Mathematical
Inequalities
& Applications
Volume 19, Number 2 (2016), 439–449 doi:10.7153/mia-19-35

STARLIKENESS OF BESSEL FUNCTIONS AND THEIR DERIVATIVES

ÁRPÁD BARICZ, MURAT ÇAĞLAR AND ERHAN DENIZ

Dedicated to Aysima, Boróka, Eren and Koppány

(*Communicated by T. Erdélyi*)

Abstract. In this paper necessary and sufficient conditions are obtained for the starlikeness of Bessel functions of the first kind and their derivatives of the second and third order by using a result of Shah and Trimble about transcendental entire functions with univalent derivatives and Mittag-Leffler expansions for the derivatives of Bessel functions of the first kind, as well as some results on the zeros of these functions.

1. Introduction and the Main Results

Geometric properties of Bessel functions of the first kind J_ν, like univalence, starlikeness, spirallikeness and convexity were studied in the sixties by Brown [10, 11, 12], and also by Kreyszig and Todd [15]. Other geometric properties of Bessel functions of the first kind were studied later in the papers [2, 3, 4, 6, 7, 8, 22, 23]. Very recently, in [9] the close-to-convexity of the derivatives of Bessel functions was considered. Motivated by the above results, in this paper we make a contribution to the subject by obtaining some necessary and sufficient conditions for the starlikeness of Bessel functions of the first kind and second and third order derivatives by using a result of Shah and Trimble [20, Theorem 2] about transcendental entire functions with univalent derivatives and Mittag-Leffler expansions for the derivatives of Bessel functions of the first kind, as well as some results on the zeros of these functions. For more details on Bessel functions of the first kind we refer to the book of Watson [24].

Our first set of sharp results are about the starlikeness of order α of two normalized Bessel functions of the first kind. We note that these results naturally complement the main results of [6, 8, 22].

Mathematics subject classification (2010): 33C10, 30C45.

Keywords and phrases: Bessel functions of the first kind, close-to-convex functions, starlike functions, transcendental entire functions, zeros of Bessel functions, infinite product, derivatives of Bessel functions.

The research of Á. Baricz was supported by a research grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-RU-TE-2012-3-0190. The work of Á. Baricz was completed during his visit in September 2014 to Department of Mathematics of Kafkas University in Kars, to which this author is grateful for hospitality. The authors are very grateful to the referee for carefully reading the paper and for the corrections.

Paper MIA-19-35

THEOREM 1. *The function*

$$z \mapsto f_\nu(z) = \left(2^\nu \Gamma(\nu+1) J_\nu(z)\right)^{\frac{1}{\nu}}$$

is starlike of order $\alpha \in [0,1)$ *in* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ *if and only if* $\nu > \nu_1(\alpha)$, *where* $\nu_1(\alpha)$ *is the unique root of the equation* $(1-\alpha)\nu J_\nu(1) = J_{\nu+1}(1)$, *situated in* $(0,\infty)$. *In particular,* f_ν *is starlike in* $\mathbb{D}$ *if and only if* $\nu > \nu_1(0)$, *where* $\nu_1(0) \simeq 0.390\ldots$ *is the unique root of the equation* $\nu J_\nu(1) = J_{\nu+1}(1)$.

THEOREM 2. *The function*

$$z \mapsto g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\nu} J_\nu(z)$$

is starlike of order $\alpha \in [0,1)$ *in* $\mathbb{D}$ *if and only if* $\nu > \nu_2(\alpha)$, *where* $\nu_2(\alpha)$ *is the unique root of the equation* $(1-\alpha)J_\nu(1) = J_{\nu+1}(1)$, *situated in* $(\tilde{\nu},\infty)$, *where* $\tilde{\nu} \simeq -0.774\ldots$ *is the unique root of* $j_{\nu,1} = 1$ *and* $j_{\nu,1}$ *is the first positive zero of* J_ν. *In particular, the function* g_ν *is starlike in* $\mathbb{D}$ *if and only if* $\nu > \nu_2(0)$, *where* $\nu_2(0) \simeq -0.339\ldots$ *is the unique root of the transcendental equation* $J_\nu(1) = J_{\nu+1}(1)$.

