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ON ONE EXTENSION THEOREM DEALING WITH WEIGHTED ORLICZ–SLOBODETSKII SPACE. ANALYSIS ON LIPSCHITZ SUBGRAPH AND LIPSCHITZ DOMAIN

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Abstract. Having a given weight $\rho(x)=\tau(\mathrm{dist}(x,\partial\Omega))$ defined on Lipschitz boundary domain Ω and an Orlicz function Ψ, we construct the subordinated weight $\omega(\cdot,\cdot)$ defined on $\partial\Omega\times\partial\Omega$ and extension operator $\mathrm{Ext}^L: Lip(\partial\Omega)\mapsto Lip(\overline{\Omega})$ form Lipschitz functions defined on $\partial\Omega$ to Lipschitz functions defined on $\overline{\Omega}$, independent of τ and Ψ, in such a way that Ext^L extends to the bounded operator from the subspace of weighted Orlicz-Slobodetskii space $Y_{\omega}^{\Psi,\Psi}(\partial\Omega)$ generated by Lipschitz functions and subordinated to the weight ω to Orlicz-Sobolev space $W_{\rho}^{1,\Psi}(\Omega)$. More detailed analysis on Lipschitz subgraph is also provided. Result is new in the unweighted Orlicz setting for general function Ψ as well as in the weighted L^p setting.

1. Introduction

In this paper we deal with an extension theorems between weighted Orlicz-Slobodetskii space defined on the boundary of domain Ω to the weighted Sobolev space defined on Ω, where Ω is either a Lipschitz boundary domain or subgraph of Lipschitz function.

More precisely, let $\Omega\subseteq\mathbb{R}^n$ be the Lipschitz boundary domain, $\rho(x)=\tau(\mathrm{dist}(x,\partial\Omega))$ be the given weight defined on Ω and let Ψ be given Orlicz function. We construct:

- a linear extension operator between spaces of Lipschitz functions: $\mathrm{Ext}: Lip(\partial\Omega)\to Lip(\overline{\Omega})$, $u\mapsto\tilde{u}$, where $\tilde{u}|_{\partial\Omega}=u$;
- the transformation of weights $\rho\mapsto\omega_\rho$, where ω_ρ is defined on $\partial\Omega\times\partial\Omega$, $\omega_\rho(x,y)=\tau(|x-y|)=\tau(\mathrm{dist}(x,y))$;
- a weighted Slobodetskii type space defined on $\partial\Omega$ and subordinated to ω_ρ, $Y=Y_{\omega_\rho}^{\Psi,\Psi}(\partial\Omega)$, extending the classical definition of Orlicz-Slobodetskii space $W^{1-\frac{1}{p},p}(\partial\Omega)$ to the weighted Orlicz setting

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in such a way that the operator Ext extends to the bounded operator

$$\widetilde{\mathrm{Ext}} : Y_L \to W^{1,\Psi}_{\rho}(\Omega)$$

acting between Y_L - the completion of Lipschitz functions in Y and weighted Orlicz-Sobolev space $W^{1,\Psi}_{\rho}(\Omega)$. This extension result is a consequence of the following inequalities proven in Theorem 3.3

$$\int_{\Omega} \Psi\left(\frac{|\widetilde{u}(x)|}{\lambda}\right) \tau(\mathrm{dist}(x,\partial\Omega))\,dx \prec \int_{\partial\Omega} \Psi\left(\frac{E}{\lambda}|u(x)|\right) d\sigma(x); \tag{1.1}$$

$$\begin{aligned} &\int_{\Omega} \Psi\left(\frac{|\nabla\widetilde{u}(x)|}{\lambda}\right) \tau(\mathrm{dist}(x,\partial\Omega))\,dx \\ &\prec \int_{\partial\Omega}\int_{\partial\Omega} \Psi\left(\frac{B}{\lambda}\frac{|u(x)-u(y)|}{|x-y|}\right) \frac{\tau(|x-y|)}{|x-y|^{n-2}} d\sigma(x)d\sigma(y) + \int_{\partial\Omega} \Psi\left(\frac{D}{\lambda}|u(x)|\right) d\sigma(x), \end{aligned} \tag{1.2}$$

as well as their norm counterparts

$$\|\widetilde{u}\|_{W^{1,\Psi}_{\tau(\mathrm{dist}(x,\partial\Omega))}(\Omega)} \prec \|u\|_{Y^{\Psi,\Psi}_{\tau(\mathrm{dist}(x,y))}(\partial\Omega)},$$

with constants E,B,D independent on $u \in Lip(\partial\Omega)$, where $\tilde{u} = Ext(u) \in Lip(\overline{\Omega})$. Here $Y^{\Psi,\Psi}_{\tau(\mathrm{dist}(x,y))}(\partial\Omega)$ is the space of functions for which (1.1) and (1.2) are finite for certain λ, equipped with the related Luxemburg norm (see Section 2.3.4).

The problem of extension and trace operator in the unweighted L^p-setting has been completely solved in the late 50's of the last century (see papers by Aronszajn [2], Slobodetskii [64] and Gagliardo [19]). The rudiments in the weighted setting have appeared in papers by Nikolski [56] and Lizorkin [46]. Trace and extension operator in the unweighted Orlicz setting was investigated by Nečas ([54], Chapter II, Section 4.3), Fougéres [16, 17] and Lacroix [42] in the 60's and 70's of the last century. Further related contributions can be found in [16, 17, 42], [31], Theorem 5.1, [60, Theorem 7] and [67, Section 2.6.2]. Trace and extension operator in the weighted L^p-setting, i.e. in weighted Sobolev spaces $W^{1,p}_{\rho}(\Omega)$ has been analyzed in [40, 54, 67]. Perhaps first trace embedding and extension theorems in the weighted L^p-setting can be found in the paper by Nikolskii [56], written in 1953, before paper by Slobodetskii [64] obtained in 1958. It dealt with power measure $\mathrm{dist}(x,\partial\Omega)^2$, in the form not involving Slobodetskii type spaces directly. Extensions within measures of the form $\mathrm{dist}(x,\partial\Omega)^{\alpha}$ can be found in works by Lizorkin [46], Vasarin [70], Portnov [62], Kudryavcev [39] (Section 9), Uspenski [71], Nečas [55]. See also [1, 24, 34, 52, 57, 58, 61] for related works. For trace and extension operator in the weighted Orlicz setting we refer to paper by Lacroix [43] (providing very very abstract approach), Palmieri [59] where weight functions involved are powers of distances from the boundary, Kokilashvili [37].

See also [47, 9, 10], Theorem 9.14 in [40], Theorem 2.2 from page 291 in [54], for interesting related works.

Let us mention that our extension result in two important cases: in the weighted L^p setting and in the unweighted Orlicz setting, is new and rather general as it involves measures which can be different than powers of distances from the boundary. Moreover, we propose formulaes to transform the related weights $\rho \mapsto \omega_\rho$, as far as ρ is a function of the distance from the boundary. We also have no assumptions on generating Orlicz functions involved.

Weighted Sobolev spaces are basic tool to study degenerated PDEs. To explain our motivation we focus on the following simple example, showing certain application to the study of the nonhomogeneous boundary-value problems of elliptic type. Consider the following boundary value problem:

$$\begin{cases} -\mathrm{div}\,(\rho(x)\nabla u(x)) = f & \text{in } \ \Omega \\ u = g & \text{in } \partial\Omega, \end{cases} \tag{1.3}$$

where ρ is the given weight and assume that $g \in Y$, where Y is some function space on $\partial\Omega$. Suppose further that there exists bounded operator: $\mathrm{Ext}: Y \to W^{1,2}_\rho(\Omega)$. Then there exists $\Psi_g \in W^{1,2}_\rho(\Omega)$ such that $\Psi_g|_{\partial\Omega} = g$. Simple substitution: $v := u - \Psi_g$ allows to reduce the problem to the homogeneous equivalent one:

$$\begin{cases} Pv = f - P\Psi_g & \text{in } \ \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $Pw = -\mathrm{div}\,(\rho\nabla w)$. Let us consider $W^{1,2}_\rho(\Omega) = \{u \in L^1_{loc}(\Omega) : u, |\nabla u| \in L^p_\rho(\Omega)\}$ with the usual norm and Hilbert space $H := W^{1,2}_{\rho,0}(\Omega)$ - the completion of $C^\infty_0(\Omega)$ in $W^{1,2}_\rho(\Omega)$. Assume further that $f \in H^*$. With suitable assumptions on the admitted weight ρ one can prove existence of solutions of last equation for example by Lax Milgram theorem. In particular we also have the solution of (1.3) and boundary data interprets as $u - \Psi_g \in W^{1,2}_{\rho,0}(\Omega)$. Details are provided in [11].

For more general equations

$$\begin{cases} -\mathrm{div}\,(\rho(x)F_A(\nabla u(x))) = f & \text{in } \ \Omega \\ u = g & \text{in } \partial\Omega, \end{cases} \tag{1.4}$$

where $g \in Y$, $F_A(z) := A'(|z|)|z|^{-1}z$ and $A : [0,\infty) \to [0,\infty)$ is the given convex function (A' is the derivative of A), one needs to investigate the extension operator: $\mathrm{Ext}: Y \to W^{1,A}_\rho(\Omega)$, where $W^{1,A}_\rho(\Omega)$ is the weighted Orlicz-Sobolev space corresponding to A, and find the respective space Y.

The typical example of pde like (1.4) where $\rho(x) = \tau(\mathrm{dist}(x,\partial\Omega))$ like in our approach is the case when Ω is a ball with center at 0 and ρ is a function of $|x|$.

For some other example motivations to consider weighted Sobolev spaces we refer to books: [7, 14, 25, 40, 54], papers [15, 20, 29, 35, 48, 49, 51, 52, 65], [72], page 1146, and to their references. For motivations to consider Orlicz-Sobolev spaces we refer e.g. to [3, 5, 8, 13, 21, 22] and references therein. Moreover, in many cases the theory of existence of solutions to non-homogeneous boundary value problems like

(1.4) has been systematically undertaken but the authors are aware of the fact that this theory is avoiding the nonhomogeneous boundary problems because of lack of general trace/extension results. See e.g. [14], page 16 in the Introduction.

Our tools are based on recent paper [12], where we have introduced the key estimates for trace extension theorem (Theorem 2.10), providing analysis on cube. The passage from analysis on cube to analysis on Lipschitz boundary domain is not so automatic, because we have to care on transformation of weight which in general strongly depend on geometry of the domain. Therefore at first we provide an analysis on Lipschitz subgraph (Theorem 3.1) and obtain the estimates with certain care on constants. Our constants are not sharp, but one can observe their dependences on geometry of the domain, related weights and the dimension. In second step we obtain estimations on Lipschitz boundary domain. Let us mention that our first approach here, analysis on Lipschitz subgraph, is essentially more general if one takes into account the admitted class of weights (see Section 5). In our second approach, dealing with Lipschitz boundary domain, we restricted our attention to the special class of weights τ with some good doubling/halfing properties (Theorem 3.3), which still can be essentially more general than homogeneous type ones like $\tau(s) = s^{\alpha}$, moreover, τ can vanish or explode at 0. Our selection of weights allows us to provide analysis which is independent of geometric properties of domain Ω. We hope that more detailed analysis provided on Lipschitz subgraph, the technical step, allows to generalize extension theorems involving more general weights in some further issues.

2. Notation and preliminaries

2.1. Basic notation and general assumptions

NOTATION. Let $\Omega \subset \mathbb{R}^n$ be an open set. By $C^{\infty}(\bar{\Omega})$ we mean set of functions which have smooth extension to certain open neighborhood of $\bar{\Omega}$. If f is defined on a set $A \subseteq \mathbb{R}^n$, by $f\chi_A$ we mean the function f extended by 0 outside A. Having to norms $\|\cdot\|$ and $\|\cdot\|_1$ defined on a Banach space X, we will write $\|\cdot\| \sim \|\cdot\|_1$ if norm $\|\cdot\|$ is equivalent to $\|\cdot\|_1$ on X. When $n \in \mathbb{N}$, we denote: $Q' = (-\frac{1}{2}, \frac{1}{2})^{n-1}$, $Q = Q' \times (0,1)$, and $tA := \{tx : x \in A\}$ whenever A is an arbitrary subset of an Euclidean space. If X is a subset in an Euclidean space, by $Lip(X)$ we denote Lipschitz functions defined on X, while the notation $Lip_0(X)$ stands for Lipschitz functions with compact support in X. When $B \in \mathbb{R}^{m\times n}$ is any matrix, we denote $\|B\| = \sqrt{\sum_j m_j^2}$, where m_j's are all minors of B of highest order. By θ_k we denote the k-dimensional Hausdorff measure of the unit sphere $S^k(1) \subseteq \mathbb{R}^{k+1}$. If f_1, f_2 are two given functions defined on the same domain D, we say that $f_1 \prec f_2$ if there is constant $C > 0$ such that for every $x \in D$ we have $f_1(x) \leqslant C f_2(x)$. If we use this notation in inequalities involving some general function u, it is meant that inequalities hold with constants independent on u.

