{\lambda}(L\cap E_u, v).$$

This identity together with (3.1) and Hölder's inequality allow us to write

$$\tilde{w}^p_{p,j}(M_\lambda\cap E_u, E\cap E_u) \leqslant \tilde{w}^{p(1-\lambda)}_{p,j}(K\cap E_u, E\cap E_u)\tilde{w}^{p\lambda}_{p,j}(L\cap E_u, E\cap E_u),$$

with equality for some $\lambda \in (0,1)$ if and only if $K\cap E_u$ and $L\cap E_u$ are dilatates. The proof now concludes by using Lemma 4.2 and applying the above relation together with Hölder's inequality. □

For the classical Minkowski radial combination, the Brunn-Minkowski type inequality for L_p-mixed intersection bodies stated in the introduction is established:

THEOREM 4.7. *Let* $K,L,E \in \mathscr{S}^n$, $0 \leqslant i < n$, $0 \leqslant j < n-1$, $i,j \in \mathbb{N}$. *For all* $\lambda \in [0,1]$ *and* $q \geqslant p^2 \geqslant 1$, *we have*

$$\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}((1-\lambda)\cdot K \tilde{+} \lambda \cdot L, E)) \leqslant (1-\lambda)\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(K,E)) + \lambda \tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(L,E)). \quad (4.18)$$

When $0 \leqslant i < n-1$ *and* $0 \leqslant j < n-2$, *equality for some* $\lambda \in (0,1)$ *in* (4.18) *holds if and only if* $K = L$.

Proof. Since $p^2 \geqslant 1$, with the same argument as that of the proof of Theorem 4.5 we have

$$\begin{aligned} &\tilde{w}^p_{p,j}(M_{\lambda,1} \cap E_u, E \cap E_u) \\ \leqslant &(1-\lambda)\tilde{w}^p_{p,j}(K \cap E_u, E \cap E_u) + \lambda \tilde{w}^p_{p,j}(L \cap E_u, E \cap E_u), \end{aligned} \quad (4.19)$$

with equality for $\lambda \in (0,1)$ and $0 \leqslant j < n-2$ if and only if $K \cap E_u = L \cap E_u$.

For $q \geqslant p^2 \geqslant 1$, it follows from (4.15) and (4.19) that

$$\begin{aligned} &\tilde{W}^q_{q,i}(\mathrm{I}_{p,j}(M_{\lambda,1},E)) \leqslant C^q(n,p,q) \\ &\times \int_{S^{n-1}} [(1-\lambda)\tilde{w}^p_{p,j}(K \cap E_u, E \cap E_u) + \lambda \tilde{w}^p_{p,j}(L \cap E_u, E \cap E_u)]^{\frac{(n-i)q}{p^2}} \mathrm{d}S(u) \\ \leqslant &(1-\lambda)\tilde{W}^q_{q,i}(\mathrm{I}_p(K,E)) + \lambda \tilde{W}^q_{q,i}(\mathrm{I}_p(L,E)), \end{aligned} \quad (4.20)$$

where equality for $\lambda \in (0,1)$, $0 \leqslant i < n-1$ and $0 \leqslant j < n-2$ holds if and only if $K = L$. $\square$

REMARK 2. Taking $p = q = 1$ in Theorems 4.6 and 4.7, respectively, we obtain the corresponding Brunn-Minkowski type inequality for mixed intersection bodies.

From the proof of Theorem 4.7, we can get the following results.

COROLLARY 4.8. *Let* $K,L,E \in \mathscr{S}^n$, $0 \leqslant i < n$, $0 \leqslant j < n-1$, $i,j \in \mathbb{N}$. *Some sufficient conditions for the parameters* p,q *in order that* (4.18) *holds for all* $\lambda \in [0,1]$ *are that*

(1) $0 < p < \frac{1}{n-1}$ *and* $q < 0$,

(2) $p < 0$ *and* $q > p^2$.

Furthermore, a sufficient condition for p,q *in order that* (4.18) *holds with the opposite sign for all* $\lambda \in [0,1]$ *is that*

(3) $0 < p < \frac{1}{n-1}$ *and* $0 < nq < p^2$.

In all the cases, equality holds for some $\lambda \in (0,1)$ *if and only if* $K = L$.

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