We note that very recently Antonino and Miller [1, Example 3] as an application of the third-order differential subordinations proved that the function $z \mapsto \int_0^z J_0(t)dt$ is convex (and hence univalent) in $\mathbb{D}$. If we consider the function $w_\nu : \mathbb{D} \to \mathbb{C}$, defined by

$$w_\nu(z) = 2^\nu \Gamma(\nu+1) \int_0^z t^{-\nu} J_\nu(t)dt,$$

then in view of the relation

$$1 + \frac{z w_\nu''(z)}{w_\nu'(z)} = \frac{z g_\nu'(z)}{g_\nu(z)}$$

and the analytic characterizations of starlike and convex functions, Theorem 2 can be rewritten as follows: *The function* w_ν *is convex of order* $\alpha \in [0,1)$ *in* $\mathbb{D}$ *if and only if* $\nu > \nu_2(\alpha)$, *and in particular, the function* w_ν *is convex (and hence univalent) in* $\mathbb{D}$ *if and only if* $\nu > \nu_2(0)$. This generalizes the result of Antonino and Miller [1, Example 3] on w_0 and shows actually that if $\nu < \nu_2(0)$, then the above convexity property is no longer true.

The next set of sharp main results are based on a result of Shah and Trimble [20, Theorem 2], see Lemma 1 in the next section, and these results are natural companions of the main results in [5, 9]. We note that it would be interesting to see a common generalization of the next three theorems. Following the proof of these theorems it is clear that *the monotonicity of the zeros (with respect to the order) of the derivative (of arbitrary order greater than three) of Bessel functions of the first kind* together with Lemma 1 would be sufficient.

THEOREM 3. *The function*

$$z \mapsto 2^\nu \Gamma(\nu) z^{\frac{3}{2}-\frac{\nu}{2}} J_\nu'(\sqrt{z})$$

is starlike and all of its derivatives are close-to-convex (and hence univalent) in $\mathbb{D}$ *if and only if* $\nu \geqslant \dot{\nu}$, *where* $\dot{\nu} \simeq 0.702\ldots$ *is the unique root on* $(0,\infty)$ *of the transcendental equation*

$$(2\nu-1)J_\nu(1)+(\nu-2)J_{\nu+1}(1)=0.$$

THEOREM 4. *The function*

$$z \mapsto 2^\nu \Gamma(\nu-1) z^{2-\frac{\nu}{2}} J_\nu''(\sqrt{z})$$

is starlike and all of its derivatives are close-to-convex (and hence univalent) in $\mathbb{D}$ *if and only if* $\nu \geqslant \nu^*$, *where* $\nu^* \simeq 1.905\ldots$ *is the unique root on* $(1,\infty)$ *of the transcendental equation*

$$(2\nu^2-2\nu-3)J_\nu(1)=(\nu^2+\nu-3)J_{\nu+1}(1).$$

THEOREM 5. *The function*

$$z \mapsto 2^\nu \Gamma(\nu-2) z^{\frac{5}{2}-\frac{\nu}{2}} J_\nu'''(\sqrt{z})$$

is starlike and all of its derivatives are close-to-convex (and hence univalent) in $\mathbb{D}$ *if and only if* $\nu \geqslant \nu^\star$, *where* $\nu^\star \simeq 3.077\ldots$ *is the unique root on* $(2,\infty)$ *of the transcendental equation*

$$(2\nu^3-7\nu^2+3)J_\nu(1)+(\nu^3+\nu^2+\nu-1)J_{\nu+1}(1)=0.$$

The last main result of this paper is a common generalization of Theorems 3 and 4.

THEOREM 6. *Let* $a,b,c \in \mathbb{R}$ *such that* $c=0$ *and* $b \neq a$ *or* $c>0$ *and* $b>a$. *Moreover, suppose that* $\nu \geqslant \overline{\nu}$, *where* $\overline{\nu}=\max\{0,\nu_0\}$ *and* ν_0 *is the largest root of the quadratic* $Q(\nu)=a\nu(\nu-1)+b\nu+c$. *Assume also that the following inequalities hold*

$$Q(\nu)+4a\nu+2a+2b>0, \quad (4\nu+3)Q(\nu)>4a\nu+2a+2b. \tag{1.1}$$

Then the function

$$z \mapsto 2^\nu [Q(\nu)]^{-1}\Gamma(\nu+1) z^{1-\frac{\nu}{2}} \left(azJ_\nu''(\sqrt{z})+b\sqrt{z}J_\nu'(\sqrt{z})+cJ_\nu(\sqrt{z})\right)$$

is starlike and all of its derivatives are close-to-convex (and hence univalent) in $\mathbb{D}$ *if and only if* $\nu \geqslant \nu^\circ$, *where* ν° *is the unique root on* $(\overline{\nu},\infty)$ *of the transcendental equation*

$$(2a\nu^2-2a\nu+2b\nu-3a-b+2c)J_\nu(1)=(a\nu^2+a\nu-b\nu-3a+2b+c)J_{\nu+1}(1).$$

It is worth to mention that when $b=c=0$ and $a=1$, then Theorem 6 reduces to Theorem 4. In this case $\overline{\nu}=1$, ν° becomes $\nu^\star$ and the inequalities (1.1) become $\nu^2+3\nu+2>0$, and $4\nu^3-\nu^2-7\nu-2>0$. These inequalities give $\nu>-1$ and $\nu>1.568\ldots$, which are certainly satisfied for $\nu>\nu^\star$.