ASSUMPTIONS. If not said otherwise we assume that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain of class $\mathscr{C}^{0,1}$ described below, the symbol $d\sigma(x)$ stands for the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$. All weight functions in our considerations (non-negative measurable functions) are assumed to be integrable on domains of their definition.

2.2. Domains of class $\mathscr{C}^{0,1}$

We will be using the following definition of Lipschitz boundary domain (see e.g. [41]).

DEFINITION 2.1. We say that $\Omega \subset \mathbb{R}^n$ is of class $\mathscr{C}^{0,1}$ if the following conditions are satisfied:

(a) there is $m \in \mathbb{N}$ and decomposition

$$\partial\Omega = \cup_{r=1}^{m} \Lambda_r,$$

where $\Lambda_r \subseteq \partial\Omega$ are connected and open in $\partial\Omega$ (where the topology on $\partial\Omega$ is inherited from $\mathbb{R}^n$);

(b) there are m cartesian coordinate systems X^r defined $\mathbb{R}^n$, $(r = 1, 2, \cdots, m)$

$$X^r = (x_1^r, \cdots, x_{n-1}^r, x_n^r) = ((x^r)', x_n^r) \quad \text{where} \quad (x^r)' = (x_1^r, \cdots, x_{n-1}^r);$$

(c) there are rigid motions in $\mathbb{R}^n$, $\widetilde{A}_r(X) = \mathbb{A}_r X + C_r$, where $\mathbb{A}_r$'s are orthonormal matrices with determinant one, C_r are vectors in $\mathbb{R}^n$, $r = 1, \ldots, m$;

(d) there exist functions $\alpha_r \in Lip(aQ')$, where $aQ' = [-\frac{a}{2}, \frac{a}{2}]^{n-1}$, $r = 1, \ldots, m$, $a > 0$;

such that

(i) sets Λ_r are the rigid presentations of the Lipschitz graph of α_r under the mapping $\widetilde{A}_r^{-1}$, i. e.

$$\Lambda_r = \widetilde{A}_r^{-1}\left(\{(x', \alpha_r(x')) : x' \in aQ'\}\right);$$

(ii) there exists a constant $b > 0$ such that for every $r = 1, \cdots, m$ sets

$$U_r^+ = \widetilde{A}_r^{-1}(\{X^r = ((x^r)', x_n^r);\ (x^r)' \in aQ' \text{ and } \alpha_r((x^r)') < x_n^r < \alpha_r((x^r)') + b\})$$

are subsets of Ω, while sets

$$U_r^- = \widetilde{A}_r^{-1}(\{X_r = ((x^r)', x_n^r);\ (x^r)' \in aQ' \text{ and } \alpha_r((x^r)') - b < x_n^r < \alpha_r((x^r)')\})$$

are a subsets of $\mathbb{R}^n \setminus \Omega$.

2.3. Orlicz, Orlicz-Sobolev and Orlicz-Slobodetskii spaces equipped with weights

2.3.1. Orlicz space L_ρ^Ψ

We start with the definition of Orlicz space.

DEFINITION 2.2. The function $\Psi : [0, \infty) \to [0, \infty)$ is called Orlicz function if it is nondecreasing, convex and satisfies conditions: $\Psi(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = +\infty$.

We will write that $\Psi \in \Delta_2$ if it satisfies the Δ_2-condition: $\Psi(2\lambda) \leqslant C\Psi(\lambda)$, for every $\lambda > 0$, with a constant C independent of λ.

We define now the Orlicz spaces. We are particularly interested in definition on domain and on its boundary.

A. Orlicz space on domain. Let Ψ be an Orlicz function and $\rho : \Omega \to (0,\infty)$ be a given weight function. The space

$$L^{\Psi}_{\rho}(\Omega) := \{f \in L^1_{loc}(\Omega) : \int_{\Omega} \Psi(s|f(x)|)\,\rho(x)\,dx < \infty \text{ for some } s > 0\}$$

is called *weighted Orlicz space* with weight ρ. It is a Banach space with the *Luxemburg norm*:

$$\|f\|_{L^{\Psi}_{\rho}(\Omega)} := \inf\left\{\lambda > 0 : \int_{\Omega} \Psi\left(\frac{|f(x)|}{\lambda}\right)\rho(x)\,dx \leqslant 1\right\}.$$

As is well known, when $\Psi(\lambda) = \lambda^p$ and $p \geqslant 1$, then $L^{\Psi}_{\rho}(\Omega) = L^p_{\rho}(\Omega)$. See e.g. [63]. The Legendre transform of Ψ is $\Psi^*(t) = \sup_{s \geqslant 0}\{st - \Psi(s)\}$, $t \geqslant 0$.

B. Orlicz space on the boundary of domain. Similarly, we define the weighted Orlicz space on the boundary of the domain:

$$L^{\Psi}_{r}(\partial\Omega) := \{f \in L^1_{loc}(\partial\Omega) : \int_{\partial\Omega} \Psi(s|f(x)|)\,r(x)\,d\sigma(x) < \infty \text{ for some } s > 0\},$$

with the norm:

$$\|f\|_{L^{\Psi}_{r}(\partial\Omega)} := \inf\left\{\lambda > 0 : \int_{\partial\Omega} \Psi\left(\frac{|f(x)|}{\lambda}\right) r(x)\,d\sigma(x) \leqslant 1\right\},$$

where $r : \partial\Omega \to (0,\infty)$ is a given weight function defined on the boundary of Ω.

The same notation will be used for vector functions, $u : \Omega \to \mathbb{R}^m$, with the formal difference that instead of $|u(x)|$ we shall work with the Euclidean norm of the vector $u(x)$.

2.3.2. Information about classical Besov spaces $B^{p,q}_s$

For $1 \leqslant p, q < \infty$ and $0 < s < 1$ one defines Besov space $B^{p,q}_s(\Omega)$ as the collection of all measurable functions f defined on Ω such that

$$\|f\|_{B^{p,q}_s(\Omega)} := \|f\|_{L^p(\Omega)} + \left(\int_{\Omega}\left(\int_{\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{(n+sq)p/q}}dy\right)^{q/p} dx\right)^{1/q} < \infty.$$

With the same range of parameters Besov space $B^{p,q}_s(\partial\Omega)$ is the collection of all measurable functions f defined on $\partial\Omega$ such that

$$\|f\|_{B^{p,q}_s(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\left(\int_{\partial\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{(n-1+sq)p/q}}d\sigma(y)\right)^{q/p} d\sigma(x)\right)^{1/q} < \infty.$$

We have used the (nonatomic) definition of Besov spaces (see e.g. [67] and Section 4 in [53] for discussion and overview of Besov spaces specializing on those defined on Lipschitz domains and their boundaries).

2.3.3. Orlicz-Sobolev space $W_\rho^{1,\Psi}$

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded domain, $k \in \mathbf{N}$, and $\Psi : [0,\infty) \to [0,\infty)$ be a given Orlicz function. The weighted *Orlicz-Sobolev space* with weight ρ, $W_\rho^{k,\Psi}(\Omega)$ is the linear set

$$\{u \in L^1_{loc}(\Omega) : D^\alpha u \in L^\Psi_\rho(\Omega) \text{ for every } \alpha : |\alpha| \leqslant k\},$$

equipped with the norm

$$\|u\|_{W_\rho^{k,\Psi}(\Omega)} := \sum_{\alpha:|\alpha|\leqslant k} \|D^\alpha u\|_{L^\Psi_\rho(\Omega)}.$$

Here $D^\alpha u$ means the distributional derivative of u. We will be dealing with $k=1$ only. For more information we refer e.g. [8].

Symbol $W^{1,\Phi}_{\rho,L}(\Omega)$ will denote the completion of Lipschitz functions in the norm of the space $W^{1,\Phi}_\rho(\Omega)$. For general class of weights density results for Lipschitz functions are rather missing in the literature. We refer to the related articles [4, 26, 23, 33, 45, 72].

In the special case when $\Psi(\lambda) = \lambda^p, 1 \leqslant p < \infty$, we use the standard notation: $W_\rho^{1,\Psi}(\Omega) = W_\rho^{1,p}(\Omega)$.

REMARK 2.3. $\Psi(\lambda) = \lambda^p$, $1 \leqslant p < \infty$. Classical Besov space $B_s^{p,p}(\Omega)$ can in some cases be compared with weighted Sobolev spaces $W^{1,p}_{(dist(x,\partial\Omega))^\alpha}(\Omega)$ according to the following proposition.

PROPOSITION 2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a domain of class $\mathscr{C}^{0,1}$ and $\rho(\cdot) = (dist(x,\partial\Omega))^\alpha$. Then we have*

i) *When $\alpha \in (0,p)$ and $1 \leqslant p \leqslant \infty$, we have $W_\rho^{1,p}(\Omega) \subseteq B^{p,p}_{1-\frac{\alpha}{p}}(\Omega)$;*

ii) *When $\alpha \in (-1,0), p > 1$, we have $W_\rho^{1,p}(\Omega) \supseteq B^{p,p}_{1-\frac{\alpha}{p}}(\Omega)$.*

Moreover, when $\alpha \in (0,p)$ and u is harmonic then $u \in W_\rho^{1,p}(\Omega)$ if and only if $u \in B^{p,p}_{1-\frac{\alpha}{p}}(\Omega)$.

Proof. For part i) and ii) see [53], Proposition 4.4 and [52] by taking $k=0$ and $k=1$ respectively, Appendix A. For last statement see [27], Theorem 4.1. □

2.3.4. Orlicz-Slobodetskii space $Y_\omega^{\Psi,\Phi}$

A. Orlicz-Slobodetskii space on domain. Let $\omega \in L^1(\Omega \times \Omega)$ be the given weight. Moreover, let Ψ and Φ be the given two Orlicz functions. By $Y_\omega^{\Psi,\Phi}(\Omega)$ we denote the space of all $u \in L^\Psi(\Omega)$, for which the quantity

$$I_\omega^\Phi(su,\Omega) := \int_\Omega \int_\Omega \Phi\left(\frac{s|u(x)-u(y)|}{|x-y|}\right) \frac{\omega(x,y)}{|x-y|^{n-1}}\, dxdy$$

is finite for some $s > 0$. We equip it with the norm

$$\|u\|_{Y_\omega^{\Psi,\Phi}(\Omega)} := \|u\|_{L^\Psi(\Omega)} + J_\omega^\Phi(u,\Omega),$$

involving Luxemburg-type seminorm

$$J_\omega^\Phi(u,\Omega) := \inf\left\{\lambda > 0 : I_\omega^\Phi\left(\frac{u}{\lambda},\Omega\right) \leqslant 1\right\}.$$

B. Orlicz-Slobodetskii space on the boundary of domain. The same type of space can be defined on the boundary of Ω, with a given weight $\omega(x,y) \in L^1(\partial\Omega \times \partial\Omega)$. Namely, when

$$I_\omega^\Phi(u,\partial\Omega) := \int_{\partial\Omega}\int_{\partial\Omega} \Phi\left(\frac{|u(x)-u(y)|}{|x-y|}\right) \frac{\omega(x,y)}{|x-y|^{n-2}}\, d\sigma(x) d\sigma(y),$$

we define the space

$$Y_\omega^{\Psi,\Phi}(\partial\Omega) := \left\{u \in L^\Psi(\partial\Omega) : \text{there exists } s > 0; I_\omega^\Phi(su,\partial\Omega) < \infty\right\},$$

equipped with the norm

$$\|u\|_{Y_\omega^{\Psi,\Psi}(\partial\Omega)} := \|u\|_{L^\Psi(\partial\Omega)} + J_\omega^\Phi(u,\partial\Omega),$$

where

$$J_\omega^\Phi(u,\partial\Omega) := \inf\left\{\lambda > 0 : I_\omega^\Phi\left(\frac{u}{\lambda},\partial\Omega\right) \leqslant 1\right\}.$$

In the similar way as before we define spaces: $Y_{\omega,L}^{\Psi,\Phi}(\Omega)$, $Y_{\rho,L}^{\Psi,\Phi}(\partial\Omega)$ as the completion of Lipschitz functions in the space $Y_\omega^{\Psi,\Phi}(\Omega)$ and $Y_\rho^{\Psi,\Phi}(\partial\Omega)$, respectively. See [44] for the related density result dealing with the case $\omega \equiv 1$.

REMARK 2.5. If $\omega \equiv 1$ and $\Psi(\lambda) = \Phi(\lambda) = |\lambda|^p$, $1 < p < \infty$, then we have

$$\|u\|_{Y^{\Psi,\Phi}(\partial\Omega)} \sim \|u\|_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{p+n-2}}\, d\sigma(x)\, d\sigma(y)\right)^{1/p},$$

which is the norm of u in the Slobodetskii space $W^{1-\frac{1}{p},p}(\partial\Omega)$, see e.g. [41].

Obvious modifications lead to the same type of spaces defined on manifolds (involoving Hausdorff measures), or less general but important, on parts of the boundary of Ω: $\Lambda \subseteq \partial\Omega$, when Ω is a domain of class $\mathscr{C}^{0,1}$.