Similarly, we note that when $a=c=0$ and $b=1$, then Theorem 6 reduces to Theorem 3. In this case $\overline{\nu}=0$, ν° becomes $\dot{\nu}$ and the inequalities (1.1) become $\nu+2>0$, and $4\nu^2+3\nu-2>0$. These inequalities give $\nu>-2$ and $\nu>0.425\ldots$, which are certainly satisfied for $\nu>\dot{\nu}$.

2. Proofs of the main results

In this section we present the proof of the main results of this paper. The proof of Theorems 1 and 2 are mainly based on the Mittag-Leffler expansions and some inequalities from the proof of the main result from [6].

Proof of Theorem 1. Let us denote by $j_{\nu,n}$ the nth positive zero of the function J_ν. From the proof of [6, Theorem 1] we know that for $\nu > 0$ and $r = |z| < j_{\nu,1}$ we have

$$\operatorname{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right) \geqslant \frac{rf_\nu'(r)}{f_\nu(r)} = 1 - \frac{1}{\nu}\sum_{n\geqslant 1}\frac{2r^2}{j_{\nu,n}^2 - r^2}.$$

Since $j_{\nu,1} > j_{0,1} \simeq 2.404\ldots > 1$ when $\nu > 0$, the above inequality is clearly valid when $|z| < 1$. On the other hand, the function $r \mapsto rf_\nu'(r)/f_\nu(r)$ is clearly decreasing on $(0,1) \subset (0, j_{\nu,1})$, and consequently for all $z \in \mathbb{D}$ and $\nu > 0$ we have

$$\operatorname{Re}\left(\frac{zf_\nu'(z)}{f_\nu(z)}\right) \geqslant 1 - \frac{1}{\nu}\sum_{n\geqslant 1}\frac{2}{j_{\nu,n}^2 - 1} = \frac{f_\nu'(1)}{f_\nu(1)}.$$

Since the function $\nu \mapsto j_{\nu,n}$ is increasing on $(0,\infty)$ for $n \in \mathbb{N}$ fixed (see [19, p. 236]), it follows that the function $\nu \mapsto f_\nu'(1)/f_\nu(1)$ is increasing on $(0,\infty)$, and thus $f_\nu'(1)/f_\nu(1) > \alpha$ if and only if $\nu > \nu_1(\alpha)$, where $\nu_1(\alpha)$ is the unique root of the equation

$$f_\nu'(1) = \alpha f_\nu(1) \iff \nu\alpha J_\nu(1) = J_\nu'(1) \iff (1-\alpha)\nu J_\nu(1) = J_{\nu+1}(1).$$

Here we used that

$$\frac{zf_\nu'(z)}{f_\nu(z)} = \frac{1}{\nu}\frac{zJ_\nu'(z)}{J_\nu(z)} = 1 - \frac{zJ_{\nu+1}(z)}{\nu J_\nu(z)},$$

since J_ν satisfies the recurrence relation

$$zJ_\nu'(z) = \nu J_\nu(z) - zJ_{\nu+1}(z). \tag{2.1}$$

Taking into account the fact that all of the above inequalities are sharp it follows that indeed the function f_ν is starlike of order $\alpha \in [0,1)$ in $\mathbb{D}$ if and only if $\nu > \nu_1(\alpha)$. □

Proof of Theorem 2. From the proof of [6, Theorem 1] we know that for $\nu > -1$ and $r = |z| < j_{\nu,1}$ we have

$$\operatorname{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right) \geqslant \frac{rg_\nu'(r)}{g_\nu(r)} = 1 - \sum_{n\geqslant 1}\frac{2r^2}{j_{\nu,n}^2 - r^2}.$$

Since $\nu \mapsto j_{\nu,1}$ is increasing on $(-1,\infty)$, it follows that $j_{\nu,1} > 1$ when $\nu > \tilde{\nu}$, and thus in this case the above inequality is clearly valid when $|z| < 1$. On the other hand, the function $r \mapsto rg_\nu'(r)/g_\nu(r)$ is clearly decreasing on $(0,1) \subset (0, j_{\nu,1})$, and consequently for all $z \in \mathbb{D}$ and $\nu > \tilde{\nu}$ we have

$$\operatorname{Re}\left(\frac{zg_\nu'(z)}{g_\nu(z)}\right) \geqslant 1 - \sum_{n\geqslant 1}\frac{2}{j_{\nu,n}^2 - 1} = \frac{g_\nu'(1)}{g_\nu(1)}.$$

Since the function $\nu \mapsto j_{\nu,n}$ is increasing on $(-1,\infty)$ for $n \in \mathbb{N}$ fixed (see [19, p. 236]), it follows that the function $\nu \mapsto g_\nu'(1)/g_\nu(1)$ is increasing on $(\tilde{\nu},\infty)$, and thus $g_\nu'(1)/g_\nu(1) > \alpha$ if and only if $\nu > \nu_2(\alpha)$, where $\nu_2(\alpha)$ is the unique root of the equation

$$g_\nu'(1) = \alpha g_\nu(1) \iff (1-\nu-\alpha)J_\nu(1)+J_\nu'(1) = 0 \iff (1-\alpha)J_\nu(1) = J_{\nu+1}(1).$$

Here we used that

$$\frac{zg_\nu'(z)}{g_\nu(z)} = 1-\nu+\frac{zJ_\nu'(z)}{J_\nu(z)} = 1-\frac{zJ_{\nu+1}(z)}{J_\nu(z)}.$$

Taking into account the fact that all of the above inequalities are sharp it follows that indeed the function g_ν is starlike of order $\alpha \in [0,1)$ in $\mathbb{D}$ if and only if $\nu > \nu_2(\alpha)$. □

Now, for the proof of the remaining theorems we will use the following result of Shah and Trimble [20, Theorem 2] about transcendental entire functions with univalent derivatives, which was the key tool in the proof of the main results of [5, 8].

LEMMA 1. *Let $f : \mathbb{D} \to \mathbb{C}$ be a transcendental entire function of the form*

$$f(z) = z\prod_{n\geqslant 1}\left(1-\frac{z}{z_n}\right),$$

where all z_n have the same argument and satisfy $|z_n| > 1$. If f is univalent in $\mathbb{D}$, then

$$\sum_{n\geqslant 1}\frac{1}{|z_n|-1} \leqslant 1.$$

Moreover, the above inequality holds if and only if f is starlike in $\mathbb{D}$ and all of its derivatives are close-to-convex there.

As we can see below the next proofs are very similar and all of them use the monotonicity of the zeros with respect to the order of the derivatives of Bessel functions of the first kind.

Proof of Theorem 3. Let us denote by $j_{\nu,n}'$ the nth positive zero of the function J_ν'. From the infinite product representation [21, p. 340]

$$J_\nu'(z) = \frac{\left(\frac{z}{2}\right)^{\nu-1}}{2\Gamma(\nu)}\prod_{n\geqslant 1}\left(1-\frac{z^2}{j_{\nu,n}'^2}\right)$$

it follows that

$$2^\nu\Gamma(\nu)z^{\frac{3}{2}-\frac{\nu}{2}}J_\nu'(\sqrt{z}) = z\prod_{n\geqslant 1}\left(1-\frac{z}{j_{\nu,n}'^2}\right)$$

and

$$-\frac{1}{2}\left(1-\nu+\frac{zJ_\nu''(z)}{J_\nu'(z)}\right) = \sum_{n\geqslant 1}\frac{z^2}{j_{\nu,n}'^2-z^2}.$$

On the other hand, applying the inequality [13, Theorem 6.3]

$$j'^2_{\nu,1} > \frac{4\nu(\nu+1)}{\nu+2},$$

where $\nu > 0$, it follows that for $n \in \{2,3,\ldots\}$ we have $j'_{\nu,n} > j'_{\nu,1} > 1$ if $\nu > \frac{-3+\sqrt{41}}{8} \simeq 0.425\ldots$. Moreover, we know that $\nu \mapsto j'_{\nu,n}$ is increasing on $(0,\infty)$ for each $n \in \mathbb{N}$ fixed (see [19, p. 236]), and thus the function

$$\nu \mapsto \sum_{n\geqslant 1} \frac{1}{j'^2_{\nu,n}-1} = -\frac{1}{2}\left(1-\nu+\frac{J''_\nu(1)}{J'_\nu(1)}\right)$$

is decreasing on $(0,\infty)$. Consequently, the inequality

$$\sum_{n\geqslant 1} \frac{1}{j'^2_{\nu,n}-1} \leqslant 1$$

is valid if and only if $\nu \geqslant \dot{\nu}$, where $\dot{\nu}$ is the unique root on $(0,\infty)$ of the equation

$$\sum_{n\geqslant 1} \frac{1}{j'^2_{\nu,n}-1} = 1 \iff (3-\nu)J'_\nu(1)+J''_\nu(1) = 0. \tag{2.2}$$