REMARK 2.6. Weighted Slobodetskii spaces can in some cases be identified with the classical Besov spaces. When $p = q$ we have $Y_{|x-y|^\alpha}^{\lambda^p,\lambda^p}(\partial\Omega) = B^{p,p}_{1-\frac{1}{p}-\frac{\alpha}{p}}(\partial\Omega)$ whenever $0 < 1 - \frac{1}{p} - \frac{\alpha}{p} < 1$.

2.3.5. Equivalence of norms

We will be using the following statement (see e.g. [6], Proposition 2).

PROPOSITION 2.7. *Let M be a Young function and (X,μ) be the measurable space equipped with the measure μ. Then the expression*

$$\|f\|_{L^{\Psi}(X,\mu),\alpha} := \inf\left\{\lambda > 0 : \int_X \Psi\left(\frac{|f(x)|}{\lambda}\right)\mu(dx) \leqslant \alpha\right\}.$$

defines a complete norm on

$$L^{\Psi}(X,\mu) := \{f \in L^1_{loc}(X) : \int_{\Omega} \Psi(s|f(x)|)\,\mu(dx) < \infty \text{ for some } s > 0\}$$

for each $\alpha \in (0,\infty)$. Moreover, all norms $\|\cdot\|_{L^{\Psi}(X,\mu),\alpha}$, where $\alpha \in (0,\infty)$, are equivalent.

2.4. Trace embedding theorem (unweighted case)

2.4.1. Admitted Orlicz pairs

We will use the following assumptions coming from paper by Kita [36].

ASSUMPTION A. We assume that $a,b : [0,\infty) \to [0,\infty)$ are continuous functions which are strictly positive on $(0,\infty)$ and such that

(a) $\int_0^1 a(s)/s\,ds < \infty$, $\int_1^{\infty} \frac{a(s)}{s}\,ds = +\infty$;

(b) $b(\cdot)$ is non-decreasing, $\lim_{s\to\infty} b(s) = +\infty$.

(c) there exist constants $c_1 > 0, s_0 \geqslant 0$ such that

$$\int_0^s \frac{a(t)}{t}\,dt \leqslant c_1 b(c_1 s) \text{ for all } s > s_0,$$

and in the case $s_0 > 0$ a mapping $s \mapsto \frac{a(s)}{s}$ is bounded when $s \neq 0$ is close to 0.

We define

$$\Phi(t) := \int_0^t a(s)\,ds \quad\text{and}\quad \Psi(t) := \int_0^t b(s)\,ds, \text{ where } t \geqslant 0.$$

2.4.2. Operator of trace

Let us recall the concept of the trace of a function and of the trace operator. Suppose that for given Orlicz-functions Φ and Ψ an inequality

$$\|u\|_{Y^{\Psi,\Phi}(\partial\Omega)} \leqslant D\|u\|_{W^{1,\Psi}(\Omega)},$$

holds for every Lipschitz function u defined on $\bar{\Omega}$. Let $u \in W^{1,\Psi}_L(\Omega)$ and consider any sequence of Lipschitz functions u_m converging to u in the norm of $W^{1,\Psi}(\Omega)$. Then $\{u_m\}$ is a Cauchy sequence in $Y^{\Psi,\Phi}(\partial\Omega)$ (norm convergence) so that it converges to some element $\widehat{u} \in Y^{\Psi,\Phi}_L(\partial\Omega)$. It is easy to check that $\widehat{u}$ is independent of the choice of Lipschitz sequence $\{u_m\}$, converging to u. It allows to extend the standard definition of the trace operator:

$$\mathrm{Tr}\, u := \lim_{m\to\infty} u_m = \widehat{u} \in Y^{\Psi,\Phi}_L(\partial\Omega). \tag{2.1}$$

In the same way we can define the trace operator in weighted case

$$\mathrm{Tr} : W^{1,\Psi}_{\rho,L}(\Omega) \longmapsto Y^{\Phi,\Psi}_{\omega,L}(\partial\Omega),$$

if we only have the inequality

$$\|u\|_{Y^{\Psi,\Phi}_\omega(\partial\Omega)} \leqslant D\|u\|_{W^{1,\Psi}_\rho(\Omega)},$$

holding within Lipschitz functions. In that case, when the sequence of Lipschitz functions $\{u_m\}$ converges to u in $W^{1,\Psi}_\rho(\Omega)$, then the sequence of restrictions $\{u_m|_{\partial\Omega}\}$ converges to some $\widehat{u}$ in $Y^{\Psi,\Phi}_\omega(\partial\Omega)$, and we have

$$\mathrm{Tr}\, u := \lim_{m\to\infty} u_m = \widehat{u} \in Y^{\Psi,\Phi}_{\omega,L}(\partial\Omega).$$

2.4.3. Trace embedding theorem (unweighted)

The following theorem was obtained in [30].

THEOREM 2.8. (embedding theorem) *Let the N-functions Φ and Ψ satisfy the Assumption A and Ω be a bounded domain of class $\mathscr{C}^{0,1}$. Then we have:*

(i) There is an inequality :

$$\|u\|_{Y^{\Psi,\Phi}(\partial\Omega)} \leqslant D\|u\|_{W^{1,\Psi}(\Omega)}, \tag{2.2}$$

with D independent of u-an arbitrary Lipschitz function defined on $\bar{\Omega}$;

(ii) The trace operator $\mathrm{Tr} : W^{1,\Psi}_L(\Omega) \mapsto Y^{\Psi,\Phi}_L(\partial\Omega)$ is well defined by (2.1) and for every $u \in W^{1,\Psi}_L(\Omega)$ we have

$$\|\,\mathrm{Tr}\, u\|_{Y^{\Psi,\Phi}(\partial\Omega)} \leqslant D\|u\|_{W^{1,\Psi}(\Omega)},$$

where D is the same as in (2.2).

REMARK 2.9. We always $\Phi(s) \prec \Psi(cs)$, with some universal constant c, see e.g. Proposition 5.1 in [32]. When Φ and Ψ satisfy the Assumption A with $b(s) = \int_0^s \frac{a(t)}{t} dt$

and one of the functions Φ^* or Ψ^* satisfies Δ_2 condition, then we have $\Phi \sim \Psi$ as Orlicz functions (i. e. there exist constants $c_1, c_2 > 0$ such that $\Phi(c_1 s) \prec \Psi(s) \prec \Phi(c_2 s)$ for every $s > 0$), see e.g. Proposition 5.1 in [32]. The same holds when there exist constants $c_1, c_2 > 0$ such that $c_2 b(c_1 s) = \int_0^s \frac{a(t)}{t} dt$ for every $t > 0$. The condition $c_2 b(c_1 s) = \int_0^s \frac{a(t)}{t} dt$ above cannot be substituted by the essentially weaker one $c_1 b(c_1 s) \geqslant \int_0^s \frac{a(t)}{t} dt$ defined in part (c) in Assumption A, to still have the property *(X): if one of the functions* Φ^* *or* Ψ^* *satisfies* Δ_2 *condition then we have* $\Phi \sim \Psi$. This is confirmed by the example $a(t) = \frac{t^2}{2}, b(t) = \exp(t) - 1$, presented to us by the referee. In this case the condition (c) in Assumption A is satisfied and we have $\Phi = \frac{1}{6}\lambda^3$, $\Phi^* = \frac{2}{3}\lambda^{3/2}$, $\Psi = \exp(t) - t - 1$, $\Psi^* = (1+\lambda)\ln(1+\lambda) - \lambda$ (see [38], Section I.2, Example 3). In particular both functions Φ^* and Ψ^* satisfy the Δ_2 condition however Φ and Ψ are not equivalent as Orlicz functions. Consequently, the condition (X) does not hold for them.

It is claimed in Remark 2.6 in [12], Remark 5.3 in [31] and Remark 3.2 in [30] that property (X) still holds with condition (c) in Assumption A. The fact that the property (X) requires stronger assumption has been overlooked there. Let us emphasize however that all three mentioned remarks have the form of the additional comment about Kita pairs (Φ, Ψ) and they are not used in any other places in the mentioned papers.

2.5. Extension theorem for cube

Let $\rho : [0,1] \to [0,\infty)$ be a given weight function, $\int_0^1 \rho(t)dt < \infty$ and let us define the following transforms (global and local) of the weight ρ:

$$\omega_\rho(z) := |z|^{n-1} \int_0^1 \frac{1}{t^n} \chi_{\{\frac{z}{t} \in (-\frac{3}{4}, \frac{3}{4})^{n-1}\}} \rho(t)\, dt, \quad z \in \mathbf{R}^{n-1}, \tag{2.3}$$

$$\omega_{\rho,\kappa}(z) := |z|^{n-1} \int_0^\kappa \frac{1}{t^n} \chi_{\{\frac{z}{t} \in (-\frac{3}{4}, \frac{3}{4})^{n-1}\}} \rho(t)\, dt, \ \kappa \in (0,1) \quad z \in \mathbf{R}^{n-1}.$$

Moreover, let $\phi_t : \mathbb{R}^{n-1} \to \mathbb{R}$ $(t > 0)$ be Lipschitz molifier function, i.e.

$$\phi_t(x) = t^{-(n-1)} \phi(x/t), \quad \phi(x_1, \ldots, x_{n-1}) = \psi(x_1) \cdot \ldots \cdot \psi(x_{n-1}), \tag{2.4}$$

where ψ is the Lipschitz one variable even function defined by

$$\psi(t) = \begin{cases} 1 & \text{when } 0 \leqslant t \leqslant \frac{1}{4}, \\ -2t + \frac{3}{2} & \text{when } \frac{1}{4} \leqslant t \leqslant \frac{3}{4}, \\ 0 & \text{when } \quad t > \frac{3}{4}. \end{cases}, \quad \text{for } t \geqslant 0.$$

In particular $0 \leqslant \phi \leqslant 1$, $\operatorname{supp}\phi \subseteq 3Q'$, $\phi \equiv 1$ in $\frac{1}{2}Q'$, and $\int_{\mathbb{R}^{n-1}} \phi dx = 1$.

We will deal with weighted Sobolev space $W^{1,\Psi}_{\tilde{\rho}}(Q)$ where $\tilde{\rho}(x',t) = \rho(t)$. The following result was obtained in [12].

THEOREM 2.10. *Let* Ψ *be the given Orlicz function,* $n \geqslant 2$*,* $Q' = (-\frac{1}{2}, \frac{1}{2})^{n-1}$*,* $Q = Q' \times (0,1)$*,* $v : Q' \to \mathbf{R}$ *be Lipschitz and compactly supported in* $(1-d)Q'$*, where* $d \in (0,1)$ *and*

$$\widetilde{v}(x,t) = v * \phi_t(x) := \begin{cases} \int_{\mathbb{R}^{n-1}} v(y)\phi_t(x-y)\,dy & \text{when } t > 0, \\ v(x) & \text{when } t = 0, \end{cases}$$

where ϕ_t *is as in* (2.4)*. Moreover, let* $\rho : [0,1] \to [0,\infty)$ *be a given weight function,* $\int_0^1 \rho(t)dt = C(\rho) < \infty$ *and* $\tilde{\rho}(x',t) = \rho(t)$*. Then we have:*

(i)

$$\int_{Q'} \int_0^1 \Psi(|\widetilde{v}(x,t)|)\,\rho(t)\,dxdt \leqslant C(\rho) \int_{Q'} \Psi(|v(x)|)\,dx$$

and there exists constant $\tilde{B}_1$ *independent of* u *such that*

$$\|\widetilde{v}\|_{L^{\Psi}_{\tilde{\rho}}(Q)} \leqslant \tilde{B}_1 \|v\|_{L^{\Psi}(Q')}.$$

(ii)

$$\int_{Q'} \int_0^1 \Psi(|\nabla\widetilde{v}|)\,\rho(t)\,dtdx \leqslant L \int_{x\in Q'} \int_{y\in Q'} \Psi\left(\frac{I|v(y)-v(x)|}{|x-y|}\right) \frac{\omega_\rho(x-y)}{|x-y|^{n-2}}\,dydx + \frac{C(\rho)}{2} \int_{Q'} \Psi(J|v(x)|)\,dx, \tag{2.5}$$

where $I = \frac{5n}{2}\left(\frac{3}{2}\right)^n \sqrt{\frac{n-1}{3}}$*,* $J = \left(\frac{3}{2d}\right)^{n-1} \frac{n+7}{2}$*,* $L = \frac{1}{2}\left(\frac{4}{3}\right)^n \frac{e}{\sqrt{n-1}}$*,* e *is the Euler number,* ω_ρ *is defined by* (2.3) *and* $\nabla\widetilde{v}$ *denotes full gradient of* $\widetilde{v}$ *and there exists constant* $\tilde{B}_2$ *independent of* u *such that*

$$\|\nabla\widetilde{v}\|_{L^{\Psi}_{\tilde{\rho}}(Q)} \leqslant \tilde{B}_2 \|v\|_{Y^{\Psi,\Psi}_{\omega_\rho}(Q')}.$$

(iii) there exists a constant $\tilde{B}_3$ *independent of* u *such that*

$$\|\widetilde{v}\|_{W^{1,\Psi}_{\tilde{\rho}}(Q)} \leqslant \tilde{B}_3 \|v\|_{Y^{\Psi,\Psi}_{\omega_\rho}(Q')}.$$

REMARK 2.11. It is easy to check that $\tilde{v}$ is Lipschitz in space direction whenever v is Lipschitz. What is less trivial is its Lipschitzity in the t direction as well. For this verification we only give the hint, leaving the details to the reader. We have

$$\begin{aligned} \left|\frac{\tilde{v}(x,t)-\tilde{v}(x,0)}{t}\right| &\leqslant \int_{\{\frac{|y|}{t} \leqslant \frac{3}{2}\sqrt{n-1}\}} \left|\frac{v(x-y)-v(x)}{t}\right| \phi_t(y)dy \\ &\leqslant \|\nabla v\|_\infty \frac{3}{2}\sqrt{n-1} \int_{\mathbb{R}^{n-1}} \phi_t(y)dy = \|\nabla v\|_\infty \frac{3}{2}\sqrt{n-1}. \end{aligned}$$

REMARK 2.12. Weight ω_ρ can be substituted by smaller weight function $\omega_{\rho,\frac{d}{3}}$ with the same constants, see Remark 4.3 in [12].