Since J_ν satisfies the Bessel differential equation, it follows that

$$z^2 J''_\nu(z) + zJ'_\nu(z) + (z^2-\nu^2)J_\nu(z) = 0,$$

and then

$$J''_\nu(1) = (\nu^2-1)J_\nu(1) - J'_\nu(1) = (\nu^2-\nu-1)J_\nu(1) + J_{\nu+1}(1),$$

where we used the recurrence relation (2.1). Consequently, equation (2.2) is equivalent to

$$(2\nu-1)J_\nu(1) + (\nu-2)J_{\nu+1}(1) = 0.$$

Thus, by applying Lemma 1 the assertion of the theorem follows. □

Proof of Theorem 4. Let us denote by $j''_{\nu,n}$ the nth positive zero of the function J''_ν. By using the infinite product representation [21, p. 340]

$$J''_\nu(z) = \frac{\left(\frac{z}{2}\right)^{\nu-2}}{4\Gamma(\nu-1)} \prod_{n\geqslant 1}\left(1-\frac{z^2}{j''^2_{\nu,n}}\right)$$

it follows that

$$2^\nu \Gamma(\nu-1) z^{2-\frac{\nu}{2}} J''_\nu(\sqrt{z}) = z\prod_{n\geqslant 1}\left(1-\frac{z}{j''^2_{\nu,n}}\right)$$

and

$$-\frac{1}{2}\left(2-\nu+\frac{zJ'''_\nu(z)}{J''_\nu(z)}\right) = \sum_{n\geqslant 1}\frac{z^2}{j''^2_{\nu,n}-z^2}.$$

On the other hand, applying the inequality [13, Theorem 8.1]

$$j_{\nu,1}''^2 > \frac{4\nu(\nu-1)}{\nu+2},$$

where $\nu > 1$, it follows that for $n \in \{2,3,\ldots\}$ we have $j_{\nu,n}'' > j_{\nu,1}'' > 1$ if $\nu > \frac{5+\sqrt{57}}{8} \simeq 1.568\ldots$. Moreover, we know that $\nu \mapsto j_{\nu,n}''$ is increasing on $(1,\infty)$ for each $n \in \mathbb{N}$ fixed (see [18, 25]), and thus the function

$$\nu \mapsto \sum_{n\geqslant 1} \frac{1}{j_{\nu,n}''^2 - 1} = -\frac{1}{2}\left(2 - \nu + \frac{J_\nu'''(1)}{J_\nu''(1)}\right)$$

is decreasing on $(1,\infty)$. Consequently, the inequality

$$\sum_{n\geqslant 1} \frac{1}{j_{\nu,n}''^2 - 1} \leqslant 1$$

is valid if and only if $\nu \geqslant \nu^*$, where ν^* is the unique root in $(1,\infty)$ of the equation

$$\sum_{n\geqslant 1} \frac{1}{j_{\nu,n}''^2 - 1} = 1 \iff (4-\nu)J_\nu''(1) + J_\nu'''(1) = 0. \tag{2.3}$$

Since J_ν satisfies the Bessel differential equation, it follows that

$$z^2 J_\nu'''(z) + 3zJ_\nu''(z) + (z^2 + 1 - \nu^2)J_\nu'(z) + 2zJ_\nu(z) = 0,$$

and then

$$J_\nu'''(1) = (1-3\nu^2)J_\nu(1) + (\nu^2+1)J_\nu'(1) = (\nu^3 - 3\nu^2 + \nu + 1)J_\nu(1) - (\nu^2+1)J_{\nu+1}(1),$$

where we used the recurrence relation $zJ_\nu'(z) = \nu J_\nu(z) - zJ_{\nu+1}(z)$. Consequently, equation (2.3) is equivalent to

$$(2\nu^2 - 2\nu - 3)J_\nu(1) = (\nu^2 + \nu - 3)J_{\nu+1}(1).$$

Thus, by applying Lemma 1 the assertion of the theorem follows. □

Proof of Theorem 5. Let us denote by $j_{\nu,n}'''$ the nth positive zero of the function J_ν'''. From the infinite product representation [21, p. 340]

$$J_\nu'''(z) = \frac{\left(\frac{z}{2}\right)^{\nu-3}}{8\Gamma(\nu-2)} \prod_{n\geqslant 1}\left(1 - \frac{z^2}{j_{\nu,n}'''^2}\right)$$

it follows that

$$2^\nu \Gamma(\nu-2) z^{\frac{5}{2}-\frac{\nu}{2}} J_\nu'''(\sqrt{z}) = z \prod_{n\geqslant 1}\left(1 - \frac{z}{j_{\nu,n}'''^2}\right)$$

and

$$-\frac{1}{2}\left(3-\nu+\frac{zJ_\nu''''(z)}{J_\nu'''(z)}\right)=\sum_{n\geqslant 1}\frac{z^2}{j_{\nu,n}'''^2-z^2}.$$

On the other hand, taking into account that the function $\nu\mapsto j_{\nu,1}'''$ is increasing on $(2,\infty)$ (see [14, 17]) it follows that for $\nu>3$ we have $j_{\nu,1}'''>j_{3,1}'''\simeq 1.376\ldots>1$. Thus, for $n\in\{2,3,\ldots\}$ we have $j_{\nu,n}'''>j_{\nu,1}'''>1$ if $\nu>3$. We would like to mention here that we approximated the zero $j_{3,1}'''$ by using the mathematical software Matlab by taking into account that $j_{3,1}'''$ is actually the first positive zero of the equation

$$\left((1-\nu)z^2+\nu^3-3\nu^2+2\nu\right)J_\nu(z)=\left((2+\nu^2)z-z^3\right)J_{\nu+1}(z)$$

when $\nu=3$. Appealing again to the fact that $\nu\mapsto j_{\nu,n}'''$ is increasing on $(2,\infty)$ for each $n\in\mathbb{N}$ fixed (see [14, 17]), we obtain that the function

$$\nu\mapsto\sum_{n\geqslant 1}\frac{1}{j_{\nu,n}'''^2-1}=-\frac{1}{2}\left(3-\nu+\frac{J_\nu''''(1)}{J_\nu'''(1)}\right)$$

is decreasing on $(2,\infty)$. Consequently, the inequality

$$\sum_{n\geqslant 1}\frac{1}{j_{\nu,n}'''^2-1}\leqslant 1$$

is valid if and only if $\nu\geqslant\nu^\star$, where $\nu^\star$ is the unique root on $(2,\infty)$ of the equation

$$\sum_{n\geqslant 1}\frac{1}{j_{\nu,n}'''^2-1}=1\quad\Longleftrightarrow\quad(5-\nu)J_\nu'''(1)+J_\nu''''(1)=0.\tag{2.4}$$

Since J_ν satisfies the Bessel differential equation, it follows that

$$z^2J_\nu''''(z)+5zJ_\nu'''(z)+(z^2+4-\nu^2)J_\nu''(z)+4zJ_\nu'(z)+2J_\nu(z)=0,$$

and then

$$\begin{aligned}J_\nu''''(1)&=(\nu^4+9\nu^2-2)J_\nu(1)-(6\nu^2+4)J_\nu'(1)\\&=(\nu^4-6\nu^3+9\nu^2-4\nu-2)J_\nu(1)+(6\nu^2+4)J_{\nu+1}(1).\end{aligned}$$

Consequently, equation (2.4) is equivalent to

$$(2\nu^3-7\nu^2+3)J_\nu(1)+(\nu^3+\nu^2+\nu-1)J_{\nu+1}(1)=0.$$

Applying again Lemma 1 the assertion of the theorem follows. □

Proof of Theorem 6. Let us consider the power series

$$\begin{aligned}&2^\nu\Gamma(\nu+1)z^{-\nu}\left(az^2J_\nu''(z)+bzJ_\nu'(z)+cJ_\nu(z)\right)\\&\quad=\sum_{n\geqslant 0}\frac{(2n+\nu)(2n+\nu-1)a+(2n+\nu)b+c}{4^nn!(\nu+1)_n}(-1)^nz^{2n},\end{aligned}$$

where $(a)_n = a(a+1)\dots(a+n-1) = \Gamma(a+n)/\Gamma(a)$. Using the fact that for $\tau > 0$ the quotient $\log\Gamma(n+\tau)/(n\log n)$ tends to 1 as n tends to infinity, we obtain that the growth order of the above entire function is

$$\rho = \lim_{n\to\infty} \frac{n\log n}{n\log 4 + \log\Gamma(n+1) + \log\Gamma(n+\nu+1) - \log Q_n(\nu)} = \frac{1}{2},$$

where $Q_n(\nu) = (2n+\nu)(2n+\nu-1)a + (2n+\nu)b + c$. Thus, if $\lambda_{\nu,n}$ denotes the nth positive zero of the function $z \mapsto az^2J''_\nu(z) + bzJ'_\nu(z) + cJ_\nu(z)$, then by applying Hadamard's theorem [16, p. 26] we obtain