3. Formulation of main results

Our main results deal with two cases: the case when Ω is Lipschitz subgraph and more general case, when Ω is a Lipschitz boundary domain. In this section we only formulate and discuss them, postponing their proofs to the preceding sections.

3.1. Inequalities on Lipschitz subgraph

For our considerations we deal with the following assumption.

ASSUMPTION B. There are given:

(a) a Lipschitz function $\alpha : \overline{aQ'} \to \mathbf{R}$ and sets

$$V^+ := \{(x', x_n) \in aQ' \times \mathbf{R} : \alpha(x') < x_n < \alpha(x') + b\},$$
$$S := \{(x', x_n) \in aQ' \times \mathbf{R} : \alpha(x') = x_n\},$$

where $a, b > 0$, being the subset of the subgraph of α and the graph of α, respectively;

(b) a bilipschitz mapping

$$\widetilde{\alpha} : \overline{Q_{a,b}} := \overline{aQ'} \times [0,b] \mapsto V^+, \quad \widetilde{\alpha}(x', x_n) = (x', \alpha(x') + x_n),$$
$$\widetilde{\alpha}^{-1}(y', y_n) = (y', y_n - \alpha(y'));$$

(c) a linear mapping

$$T : \overline{Q_{a,b}} := \overline{aQ'} \times [0,b] \to \overline{Q} = \left[-\frac{1}{2}, \frac{1}{2}\right]^{n-1} \times [0,1], \ T(x', x_n) = \left(\frac{x'}{a}, \frac{x_n}{b}\right);$$

(d) a weight ρ defined on V^+ which is of the form $\rho(x) = \tau(\mathrm{dist}(x, S))$, involving continuous function $\tau : (0,\infty) \to (0,\infty)$ and the following transforms of τ

$$\tilde{\tau}(t) = \tau_{e_1,e_2}(t) := \sup\{\tau(s) : s \in (e_1 t, e_2 t)\}, \text{ where } t > 0 \text{ and}$$
$$e_1 = \frac{1}{(1+a_0)b}, \ e_2 = \frac{1}{b}, \ a_0 = \|\nabla\alpha\|_\infty, \ \int_0^1 \tilde{\tau}(s)ds < \infty,$$
$$\widetilde{\omega}(s) = \widetilde{\omega}_{\tilde{\tau}}(s) := s^{n-1} \int_{\min\{s,1\}}^1 \frac{1}{t^n} \tilde{\tau}(t)\, dt;$$

(e) a number $d > 0$ and the nonempty set $Lip_d(S)$ consisting of all Lipschitz function $g : S \to \mathbb{R}$ which are compactly supported in S (treated as an open $n-1$ dimensional submanifold in $\mathbb{R}^n$) and such that

$$\mathrm{dist}\left(\mathrm{supp}\left(g \circ \tilde{\alpha} \circ T^{-1}|_{Q' \times \{0\}}\right), \partial Q' \times \{0\}\right) \geqslant d. \tag{3.1}$$

Main statement of this subsection reads as follows.

THEOREM 3.1. *Suppose that Assumption B is satisfied and* Ψ *is an Orlicz function. Then there exists a linear operator* $Ext_d^L : Lip_d(S) \to Lip(\overline{V^+})$ *such that when* $g \in Lip_d(S)$ *then* $\widetilde{g} := Ext_d^L(g)$ *satisfies* $\tilde{g} = g$ *on* S *and*

i) *For every* $\lambda > 0$ *we have*

$$\int_{V^+} \Psi\left(\frac{|\widetilde{g}(x)|}{\lambda}\right) \rho(x)\, dx \leqslant C_1 \int_S \Psi\left(\frac{|g(x)|}{\lambda}\right) d\sigma(x);$$
$$\int_{V^+} \Psi\left(\frac{|\nabla \widetilde{g}(x)|}{\lambda}\right) \rho(x) dx \leqslant B_0 \int_S \int_S \Psi\left(\frac{B_2}{\lambda} \frac{|g(x)-g(y)|}{|x-y|}\right) \frac{\widetilde{\omega}(B_4|x-y|)}{|x-y|^{n-2}} d\sigma(x) d\sigma(y)$$
$$+ B_1 \int_S \Psi\left(\frac{B_3}{\lambda}|g(x)|\right) d\sigma(x),$$

with constants $C_1, B_0, B_1, B_2, B_3, B_4$ *independent of* g *but depending on* S, ρ *and* d.

ii) *There exist constants* $\widetilde{B}_1$, $\widetilde{B}_2$ *independent of* g *such that*

$$\|\widetilde{g}\|_{L^\Psi_\rho(V^+)} \leqslant \widetilde{B}_1 \|g\|_{L^\Psi(S)}, \tag{3.2}$$
$$\|\nabla \widetilde{g}\|_{L^\Psi_\rho(V^+)} \leqslant \widetilde{B}_2 \|g\|_{Y^{\Psi,\Psi}_\omega(S)},$$

where $\omega(x,y) = \widetilde{\omega}(B_4|x-y|)$.

REMARK 3.2. Constants C_1, B_0, B_2, B_3, B_4 in our estimations are described in table below.

Constant	
C_1	$b \cdot \int_0^1 \tilde{\tau}(t) dt$
B_0	$\frac{1}{2}\left(\frac{4}{3}\right)^n \frac{e}{\sqrt{n-1}} \cdot b \cdot a^{n-2} c_n^{-(n-1)} \cdot a_1^{2n-3}$
B_1	$2^{-1} \cdot b \cdot \int_0^1 \tilde{\tau}(t) dt$
B_2	$\frac{5n}{2}\left(\frac{3}{2}\right)^n \sqrt{\frac{n-1}{3}} \cdot a \cdot \max\{\frac{1}{a}, \frac{1}{b}\} \cdot (1+a_0)$
a_p	$\sqrt{p + \|\nabla \alpha\|_\infty^2},\ p \geqslant 0$
B_3	$\left(\frac{3}{2d}\right)^{n-1} \frac{n+7}{2} \cdot \max\{\frac{1}{a}, \frac{1}{b}\} \cdot (1+a_0)$
B_4	$c_n \frac{1}{a \cdot a_1}$
c_n	$\frac{4}{3\sqrt{n-1}}$
d	$\mathrm{dist}\left(\mathrm{supp}\left(g \circ \tilde{\alpha} \circ T^{-1}\|_{Q' \times \{0\}}\right), \partial Q' \times \{0\}\right) \geqslant d$

Proof of Theorem 3.1 will be provided in Section 4.3.

3.2. Inequalities on Lipschitz boundary domains

As a direct consequence we obtain the following theorem, which generalizes Theorem 3.1 to Lipschitz boundary domains when we select the admitted measures. The proof of this statement will be provided in Section 6.

THEOREM 3.3. *Suppose that Ψ is Orlicz function, Ω is a domain of class $\mathscr{C}^{0,1}$, $\rho(x) = \tau(\mathrm{dist}(x,\partial\Omega))$ and $\tau : (0,\infty) \to \mathbb{R}_+$ is continuous, monotonic, $\int_0^1 \tau(t)dt < \infty$, τ satisfies one of the following conditions i) or ii) for small arguments:*

i) *τ is nondecreasing, absolutely continuous, satisfies the Δ_2 condition and $s\tau'(s) \leqslant F\cdot\tau(s)$, where $\frac{F}{n-1} < 1$;*

ii) *τ is nonincreasing and τ satisfies $\Delta_{\frac{1}{2}}$-condition i.e. $\tau(\frac{1}{2}s) < c\tau(s)$, where c is independent of s.*

Then there exists a linear operator $Ext^L : Lip(\partial\Omega) \to Lip(\bar{\Omega})$ such that when $u : \partial\Omega \to \mathbb{R}$ is Lipschitz then the function $\widetilde{u} := Ext^L(u) \in Lip(\bar{\Omega})$ is such that $\tilde{u} = u$ on $\partial\Omega$ and

a) *For every $\lambda > 0$ we have*

$$\int_{\Omega} \Psi\left(\frac{|\widetilde{u}(x)|}{\lambda}\right) \tau(\mathrm{dist}(x,\partial\Omega))\,dx \prec \int_{\partial\Omega} \Psi\left(\frac{E|u(x)|}{\lambda}\right) d\sigma(x); \tag{3.3}$$

$$\begin{aligned}\int_{\Omega} \Psi\left(\frac{|\nabla\widetilde{u}(x)|}{\lambda}\right) \tau(\mathrm{dist}(x,\partial\Omega))\,dx \\ \prec \int_{\partial\Omega}\int_{\partial\Omega} \Psi\left(\frac{B}{\lambda}\frac{|u(x)-u(y)|}{|x-y|}\right) \frac{\tau(|x-y|)}{|x-y|^{n-2}} d\sigma(x)d\sigma(y) \\ + \int_{\partial\Omega} \Psi\left(\frac{D}{\lambda}|u(x)|\right) d\sigma(x),\end{aligned} \tag{3.4}$$

where constants E,B,D, as well as constants involved in the inequalities "$\prec$", are independent of u.

b) *We have*

$$\|\widetilde{u}\|_{L^{\Psi}_{\rho}(\Omega)} \prec \|u\|_{L^{\Psi}(\partial\Omega)}, \tag{3.5}$$

$$\|\nabla\widetilde{u}\|_{L^{\Psi}_{\rho}(\Omega)} \prec \|u\|_{Y^{\Psi,\Psi}_{\omega}(\partial\Omega)}, \tag{3.6}$$

where $\omega(x,y) = \tau(|x-y|)$, $\rho(x) = \tau(\mathrm{dist}(x,\partial\Omega))$, constants involved in the inequalities "$\prec$" are independent of u.

REMARK 3.4. We recall that the notation $\prec$ means that inequalities hold up to some constants which are not dependent on u but only on geometric properties of the domain, properties of function τ and the choice of an extension operator.

REMARK 3.5. As a consequence we obtain that under the assumptions of Theorem 3.3 there exists a bounded linear operator $\mathrm{Ext}: Y^{\Psi,\Psi}_{\omega,L}(\partial\Omega) \to W^{1,\Psi}_{\rho,L}(\Omega)$ such that inequalities (3.5) and (3.6) hold with $\tilde{u} = \mathrm{Ext}(u)$ where $u \in Y^{\Psi,\Psi}_{\omega,L}(\partial\Omega)$. The equation $\tilde{u}|_{\partial\Omega} = u$ is interpreted in the following sense: when u_m's converge to u in $Y^{\Psi,\Psi}_{\omega}(\partial\Omega)$ and u_m-s are Lipschitz, then $\mathrm{Ext}(u_m)$ are Lipschitz, converge to $\tilde{u}$ in $W^{1,\Psi}_{\rho}(\Omega)$ and their restrictions to $\partial\Omega$ converge to u in $Y^{\Psi,\Psi}_{\omega}(\partial\Omega)$.

REMARK 3.6. If we relax the Lipschitzity assumptions on u all the estimates obtained in this paper are valid, provided that their right hand sides are finite. In such a case the constructed ("extension") operator is still linear and continuous from $Y^{\Psi,\Psi}_{\rho(|x-y|)}(\partial\Omega)$ to $W^{1,\Psi}_{\rho(\mathrm{dist}(\cdot,\partial\Omega))}(\Omega)$ but we do not understand then how to interpret the fact that u is the restriction of $\tilde{u}$ to $\partial\Omega$.

3.3. Links with literature

We are now to discuss links of our results with those existing in the literature.

REMARK 3.7. (Extensions to weighted Sobolew spaces with power weights) When $\Psi(\lambda) = \lambda^p$ the following result has been obtained by Mitrea and Taylor [52], Proposition 4.1. We present it as example result and some other related results have been already listed in the Introduction. In the formulation $B^{p,p}_s(\partial\Omega)$ are the related Besov spaces defined on $\partial\Omega$ (see Remark 2.6).

PROPOSITION 3.8. *Assume that the metric tensor on M has continuous coefficients and fix a Lipschitz subdomain Ω of M. Also, let $1 < p < \infty$, $0 < s < 1$, $s = 1 - \frac{1}{p} - \frac{\alpha}{p}$. Then the trace operator:*

$$\mathrm{Tr}: W^{1,p}_{\mathrm{dist}(x,\partial\Omega)^\alpha}(\Omega) \to B^{p,p}_s(\partial\Omega)$$

is well defined and bounded. Furthermore, this operator is onto has a continuous right inverse. In particular, there exists an extension operator

$$\mathrm{Ext}: B^{p,p}_s(\partial\Omega) \to W^{1,p}_{\mathrm{dist}(x,\partial\Omega)^\alpha}(\Omega),$$

which is linear and bounded and such that $\mathrm{Tr} \circ \mathrm{Ext} = I$.

A classical result dealing with $\Omega = \mathbb{R}^n_+$ was obtained by Uspienski [71] and can be adopted to domains of class $\mathscr{C}^{0,1}$ as we do in details in this paper (under this constraints the procedure is standard).

We already know (Remark 2.6) that when $\Omega \in \mathscr{C}^{0,1}$ then we have $Y^{\lambda^p,\lambda^p}_{|x-y|^\alpha}(\partial\Omega) = B^{p,p}_{1-\frac{1}{p}-\frac{\alpha}{p}}(\partial\Omega)$ when $0 < 1 - \frac{1}{p} - \frac{\alpha}{p} < 1$. Therefore second part of statement above gives extension operator

$$\mathrm{Ext}: Y^{\lambda^p,\lambda^p}_{|x-y|^\alpha}(\partial\Omega) \to W^{1,p}_{\mathrm{dist}(x,\partial\Omega)^\alpha}(\Omega),$$

whenever $-1<\alpha<p-1$. In our approach the extension is independent of the choice of convex function $\Psi(\lambda)=\lambda^p$. This forces the restriction on parameter: $-1<\alpha<n-1$ (the conditions i) and ii)), which is different than in the statement above. As $n\geqslant 2$, it works always when $-1<\alpha<1$, but in general sets $(-1,p-1)$ and $(-1,n-1)$ are independent.

For other related results see also [61].

REMARK 3.9. (Extensions within classical Besov spaces) The following theorem is known in the theory of classical Besov spaces (see e.g. Theorem 2.1 in [50] and older source, Theorem 3 from Chapter 6.2.1 in [28]).

PROPOSITION 3.10. *Let* $\Omega\subseteq\mathbb{R}^n$ *be a bounded domain of class* $\mathscr{C}^{0,1}$ *and let* $1>\widetilde{\alpha}-\frac{1}{p}>(n-1)(\frac{1}{p}-1)_+$, $0<q\leqslant\infty$. *Then the trace operator:*

$$\mathrm{Tr}: B^{p,q}_{\widetilde{\alpha}}(\Omega)\to B^{p,q}_{\widetilde{\alpha}-\frac{1}{p}}(\partial\Omega)$$

is well defined and bounded. Moreover, this operator is onto and has a continuous right inverse. In particular, there exist an extension operator

$$\mathrm{Ext}: B^{p,q}_{\widetilde{\alpha}-\frac{1}{p}}(\partial\Omega)\to B^{p,q}_{\widetilde{\alpha}}(\Omega), \tag{3.7}$$

which is linear and bounded and such that $\mathrm{Tr}\circ\mathrm{Ext}=I$.

We are interested in the situation $p=q$ only. According to Remark 2.6, we have $Y^{\lambda^p,\lambda^p}_{|x-y|^\alpha}(\partial\Omega)=B^{p,p}_{1-\frac{1}{p}-\frac{\alpha}{p}}(\partial\Omega)$ whenever $0<1-\frac{1}{p}-\frac{\alpha}{p}<1$, equivalently $\alpha\in(-1,p-1)$. Applying (3.7) with $\widetilde{\alpha}=1-\frac{\alpha}{p}$, we obtain that existence of continuous extension operator

$$\mathrm{Ext}: Y^{\lambda^p,\lambda^p}_{|x-y|^\alpha}(\partial\Omega)\to B^{p,p}_{1-\frac{\alpha}{p}}(\Omega).$$

Next, by Remark 2.3 we observe that for $\alpha\in(-1,0)$, $B^{p,p}_{1-\frac{\alpha}{p}}(\Omega)\subseteq W^{1,p}_{dist(\cdot,\partial\Omega)^\alpha}(\Omega)$. It implies that, under this constraints, Proposition 3.8 and a special variant of our Theorem 3.3 can be deduced from Proposition 3.10 which deals with the unweighted Besov spaces. On the other hand, when $\alpha\in(0,p)$ we have by Remark 2.3 that $W^{1,p}_{dist(\cdot,\partial\Omega)^\alpha}(\Omega)\subseteq B^{p,p}_{1-\frac{\alpha}{p}}(\Omega)$.

Therefore for that range of parameters Proposition 3.10 follows from Theorem 3.3 (with $\Psi=\lambda^p$, $\rho=\mathrm{dist}(x,\partial\Omega)^\alpha$) as well as from Proposition 3.8.

REMARK 3.11. (Extensions to weighted Sobolew spaces with Muckenhoput weights) We are interested now in extension results dealing with Muckenhoupt weights.

Recall that a positive function $\omega\in L^1_{loc}(\mathbb{R}^n)$ belongs to Muchenhoupt class $\mathscr{A}_p$ where $1<p<\infty$, if there exists a constant $0<A<\infty$ such that for all cubes $Q\subseteq\mathbb{R}^n$

$$\left(\frac{1}{|Q|}\int_Q\omega(x)\,dx\right)\cdot\left(\frac{1}{|Q|}\int_Q\omega(x)^{\frac{-1}{p-1}}\,dx\right)^{p-1}\leqslant A.$$

This is exactly the class of weights for which Hardy Littlewood maximal function $\mathscr{M}f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy$ is bounded operator as an operator from $L^p_\omega(\mathbb{R}^n)$ to $L^p_\omega(\mathbb{R}^n)$ (see [66], Theorem 1, Chapter V). When $E \subseteq \mathbb{R}^n$ is a subset we say that $\omega \in A_p(E)$ if it is a restriction of some weight from the class $A_p(\mathbb{R}^n)$ to E. We have the following remarks.

1) In paper [69] by Tulyanev, Theorem 3.1, one finds the following statement. We present its simplified variant and omit the details.

PROPOSITION 3.12. ([69]) *Let $1 < p < \infty$ and let $\gamma \in A_p((0,1))$ and we consider weight function $\rho(x,t) = \gamma(x)$ where $(x,t) \in (0,1) \times (0,1)$. Then*

$$\mathrm{Tr}|_{y=0} W^{1,p}(Q,\gamma(x)) = \widetilde{B}_p^{1-\frac{1}{p}}((0,1),\gamma).$$

The author uses the following definition of weighted Besov type space where $0 < s < 1$:

$$\|\phi\|_{\widetilde{B}^s_p((0,1),\gamma)} := \left(\int_0^1 \gamma(x)|\phi(x)|^p dx \right)^{\frac{1}{p}} + \left(\int_0^1 \frac{1}{z^{ps}} \int_0^1 \gamma(x) \frac{1}{z^p} \{ \int_{-z}^{z} |(\Delta^1 \phi)(x,t)| dt \}^p dx \, \frac{dz}{z} \right)^{\frac{1}{p}},$$

where $(\Delta^1\phi)(x,t) := (-1)(\phi(x) - \phi(x+t))$ if $[x, x+t] \subseteq (0,1)$ and $(\Delta^1\phi)(x,t) := 0$ otherwise.

Let us note that in the above approach the weight function $\rho = \rho(x,t)$ defined on cube depends on longitudinal coordinate x only and it is not dependent on t. In our case ρ depends on the transversal coordinate t only. Therefore the approach presented there is different and functions spaces appearing are also different. Unified approach linking the two issues would be of interest.

For other related issues in this direction we refer to papers Krbec [1], Tyulenev [68].

2) In paper by Frazier and Roudenko [18] the authors study trace and extension results between weighted Besov spaces $B^{\alpha,q}_p(W)$ where W is a matrix weight on $\mathbb{R}^n$ being p-admissible, in particular satisfying the matrix A_p condition (we omit the detailed formulation). Among other results, the authors find the necessary conditions on two p-admissible weights V, W which allow to define the continuous liner map

$$\mathrm{Ext} : \dot{B}_p^{\alpha-\frac{1}{p},q}(W) \to \dot{B}^{\alpha,q}_p(V),$$

between Besov type spaces (Theorem 1.3 in [18]) of vectorial functions. Those spaces are defined by using the atomic decomposition constructed by using suitable molifier ϕ belonging to the Schwartz space from class $\mathscr{A}$:

$$\|f\|_{\dot{B}^{\alpha q}_p(W)} = \left\| \left\{ 2^{\nu\alpha} \|\phi_\nu * \vec{f}\|_{L^p(W)} \right\}_\nu \right\|_{l^q},$$

where $\phi_\nu(x) = 2^{\nu n}\phi(2^\nu x)$ for $\nu \in \mathbb{Z}$, $\overrightarrow{f} = (f_1,\cdots,f_m)^T$ and $\phi_\nu * \overrightarrow{f} = (\phi_\nu * f_1, \phi_\nu * f_2, \cdots, \phi_\nu * f_m)^T$.

Let us mention that even in the scalar case and situation: $\Psi(\lambda) = \lambda^p$, $\alpha = 1$, $q = p$ our statements seem to deal with different spaces and admit different class of weights. Our measures depend only on the distance from the boundary but are not necessary in Muckenhopt class. Therefore the approaches seem to be different.

REMARK 3.13. The definition of Orlicz-Slobodetskii type space, we used, does not involve atomic decompositions. Those spaces appeared in papers by Fougéres [16, 17] Lacroix [42, 43] for needs of trace type theorems expressed in Orlicz setting. We think that such a (nonatomic) definition is more convenient when one studies Besov type spaces dealing with general class of Orlicz functions and measures.

3.4. Examples of admissible weights

Below we present several examples of weights $\tau(t)$ which obey assumptions of Theorems 3.1 and 3.3.

EXAMPLE 3.14. Let $\rho(x) = \tau(\mathrm{dist}(x,\partial\Omega))$. The following functions τ are admitted to Theorems 3.1 and 3.3. Easy verification is left to the reader.

(a) $\tau \equiv 1$, we retrieve the classical (unweighted) result;

(b) $\tau(t) = t^\alpha$, $-1 < \alpha < n-1$;

(c) $\tau(t) = t^\alpha \left(\ln\left(2+\frac{1}{t}\right)\right)^\beta$, $-1 < \alpha < n-1, \beta > 0$;

(d) $\tau(t) = (\log(2+\frac{1}{t}))^{-\alpha}, \alpha > 0$;

(e) $\tau(t) = 1 - e^{\alpha t}$, $\alpha < 0$, $n > 2$.

4. Inequalities on Lipschitz subgraph

4.1. Construction of extension

Having two bilipschitz dipheomorphisms T and $\tilde{\alpha}^{-1}$ such that:

$$\overline{V^+} \overset{\tilde{\alpha}^{-1}}{\mapsto} \overline{Q_{a,b}} \overset{T}{\mapsto} \overline{Q},$$

we define the bi-Lipschitz dipheomorphism $\tilde{\phi}$ by expression

$$\tilde{\phi} := \tilde{\alpha} \circ T^{-1} : \overline{Q} \mapsto \overline{V^+}, \text{ so that } \tilde{\phi}^{-1} : \overline{V^+} \mapsto \overline{Q}.$$

Note that

$$\tilde{\phi}|_{\overline{Q'}\times\{0\}} : \overline{Q'} \times \{0\} \mapsto \overline{S} \text{ and } \tilde{\phi}^{-1}|_{\overline{S}} : \overline{S} \mapsto \overline{Q'} \times \{0\}$$

are also Lipschitz, therefore $\tilde{\phi}|_{\overline{Q'}\times\{0\}}$ is bi-Lipschitz dipheomorphism.

Having Lipschitz function

$$g : \overline{S} \to \mathbf{R}$$

as in Assumption B, (3.1), we define the subordinated function $h : \overline{Q'} \times \{0\} \to \mathbf{R}$ by the formulae

$$h(x) = g(\tilde{\phi}(x)).$$

Let us note that h is supported in $(1-d)Q' \times \{0\}$. Therefore we can extend it by 0 to the whole $\mathbb{R}^{n-1} \times \{0\}$. We denote such extension by the same expression.

We extend function h to $\mathbb{R}^{n-1} \times [0,\infty)$ by formulae:

$$\widetilde{h}(x,t) = (h(\cdot,0) * \phi_t)(x) := \begin{cases} \int_{Q'} h(y,0)\phi_t(x-y)\,dy & \text{when } t > 0, \\ h(x,0) & \text{when } t = 0. \end{cases}$$

where $\{\phi_t\}$ is Lipschitz molifier as in (2.4).