$$az^2J''_\nu(z) + bzJ'_\nu(z) + cJ_\nu(z) = \frac{Q(\nu)z^\nu}{2^\nu\Gamma(\nu+1)} \prod_{n\geqslant 1}\left(1 - \frac{z^2}{\lambda^2_{\nu,n}}\right),$$

and consequently

$$\frac{2^\nu\Gamma(\nu+1)}{Q(\nu)} z^{1-\frac{\nu}{2}} \left(azJ''_\nu(\sqrt{z}) + b\sqrt{z}J'_\nu(\sqrt{z}) + cJ_\nu(\sqrt{z})\right) = z\prod_{n\geqslant 1}\left(1 - \frac{z}{\lambda^2_{\nu,n}}\right),$$

$$-\frac{1}{2}\left(-\nu + z\cdot\frac{az^2J'''_\nu(z) + (2a+b)zJ''_\nu(z) + (b+c)J'_\nu(z)}{az^2J''_\nu(z) + bzJ'_\nu(z) + cJ_\nu(z)}\right) = \sum_{n\geqslant 1}\frac{z^2}{\lambda^2_{\nu,n} - z^2}.$$

Here we used the fact that when $\nu \geqslant \overline{\nu}$, where $\overline{\nu} = \max\{0, \nu_0\}$ and ν_0 is the largest root of the quadratic $Q(\nu) = a\nu(\nu-1) + b\nu + c$, the zeros of the function $z \mapsto az^2J''_\nu(z) + bzJ'_\nu(z) + cJ_\nu(z)$ are real, according to [13, Theorem 7.1].

On the other hand, by using the inequalities in (1.1) together with [13, Eq. (8.2)]

$$\lambda_{\nu,1} > \frac{4(\nu+1)Q(\nu)}{Q(\nu) + 4a\nu + 2a + 2b},$$

it follows that for $n \in \{2,3,\dots\}$ we have $\lambda_{\nu,n} > \lambda_{\nu,1} > 1$. Moreover, we know [18, Theorem 1] that for $a,b,c \in \mathbb{R}$ such that $c = 0$ and $b \neq a$ or $c > 0$ and $b > a$ we have that $\nu \mapsto \lambda_{\nu,n}$ is increasing on $(0,\infty)$ for fixed $n \in \mathbb{N}$. Consequently, under the assumptions, the function

$$\nu \mapsto -\frac{1}{2}\left(-\nu + \frac{aJ'''_\nu(1) + (2a+b)J''_\nu(1) + (b+c)J'_\nu(1)}{aJ''_\nu(1) + bJ'_\nu(1) + cJ_\nu(1)}\right) = \sum_{n\geqslant 1}\frac{1}{\lambda^2_{\nu,n} - 1}$$

is decreasing on $(0,\infty)$. Thus, the inequality

$$\sum_{n\geqslant 1}\frac{1}{\lambda^2_{\nu,n} - 1} \leqslant 1$$

is valid if and only if $\nu \geqslant \nu^\circ$, where ν° is the unique root on $(\overline{\nu},\infty)$ of the equation

$$\sum_{n\geqslant 1}\frac{1}{\lambda^2_{\nu,n} - 1} = 1,$$

or equivalently

$$aJ_\nu'''(1)+(4a-a\nu+b)J_\nu''(1)+(3b+c-b\nu)J_\nu'(1)-(\nu-2)cJ_\nu(1)=0.$$

Using the expressions for $J_\nu'''(1)$, $J_\nu''(1)$ and $J_\nu'(1)$ from the above proofs, the above equation is equivalent to

$$(2a\nu^2-2a\nu+2b\nu-3a-b+2c)J_\nu(1)=(a\nu^2+a\nu-b\nu-3a+2b+c)J_{\nu+1}(1).$$

Applying Lemma 1 completes the proof. □

REFERENCES

[1] J. A. ANTONINO AND S. S. MILLER, *Third-order differential inequalities and subordinations in the complex plane*, Complex Var. Elliptic Equ. **56**, 5 (2011), 439–454.

[2] Á. BARICZ, *Geometric properties of generalized Bessel functions of complex order*, Mathematica **48**, 71 (2006), 13–18.

[3] Á. BARICZ, *Geometric properties of generalized Bessel functions*, Publ. Math. Debrecen **73**, (2008), 155–178.

[4] Á. BARICZ, *Generalized Bessel Functions of the First Kind*, Lecture Notes in Mathematics, vol. 1994, Springer-Verlag, Berlin, 2010.