The extension of g to V^+ is defined by

$$\widetilde{g}(\cdot) := \widetilde{h}(\tilde{\phi}^{-1}(\cdot)) : V^+ \to \mathbb{R}. \tag{4.1}$$

We will also deal with the following constants:

Constant		Constant	
a_p	$\sqrt{p + \|\nabla\alpha\|_\infty^2},\ p \geqslant 0$	G	$\max\{\frac{1}{a}, \frac{1}{b}\}$
$c(\tilde{\tau})$	$\int_0^1 \tilde{\tau}(t)dt$	c_n	$\frac{4}{3\sqrt{n-1}}$

(4.2)

In the remaining part of this section we prove that the proposed extension satisfies assertion of Theorem 3.1. This will be done in the sequence of substeps presented in the preceding subsections which end up with the proof of Theorem 3.1.

4.2. Auxilary tools

4.2.1. Reduction to the analysis on unit cube

This will be done with help of two lemmas. Proof of the first lemma is presented in the Appendix for readers convenience, while the proof of second statement is abbreviated as it is rather standard.

Before their formulations let us introduce the notation which will be used also later.

AUXILLIARY FUNCTIONS. We will deal with the following functions defined on $[0,\infty)$, subordinated to analysis on abstract Lipschitz subgraph, rectangle and cube, respectively, assuming that ρ and ω are given weights:

$$
\begin{aligned}
&\mathscr{I}_1(s) := \int_S \Psi(s|g(x)|)\, d\sigma(x),\\
&\mathscr{J}_1(s) := \int_S\int_S \Psi\left(s\frac{|g(x)-g(y)|}{|x-y|}\right)\frac{\omega(|x-y|)}{|x-y|^{n-2}}d\sigma(x)d\sigma(y),\\
&\mathscr{K}_1(s) := \int_{V^+} \Psi(s|\widetilde{g}(y)|)\,\rho(y)dy,\\
&\mathscr{L}_1(s) := \int_{V^+} \Psi(s|\nabla\widetilde{g}(y)|)\,\rho(y)dy; \qquad (4.3)\\
&\mathscr{I}_2(s) := \int_{aQ'\times\{0\}} \Psi(s|f(x)|)\, d\sigma(x),\\
&\mathscr{J}_2(s) := \int_{aQ'\times\{0\}}\int_{aQ'\times\{0\}} \Psi\left(s\frac{|f(x)-f(y)|}{|x-y|}\right)\frac{\omega(|\tilde{\alpha}(x)-\tilde{\alpha}(y)|)}{|x-y|^{n-2}}d\sigma(x)d\sigma(y),\\
&\mathscr{K}_2(s) := \int_{Q_{a,b}} \Psi\left(s|\widetilde{f}(x)|\right)\rho(\widetilde{\alpha}(x))dx,\\
&\mathscr{L}_2(s) := \int_{Q_{a,b}} \Psi\left(s|\nabla\widetilde{f}(x)|\right)\rho(\widetilde{\alpha}(x))dx;\\
&\mathscr{I}_3(s) := \int_{Q'\times\{0\}} \Psi(s|h(x)|)\, d\sigma(x),\\
&\mathscr{J}_3(s) := \int_{Q'\times\{0\}}\int_{Q'\times\{0\}} \Psi\left(s\frac{|h(x)-h(y)|}{|x-y|}\right)\frac{\omega(|\tilde{\alpha}\circ T^{-1}(x)-\tilde{\alpha}\circ T^{-1}(y)|)}{|x-y|^{n-2}}d\sigma(x)d\sigma(y),\\
&\mathscr{K}_3(s) := \int_Q \Psi\left(s|\widetilde{h}(x)|\right)\rho(\widetilde{\alpha}\circ T^{-1}(x))dx,\\
&\mathscr{L}_3(s) := \int_Q \Psi\left(s|\nabla\widetilde{h}(x)|\right)\rho(\widetilde{\alpha}\circ T^{-1}(x))dx;
\end{aligned}
$$

where weight ω will be indicated later.

We have the following results. Their proofs are given in the Appendix.

LEMMA 4.1. (reduction to the analysis on rectangles) *Let* $g: S\to\mathbf{R}$, $f: aQ'\times\{0\}\to\mathbf{R}$ *and* $\widetilde{g}: V^+\to\mathbf{R}$, $\widetilde{f}: Q_{a,b}\to\mathbf{R}$ *be related to* g *and* f *via:*

$$g(x) = f(\widetilde{\alpha}^{-1}(x)), \quad \widetilde{g}(x) = \widetilde{f}(\widetilde{\alpha}^{-1}(x)),$$

where $\tilde{\alpha}, S, V^+$ *are as in Assumption B.*

Then for any number $s\in\mathbf{R}_+$ *we have*

$$\mathscr{I}_2(s) \leqslant \mathscr{I}_1(s), \tag{4.4}$$

$$\mathscr{J}_2(s) \leqslant a_1^{n-2}\mathscr{J}_1(s), \tag{4.5}$$

$$\mathscr{K}_2(s) = \mathscr{K}_1(s), \tag{4.6}$$

$$\mathscr{L}_2(s(1+a_0)) \geqslant \mathscr{L}_1(s). \tag{4.7}$$

LEMMA 4.2. (reduction to the analysis on unit cube) *Let* $h : Q' \times \{0\} \to \mathbf{R}$, $f : aQ' \times \{0\} \to \mathbf{R}$ *and* $\widetilde{h} : Q \to \mathbf{R}, \widetilde{f} : Q_{a,b} \to \mathbf{R}$ *be related to* h *and* f *via:*

$$h(x) = f(T^{-1}(x)), \quad \widetilde{h}(x) = \widetilde{f}(T^{-1}(x)),$$

where T *is as in Assumption B.*

Then for any constant $s \in \mathbf{R}_+$ *we have*

$$\begin{aligned}
\mathscr{I}_3(s) &= \frac{1}{a^{n-1}} \mathscr{I}_2(s) \\
\mathscr{J}_3\left(\frac{s}{a}\right) &= \frac{1}{a^n} \mathscr{J}_2(s), \\
\mathscr{K}_3(s) &= \frac{1}{a^{n-1}b} \mathscr{K}_2(s), \\
\mathscr{L}_3(sG) &\geqslant \frac{1}{a^{n-1}b} \mathscr{L}_2(s), \text{ where } G := \max\left\{\frac{1}{a}, \frac{1}{b}\right\}.
\end{aligned} \tag{4.8}$$

4.2.2. Inequalities on Lipschitz subgraph under technical assumption

In this subsection we will be dealing with the following technical assumption.

ASSUMPTION C.

(a) the weight ρ_1 is defined on $(0,1)$ and weight ρ is defined on V^+. They satisfy the following inequality:

$$\rho(\widetilde{\alpha} \circ T^{-1}(x',t)) \leqslant \rho_1(t), \tag{4.9}$$

for almost every $(x',t) \in Q$;

(b) we deal with the following weight function $\tilde{\omega}_{\rho_1}$, defined on $\mathbf{R}_+$, being the transformation of ρ_1:

$$\widetilde{\omega}_{\rho_1}(s) := s^{n-1} \int_{\min\{s,1\}}^1 \frac{1}{t^n} \rho_1(t)\, dt \tag{4.10}$$

and its renormalizations:

$$\widetilde{\omega}_{\rho_1}^C(s) := C^{-(n-1)} \widetilde{\omega}_{\rho_1}(Cs) = s^{n-1} \int_{\min\{Cs,1\}}^1 \frac{1}{t^n} \rho_1(t)\, dt \tag{4.11}$$

involving parameter C which we will establish later.

We have the following result.

LEMMA 4.3. *Assume that* Ψ *is an Orlicz function,* $g : S \to \mathbf{R}$ *is Lipschitz and satisfies* (3.1) *with given constant* d, $\widetilde{g} : V^+ \to \mathbf{R}$ *is defined by* (4.1) *and Assumptions B, C are satisfied. Then we have*

i) *For every $\lambda > 0$ we have*

$$\int_{V^+} \Psi\left(\frac{|\widetilde{g}(x)|}{\lambda}\right) \rho(x)\,dx \leqslant C_1 \int_S \Psi\left(\frac{|g(x)|}{\lambda}\right) d\sigma(x); \tag{4.12}$$

$$\int_{V^+} \Psi\left(\frac{|\nabla\widetilde{g}(x)|}{\lambda}\right) \rho(x)dx \leqslant D_0 \int_S\int_S \Psi\left(\frac{D_2}{\lambda}\frac{|g(x)-g(y)|}{|x-y|}\right) \frac{v(|x-y|)}{|x-y|^{n-2}} d\sigma(x)d\sigma(y)$$
$$+ D_1 \int_S \Psi\left(\frac{D_3}{\lambda}|g(x)|\right) d\sigma(x). \tag{4.13}$$

where $v(s) = \widetilde{\omega}_{\rho_1}^{D_4}(s)$ (see (4.11)). Constants $C_1, D_0, D_1, D_2, D_3, D_4$, are independent of g but dependent only on S, ρ, b, d.

ii) *There exists a constant $\widetilde{D}$ independent of g but dependent only on S, ρ, b, d such that*

$$\|\widetilde{g}\|_{W_\rho^{1,\Psi}(V^+)} \leqslant \widetilde{D}\|g\|_{Y_v^{\Psi,\Psi}(S)}.$$

REMARK 4.4. Constants in our estimations (4.12) and (4.13) are described in table below.

Constant	
C_1	$b \cdot \int_0^1 \tilde{\tau}(t)dt$
D_0	$\frac{1}{2}\left(\frac{4}{3}\right)^n \frac{e}{\sqrt{n-1}} a_1^{n-2} \cdot b \cdot a^{-1}$
D_1	$2^{-1} \cdot b \cdot \int_0^1 \rho_1(t)dt$
D_2	$\frac{5n}{2}\left(\frac{3}{2}\right)^n \sqrt{\frac{n-1}{3}} \cdot a \cdot \max\{\frac{1}{a}, \frac{1}{b}\} \cdot (1+a_0)$
D_3	$\left(\frac{3}{2d}\right)^{n-1} \frac{n+7}{2} \cdot \max\{\frac{1}{a}, \frac{1}{b}\} \cdot (1+a_0)$
D_4	$\frac{4}{3\sqrt{n-1}} \frac{1}{a \cdot a_1}$
d	see (3.1)
a_p	$\sqrt{p + \|\nabla\alpha\|_\infty^2},\ p \geqslant 0$

Proof. We use the notation from Subsection 4.1 and of (4.3).

Proof of (4.12). According to Theorem 2.10, part (ii), we have

$$\mathscr{K}\left(\frac{1}{\lambda}\right) := \int_{Q'}\int_0^1 \Psi\left(\frac{|\widetilde{h}(x,t)|}{\lambda}\right) \rho_1(t)\,dxdt \leqslant c(\rho_1) \int_{Q'} \Psi\left(\frac{|h(x,0)|}{\lambda}\right) dx$$
$$= c(\rho_1)\mathscr{I}_3\left(\frac{1}{\lambda}\right), \tag{4.14}$$

where $c(\rho_1) := \int_0^1 \rho_1(t)dt$.

Using condition (4.9) and a chain of inequalities resulting from Lemmas 4.2, 4.1, we get

$$\mathscr{K}\left(\frac{1}{\lambda}\right) \geqslant \int_Q \Psi\left(\frac{|\widetilde{h}(x)|}{\lambda}\right)\rho(\widetilde{\alpha}(T^{-1}(x)))dx = \mathscr{K}_3\left(\frac{1}{\lambda}\right) \stackrel{Lemma\ 4.2}{=} \frac{1}{a^{n-1}b}\mathscr{K}_2\left(\frac{1}{\lambda}\right) \stackrel{Lemma\ 4.1}{\geqslant} \frac{1}{a^{n-1}b}\mathscr{K}_1\left(\frac{1}{\lambda}\right). \tag{4.15}$$

On the other hand,

$$c(\rho_1)\mathscr{I}_3\left(\frac{1}{\lambda}\right) \stackrel{Lemma\ 4.2}{=} \frac{c(\rho_1)}{a^{n-1}}\mathscr{I}_2\left(\frac{1}{\lambda}\right) \stackrel{Lemma\ 4.1}{\leqslant} \frac{c(\rho_1)}{a^{n-1}}\mathscr{I}_1\left(\frac{1}{\lambda}\right). \tag{4.16}$$

We obtain from (4.14),(4.16) and (4.15):

$$\frac{c(\rho_1)}{a^{n-1}}\mathscr{I}_1\left(\frac{1}{\lambda}\right) \geqslant c(\rho_1)\mathscr{I}_3\left(\frac{1}{\lambda}\right) \geqslant \mathscr{K}\left(\frac{1}{\lambda}\right) \geqslant \frac{1}{a^{n-1}b}\mathscr{K}_1\left(\frac{1}{\lambda}\right).$$

From there inequality (4.12) follows. □

Proof of (4.13). At first we note that when $\frac{z}{t} \in \frac{3}{2}Q'$, we have $\frac{|z|}{t} \leqslant \sqrt{n-1}\frac{3}{4} = \frac{1}{c_n}$ (see (4.2)), so that $c_n|z| \leqslant t$. Using formulae (2.3) involving weight ρ_1 and formulae (4.10) for $\tilde{\omega}_{\rho_1}$, we get

$$\omega_{\rho_1}(z) \stackrel{(2.3)}{\leqslant} |z|^{n-1}\int_{\min\{c_n|z|,1\}}^{1}\frac{1}{t^n}\rho_1(t)dt \stackrel{(4.11)}{=} \tilde{\omega}_{\rho_1}^{c_n}(|z|) =: V_1(|z|),$$

and so we can apply Theorem 2.10, inequality (2.5), to get

$$\begin{aligned}\mathscr{L}\left(\frac{1}{\lambda}\right) &:= \int_{Q'}\int_0^1 \Psi\left(\frac{|\nabla\widetilde{h}(x,t)|}{\lambda}\right)\rho_1(t)\,dtdx \\ &\leqslant L\int_{x\in Q'\times\{0\}}\int_{y\in Q'\times\{0\}}\Psi\left(\frac{I|h(x)-h(y)|}{\lambda|x-y|}\right)\frac{V_1(|x-y|)}{|x-y|^{n-2}}d\sigma(x)d\sigma(y) \\ &\quad + \frac{c(\rho_1)}{2}\int_{Q'\times\{0\}}\Psi\left(\frac{J}{\lambda}|h(x)|\right)d\sigma(x), \\ &=: L\mathscr{J}\left(\frac{I}{\lambda}\right) + \frac{c(\rho_1)}{2}\mathscr{I}_3\left(\frac{J}{\lambda}\right),\end{aligned} \tag{4.17}$$

where $I = \frac{5n}{2}\left(\frac{3}{2}\right)^n\sqrt{\frac{n-1}{3}}$, $J = \left(\frac{3}{2d}\right)^{n-1}\frac{n+7}{2}$, $L = \frac{1}{2}\left(\frac{4}{3}\right)^n\frac{e}{\sqrt{n-1}}$, $d \leqslant \mathrm{dist}(\mathrm{supp} h, \partial(Q'\times\{0\}))$ (and $h = u\circ\tilde{\phi}$), $c(\rho_1) = \int_0^1\rho_1(t)dt$.

The condition (4.9) and Lemmas 4.2 and 4.1 imply a chain of inequalities:

$$\mathscr{L}\left(\frac{1}{\lambda}\right) \geqslant \int_{Q'}\int_0^1\Psi\left(\frac{|\nabla\widetilde{h}(x,t)|}{\lambda}\right)\rho(\widetilde{\alpha}\circ T^{-1}(x))dx = \mathscr{L}_3\left(\frac{1}{\lambda}\right) \stackrel{Lemma\ 4.2}{\geqslant} \frac{1}{a^{n-1}b}\mathscr{L}_2\left(\frac{1}{G\lambda}\right) \stackrel{Lemma\ 4.1}{\geqslant} \frac{1}{a^{n-1}b}\mathscr{L}_1\left(\frac{1}{G(1+a_0)\lambda}\right). \tag{4.18}$$

Now we will deal with first term in (4.17).

We observe that for any $x,y \in Q' \times \{0\}$ we have

$$\frac{1}{aa_1}|\tilde{\alpha}\circ T^{-1}(x)-\tilde{\alpha}\circ T^{-1}(y)| \leqslant |x-y| \leqslant \frac{1}{a}|\tilde{\alpha}\circ T^{-1}(x)-\tilde{\alpha}\circ T^{-1}(y)|.$$

Indeed, we have for $x=(x',0)$, $y=(y',0)$:

$$|\tilde{\alpha}\circ T^{-1}(x)-\tilde{\alpha}\circ T^{-1}(y)| = |\left(ax',\alpha(ax')\right)-\left(ay',\alpha(ay')\right)| \leqslant a\sqrt{1+\|\nabla\alpha\|_\infty^2}|x-y|, \text{ and}$$

$$\begin{aligned}|x-y| &= \frac{1}{a}|(ax',0)-(ay',0)| \leqslant \frac{1}{a}|(ax',\alpha(ax'))-(ay',\alpha(ay'))| \\ &= \frac{1}{a}|\tilde{\alpha}(T^{-1}(x))-\tilde{\alpha}(T^{-1}(y))|.\end{aligned}$$

In particular $|x-y| \in (e_1 s, e_2 s)$ where $s=|\tilde{\alpha}\circ T^{-1}(x)-\tilde{\alpha}\circ T^{-1}(y)|$ and $e_1 = \frac{1}{aa_1}$, $e_2=\frac{1}{a}$.

We note that

$$\begin{aligned}\sup_{r\in(e_1 s, e_2 s)} V_1(r) &= \sup_{r\in(e_1 s,e_2 s)} r^{n-1}\int_{\min\{c_n r,1\}}^1 \frac{1}{t^n}\rho_1(t)dt \leqslant (e_2 s)^{n-1}\int_{\min\{c_n e_1 s,1\}}^1 \frac{1}{t^n}\rho_1(t)dt \\ &= \frac{1}{a^{n-1}}\tilde{\omega}_{\rho_1}^{\frac{c_n}{aa_1}}(s) =: V_2(s).\end{aligned}$$

Therefore

$$V_1(|x-y|) \leqslant V_2(|\tilde{\alpha}\circ T^{-1}(x)-\tilde{\alpha}\circ T^{-1}(y)|).$$

Applying Lemmas 4.2 and 4.1 with $\omega = V_2$, we get

$$\begin{aligned}\mathscr{J}\left(\frac{I}{\lambda}\right) &\leqslant \mathscr{J}_3\left(\frac{I}{\lambda}\right) = \mathscr{J}_3\left(\frac{(\frac{I}{\lambda})a}{a}\right) \\ &\overset{Lemma\ 4.2}{=} \frac{1}{a^n}\mathscr{J}_2\left(\frac{I}{\lambda}a\right) \overset{Lemma\ 4.1}{\leqslant} \frac{a_1^{n-2}}{a^n}\mathscr{J}_1\left(\frac{I}{\lambda}a\right).\end{aligned} \tag{4.19}$$

We also have

$$\begin{aligned}\mathscr{I}_3\left(\frac{J}{\lambda}\right) &= \int_{Q'\times\{0\}}\Psi\left(\frac{J}{\lambda}|h(x)|\right)d\sigma(x) \\ &\overset{Lemma\ 4.2}{=} \frac{1}{a^{n-1}}\mathscr{I}_2\left(\frac{J}{\lambda}\right) \overset{Lemma\ 4.1}{\leqslant} \frac{1}{a^{n-1}}\mathscr{I}_1\left(\frac{J}{\lambda}\right).\end{aligned} \tag{4.20}$$

We derive from (4.17),(4.19) and (4.20):

$$\mathscr{L}\left(\frac{1}{\lambda}\right) \leqslant L\mathscr{J}\left(\frac{I}{\lambda}\right)+\frac{c(\rho_1)}{2}\mathscr{I}_3\left(\frac{J}{\lambda}\right) \leqslant L\frac{a_1^{n-2}}{a^n}\mathscr{J}_1\left(\frac{I}{\lambda}a\right)+\frac{c(\rho_1)}{2}\frac{1}{a^{n-1}}\mathscr{I}_1\left(\frac{J}{\lambda}\right),$$

which together with (4.18) gives

$$\frac{1}{a^{n-1}b}\int_{V^+}\Psi\left(\frac{|\nabla\widetilde{g}(x)|}{G(1+a_0)\lambda}\right)\rho(x)dx$$
$$\leqslant L\frac{a_1^{n-2}}{a^n}\int_S\int_S\Psi\left(\frac{Ia}{\lambda}\frac{|g(x)-g(y)|}{|x-y|}\right)\frac{V_2(|x-y|)}{|x-y|^{n-2}}d\sigma(x)d\sigma(y)$$
$$+\frac{c(\rho_1)}{2}\frac{1}{a^{n-1}}\int_S\Psi\left(\frac{J}{\lambda}|g(x)|\right)d\sigma(x).$$

Multiplying both sides of the inequality by $a^{n-1}b$, then substituting $\lambda=\widetilde{\lambda}\,(G(1+a_0))^{-1}$ gives the assertion. The proof of part ii) follows from Lemma 2.7. □

REMARK 4.5. Using estimates (4.17),(4.18),(4.20), we obtain a more precise inequality:

$$\int_{V^+}\Psi\left(\frac{|\nabla\widetilde{g}(x)|}{G(1+a_0)\lambda}\right)\rho(x)dx$$
$$\leqslant La^{n-1}b\int_{x\in Q'\times\{0\}}\int_{y\in Q'\times\{0\}}\Psi\left(\frac{I|ug\circ\tilde{\phi}(x)-g\circ\tilde{\phi}(y)|}{\lambda|x-y|}\right)\frac{V_1(|x-y|)}{|x-y|^{n-2}}d\sigma(x)\,d\sigma(y)$$
$$+\frac{c(\rho_1)b}{2}\int_\Lambda\Psi\left(\frac{J}{\lambda}|g(x)|\right)d\sigma(x),$$

recalling that $\tilde{\phi}(x)=\tilde{\alpha}\circ T^{-1}(x)$, $V_1(s)=\tilde{\omega}_{\rho_1}^{C_n}(s)$.

4.3. Proof of Theorem 3.1

Our first goal is to find estimate (4.9) in case when ρ is of the form

$$\rho(x)=\tau(\mathrm{dist}(x,S))$$

involving function τ. This will be done with help of the following lemma which is the crucial observation for this issue.

LEMMA 4.6. *Let parts (a)-(d) in Assumption B be satisfied. Then we have:*

$$\rho(\tilde{\alpha}\circ T^{-1}(x',t))\leqslant\tau_{e_1,e_2}(t)=\tilde{\tau}(t).$$

Proof. It suffices to prove the inequality:

$$\frac{t}{(1+a_0)b}\leqslant\mathrm{dist}(\tilde{\alpha}\circ T^{-1}(x',t),S)\leqslant\frac{t}{b}. \tag{4.21}$$

We observe that

$$\mathrm{dist}(\tilde{\alpha}\circ T^{-1}(x',t),S)\leqslant\mathrm{dist}(\tilde{\alpha}\circ T^{-1}(x',t),\tilde{\alpha}\circ T^{-1}(x',0))$$
$$=\left|\left(\tilde{\alpha}\left(\frac{x'}{a},\frac{t}{b}\right)\right)-\left(\tilde{\alpha}\left(\frac{x'}{a},0\right)\right)\right|$$
$$=\left|\left(\frac{x'}{a},\alpha\left(\frac{x'}{a}\right)+\frac{t}{b}\right)\right)-\left(\frac{x'}{a},\alpha\left(\frac{x'}{a}\right)\right)\right|=\frac{t}{b}.$$

This gives the second inequality in (4.21). To derive the first inequality we compute that having an arbitrary $\bar{x} = (\bar{x}', \bar{x}_n), \bar{y} = (\bar{y}', \bar{y}_n) \in V^+$, we have

$$\begin{aligned}|\tilde{\alpha}^{-1}(\bar{x}) - \tilde{\alpha}^{-1}(\bar{y})| &= |\tilde{\alpha}^{-1}(\bar{x}', \bar{x}_n) - \tilde{\alpha}^{-1}(\bar{y}', \bar{y}_n)| \\ &= |(\bar{x}', \bar{x}_n - \alpha(\bar{x}')) - (\bar{y}', \bar{y}_n - \alpha(\bar{y}'))| \\ &\leqslant |\bar{x} - \bar{y}| + \|\nabla\alpha\|_\infty |\bar{x} - \bar{y}| = (1 + a_0)|\bar{x} - \bar{y}|.\end{aligned}$$

Consequently

$$|\bar{x} - \bar{y}| \geqslant \frac{1}{1 + a_0} |\tilde{\alpha}^{-1}(\bar{x}) - \tilde{\alpha}^{-1}(\bar{y})|.$$

Applying the above inequality to $\bar{x} = \tilde{\alpha} \circ T^{-1}(x', t)$, $\bar{y} = \tilde{\alpha} \circ T^{-1}(y', 0)$, where $\bar{y}$ is such that $\mathrm{dist}(\bar{x}, S) = \mathrm{dist}(\bar{x}, \bar{y})$, we get

$$\begin{aligned}|\bar{x} - \bar{y}| &= \mathrm{dist}(\tilde{\alpha} \circ T^{-1}(x', t), S) \geqslant \frac{1}{1 + a_0} |T^{-1}(x', t) - T^{-1}(y', 0)| \\ &= \frac{1}{1 + a_0} \left| \left(\frac{x'}{a}, \frac{t}{b} \right) - \left(\frac{y'}{a}, 0 \right) \right| \geqslant \frac{1}{1 + a_0} \frac{t}{b}.\end{aligned}$$

This finishes the proof of the lemma. □

Proof of Theorem 3.1. The proof of part i) follows directly from Lemmas 4.3 and 4.6 with $\rho_1 = \tilde{\tau}$, where $\tilde{\tau}$ is as in Assumption B, ρ_1 is as in Assumption C. To obtain part ii) we use part i) and Proposition 2.7. □

5. Selection of admitted weights

Theorem 3.1 shows that the measure transformation ω can strongly depend on geometric properties of the domain Ω. In this section we show that it is not always the case. In some cases the transformed measure $\tilde{\omega}$ can be compared with function τ which defines measure ρ.