[5] Á. BARICZ, E. DENIZ AND N. YAĞMUR, *Close-to-convexity of normalized Dini functions*, Math. Nachr. doi 10.1002/mana.201500009 (in press).

[6] Á. BARICZ, P. A. KUPÁN AND R. SZÁSZ, *The radius of starlikeness of normalized Bessel functions of the first kind*, Proc. Amer. Math. Soc. **142**, 6 (2014), 2019–2025.

[7] Á. BARICZ AND S. PONNUSAMY, *Starlikeness and convexity of generalized Bessel functions*, Integr. Transforms Spec. Funct. **21** (2010), 641–653.

[8] Á. BARICZ AND R. SZÁSZ, *The radius of convexity of normalized Bessel functions of the first kind*, Anal. Appl. **12**, 5 (2014), 485–509.

[9] Á. BARICZ AND R. SZÁSZ, *Close-to-convexity of some special functions and their derivatives*, Bull. Malay. Sci. Soc. **39**, 1 (2016), 427–437.

[10] R. K. BROWN, *Univalence of Bessel functions*, Proc. Amer. Math. Soc. **11**, 2 (1960), 278–283.

[11] R. K. BROWN, *Univalent solutions of* $W''+pW=0$, Canad. J. Math. **14** (1962), 69–78.

[12] R. K. BROWN, *Univalence of normalized solutions of* $W''(z)+p(z)W(z)=0$, Int. J. Math. Math. Sci. **5** (1982), 459–483.

[13] M. E. H. ISMAIL AND M. E. MULDOON, *Bounds for the small real and purely imaginary zeros of Bessel and related functions*, Methods Appl. Anal. **2**, 1 (1995), 1–21.

[14] C. G. KOKOLOGIANNAKI AND E. N. PETROPOULOU, *On the zeros of* $J_\nu'''(x)$, Integr. Transforms Spec. Funct. **24**, 7 (2013), 540–547.

[15] E. KREYSZIG AND J. TODD, *The radius of univalence of Bessel functions*, Illinois J. Math. **4**, (1960), 143–149.

[16] B. YA. LEVIN, *Lectures on Entire Functions*, Amer. Math. Soc.: Transl. of Math. Monographs, vol. 150, 1996.

[17] L. LORCH AND P. SZEGŐ, *Monotonicity of the zeros of the third derivative of Bessel functions*, Methods Appl. Anal. **2**, 1 (1995), 103–111.

[18] A. MCD. MERCER, *The zeros of* $az^2J_\nu''(z)+bzJ_\nu'(z)+cJ_\nu(z)$ *as functions of order*, Internat. J. Math. Math. Sci. **15**, 2 (1992), 319–322.

[19] F. W. J. OLVER, D. W. LOZIER, R. F. BOISVERT AND C. W. CLARK, EDS., *NIST Handbook of Mathematical Functions*, Cambridge Univ. Press, Cambridge, 2010.

[20] S. M. SHAH AND S. Y. TRIMBLE, *Entire functions with univalent derivatives*, J. Math. Anal. Appl. **33**, (1971), 220–229.

[21] E. A. SKELTON, *A new identity for the infinite product of zeros of Bessel functions of the first kind or their derivatives*, J. Math. Anal. Appl. **267**, (2002), 338–344.

[22] R. SZÁSZ, *On starlikeness of Bessel functions of the first kind*, In: Proceedings of the 8th Joint Conference on Mathematics and Computer Science, Komárno, Slovakia, 2010, 9 pp.

[23] R. SZÁSZ AND P. A. KUPÁN, *About the univalence of the Bessel functions*, Stud. Univ. Babeş-Bolyai Math. **54**, 1 (2009), 127–132.

[24] G. N. WATSON, *A Treatise of the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, 1995.

[25] R. WONG AND T. LANG, *On the points of inflection of Bessel functions of positive order II*, Can. J. Math. **44**, 3 (1991), 628–651.

(Received March 21, 2015)

Árpád Baricz
Department of Economics, Babeş-Bolyai University
Cluj-Napoca 400591, Romania
and
Institute of Applied Mathematics, Óbuda University
1034 Budapest, Hungary
e-mail: bariczocsi@yahoo.com

Murat Çağlar
Department of Mathematics, Faculty of Science and Letters
Kafkas University
Kars 36100, Turkey
e-mail: mcaglar25@gmail.com

Erhan Deniz
Department of Mathematics, Faculty of Science and Letters
Kafkas University
Kars 36100, Turkey
e-mail: edeniz36@gmail.com