We have the following result.

LEMMA 5.1. *Suppose that* $\tau : (0, \infty) \to \mathbb{R}_+$ *is continuous and monotonic,* $n \geqslant 2$ *and one of the following conditions hold:*

i) τ *is nondecreasing, absolutely continuous on* $(0, 1)$*, satisfies the* Δ_2 *condition on every interval* $(0, K)$ *where* $K < \infty$ *and there exists* $F < n - 1$ *such that*

$$s\tau'(s) \leqslant F \cdot \tau(s) \ a.e. \text{in some neighborhood of } 0; \tag{5.1}$$

ii) τ *is nonincreasing, and* τ *satisfies* $\Delta_{\frac{1}{2}}$*-condition for small arguments, i.e.* $\tau(\frac{1}{2}s) < C\tau(s)$ *on every interval* $(0, K)$ *with some constant* $C = C_K$ *independent of* s.

Moreover, let $\tilde{\tau}$ *and* $\tilde{\omega}$ *be defined by* (3.1) *and* (3.1)*, respectively.*

Then for any numbers e_1, e_2 *such that* $0 < e_1 \leqslant e_2$ *we have*

$$\widetilde{\tau} = \tau_{e_1, e_2} \prec \tau, \text{ and } \omega_{\tilde{\tau}}(s) \prec \tau(s).$$

REMARK 5.2. Conditions $\tau(0)=0$ and $\tau(0)=\infty$ are the only special interesting cases for our issue. Otherwise $\tau \sim Const$ (we use continuity assumption) and so that the cases reduce to the situation without weights.

REMARK 5.3. Suppose that τ is nondecreasing, $\lim_{s\to 0}\tau(s)=0$. Then we have.

a) When τ is convex (τ' is nondecreasing) in some neighborhood of zero, we have $\tau(s)\leqslant s\tau'(s)$ for almost all s, in some neighborhood of 0. When additionally τ satisfies the Δ_2 condition in some neighborhood of 0, i.e. $\tau(2s)\leqslant C\tau(s)$ in some neighborhood of 0, with s independent constant C, then we have $\tau(s)\sim s\tau'(s)$, more precisely $Cs\tau'(s)\geqslant C\tau(s)\geqslant s\tau'(s)$. Inequality (5.1) holds with $F=C$. This is because

$$C\tau(s)\geqslant\tau(2s)=\int_0^{2s}\tau'(t)dt\geqslant\int_s^{2s}\tau'(t)dt\geqslant s\tau'(s),\ \text{for almost every sufficiently small } s.$$

b) When τ is concave (τ' is nonincreasing), then we have $\tau(s)\geqslant s\tau'(s)$ for almost all s, in some neighborhood of 0. Inequality (5.1) holds with $F=1$.

When additionally τ satisfies the condition: $C\tau(s)\leqslant\tau(2s)$ in some neighborhood of 0, where $C>1$ is independent of s, then we have $s\tau'(s)\sim\tau(s)$ in some neighborhood of zero, more precisely $s\tau'(s)\leqslant\tau(s)\leqslant\frac{1}{C-1}s\tau'(s)$. This is because

$$(C-1)\tau(s)\leqslant\tau(2s)-\tau(s)=\int_s^{2s}\tau'(t)dt\leqslant s\tau'(s),\ \text{for almost every sufficiently small } s.$$

Proof of Lemma 5.1.

Proof under condition **i)**. At first we show that $\tilde\tau\prec\tau$. As τ and $\tilde\tau$ are continuous and positive, it suffices to verify the domination for small arguments only. We have

$$\widetilde{\tau}(t)=\sup_{x\in(e_1t,e_2t)}\tau(x)\leqslant\tau(2^Nt),$$

where $N=\lfloor log_2 e_2\rfloor+1$, so that $e_2t\leqslant 2^Nt$. Since τ is nondecreasing for $N\leqslant 0$, we have $2^Nt\leqslant t$, and so $\tau(2^Nt)\leqslant\tau(t)$. For $N>0$, since τ satisfies Δ_2-condition, we get,

$$\tau(2^Nt)\leqslant C^N\tau(t)=C^{\lfloor\log_2 e_2\rfloor+1}\tau(t).$$

Take $\widetilde{C}=\max\{1,C^{\lfloor\log_2 e_2\rfloor+1}\}$, to get:

$$\widetilde{\tau}\leqslant\widetilde{C}\tau.$$

Next we show that $\omega_{\tilde\tau}\prec\tau$ in some neighborhood of zero.

For this, we observe that when $\tilde\tau\leqslant\tilde C\tau$, we have (see (2.3))

$$\widetilde{\omega}_{\tilde\tau}(s)=s^{n-1}\int_{\min\{s,1\}}^1\frac{1}{t^n}\widetilde{\tau}(t)\,dt\leqslant\widetilde{C}s^{n-1}\int_{\min\{s,1\}}^1\frac{1}{t^n}\tau(t)\,dt=:\widetilde{C}\omega_\tau(s).$$

Therefore $\widetilde{\omega}_{\tilde\tau}\prec\omega_\tau$ and it suffices to show that $\omega_\tau\prec\tau$.

We compute that for $0 < s \leqslant 1$

$$\begin{aligned}
\omega_\tau(s) &= s^{n-1}\int_s^1 \frac{1}{t^n}\tau(t)\,dt = s^{n-1}\int_s^1 \left(\frac{t^{-n+1}}{-n+1}\right)'\tau(t)\,dt \\
&= s^{n-1}\left(\frac{1}{-n+1}\right)\left(\tau(1) - s^{-n+1}\tau(s)\right) - s^{n-1}\int_s^1 \left(\frac{t^{-n+1}}{-n+1}\right)\tau'(t)\,dt \\
&\leqslant \frac{1}{n-1}\tau(s) + \frac{1}{n-1}s^{n-1}\int_s^1 t^{-n+1}\tau'(t)\,dt \\
&\leqslant \frac{1}{n-1}\tau(s) + \frac{F}{n-1}s^{n-1}\int_s^1 t^{-n+1}\frac{\tau(t)}{t}\,dt \leqslant \frac{1}{n-1}\tau(s) + \frac{F}{n-1}\omega_\tau(s).
\end{aligned}$$

Therefore $(1-\frac{F}{n-1})\omega_\tau(s) \leqslant \frac{1}{n-1}\tau(s)$ and the statement follows.

Proof under condition **ii)**. At first we show that $\tilde{\tau} \prec \tau$. For this we note that $\widetilde{\tau}(t) = \sup_{x\in(e_1t,e_2t)}\tau(x) \leqslant \tau(2^N t)$, where $N = \lfloor log_2 e_1 \rfloor$. Since τ is non-increasing for $N \geqslant 0$, we have $2^N t \geqslant t$, and so $\tau(2^N t) \leqslant \tau(t)$. For $N < 0$, consider $N = -|N|$ and since τ satisfies $\Delta_{\frac{1}{2}}$-condition, we get, $\tau(2^{-|N|}t) \leqslant C^{|N|}\tau(t) = C^{-\lfloor \log_2 e_1 \rfloor}\tau(t)$. Take $\widetilde{C} = \max\{1, C^{-\lfloor \log_2 e_1 \rfloor}\}$, to get: $\widetilde{\tau} \leqslant \widetilde{C}\tau$. To show that $\omega_{\widetilde{\tau}} \prec \tau$ in some neighborhood of zero, we compute that when $0 < s \leqslant 1$ we have

$$\begin{aligned}
\omega_{\widetilde{\tau}}(s) &= s^{n-1}\int_{\min\{s,1\}}^1 \frac{1}{t^n}\widetilde{\tau}(t)\,dt \leqslant \widetilde{C}s^{n-1}\int_{\min\{s,1\}}^1 \frac{1}{t^n}\tau(t)\,dt \leqslant \widetilde{C}s^{n-1}\tau(s)\int_s^1 \frac{1}{t^n}\,dt \\
&\leqslant \widetilde{C}s^{n-1}\tau(s)\frac{1}{-n+1}(1-s^{-n+1})\,dt \leqslant \widetilde{C}s^{n-1}\tau(s)\frac{1}{n-1}s^{-n+1}\,dt \leqslant \frac{\widetilde{C}}{n-1}\tau(s).
\end{aligned}$$

This ends the proof of the lemma. □

REMARK 5.4. Note that condition **i)** depends on number n interpreted as the dimension, while the condition **ii)** is independent of the dimension.

6. Inequalities on Lipschitz boundary domains. Proof of Theorem 3.3

We are now to prove Theorem 3.3. For this we will use Theorem 3.1 and select the admitted measures using results of Section 5. The proof follows by the following steps introduced below.

Let $u \in Lip(\partial\Omega)$.

Step 1. We establish localization arguments.

We cover $\partial\Omega$ by the open sets in $\partial\Omega$, $\Lambda_1, \cdots, \Lambda_m \subseteq \partial\Omega$ and consider sets U_r^+, U_r^-, U_r, $r = 1, \dots, m$, such that $\{U_r\}_{r=1,\dots,m}$ cover $\partial\Omega$ and

$$U_r = U_r^+ \cup U_r^- \cup \Lambda_r; \quad r = 1, \cdots, m,$$

where $U_r^+ \subseteq \Omega$, $U_r^- \subseteq \mathbb{R}^n \setminus \Omega$ are described in Definition 2.1. Then we take $\phi_1, \cdots, \phi_m$, a Lipschitz partition of unity subordinated to $\{U_r\}_{r=1,\ldots,m}$. In particular, $\phi_r \in Lip_0(\mathbb{R}^n)$, $\mathrm{supp}\phi_r \subset U_r$, $0 \leqslant \phi_r \leqslant 1$ and $\sum_{r=1}^m \phi_r \equiv 1$. We define

$$u_r := \phi_r u.$$

Step 2. We extend u locally, i. e. we extend each u_r separately inside U_r^+.

For this we consider at first function $g_r(x) = u_r(\tilde{A}_r^{-1}(x)) \in Lip(S_r)$, where $S_r = \tilde{A}_r(\Lambda_r)$ is Lipschitz graph. We apply Theorem 3.1 with $g = g_r$, obtaining the extension $\tilde{g}_r$ inside set V_r^+, the subgraph of S_r. Then function $\tilde{u}_r(x) := \tilde{g}_r(\tilde{A}_r(x))$ is extension of u_r inside U_r^+ and satisfies the following inequalities holding for every $\lambda > 0$

$$\int_{U_r^+} \Psi\left(\frac{|\widetilde{u}_r(x)|}{\lambda}\right) \rho(x)\, dx \prec \int_{\Lambda_r} \Psi\left(\frac{|u_r(x)|}{\lambda}\right) d\sigma(x) \prec \int_{\partial\Omega} \Psi\left(\frac{|u(x)|}{\lambda}\right) d\sigma(x); \tag{6.1}$$

$$\int_{U_r^+} \Psi\left(\frac{|\nabla\widetilde{u}_r(x)|}{\lambda}\right) \rho(x) dx \prec \int_{\Lambda_r}\int_{\Lambda_r} \Psi\left(\frac{B_2}{\lambda}\frac{|u_r(x)-u_r(y)|}{|x-y|}\right) \frac{\tau(|x-y|)}{|x-y|^{n-2}} d\sigma(x) d\sigma(y) + \int_{\Lambda_r} \Psi\left(\frac{B_3}{\lambda}|u_r(x)|\right) d\sigma(x), \tag{6.2}$$

with constants B_2, B_3 independent of u and $r = 1, \ldots, m$, where $\rho(x) = \tau(\mathrm{dist}(x, \partial\Omega))\, dx$.

Step 3. We finish the construction. We define final extension in the following way. We take $\kappa \in Lip_0(\mathbb{R}^n)$ such that $\kappa \equiv 1$ in some neighborhood of $\partial\Omega$ and $\kappa \equiv 0$ outside $\bigcup_{r=1}^m U_r$, $0 \leqslant \kappa \leqslant 1$ (use Tietz theorem), then we set

$$\widetilde{u}(x) = \kappa(x) \cdot \left(\sum_{r=1}^m \widetilde{u}_r(x)\right).$$

We are now to prove that $\widetilde{u}$ obeys the required properties of Theorem 3.1 as an extension of u.

For abbreviation let us denote

$$\mathscr{I}(s) := \int_{\partial\Omega} \Psi(s|u(x)|)\, d\sigma(x),$$

$$\mathscr{J}(s) := \int_{\partial\Omega}\int_{\partial\Omega} \Psi\left(s\frac{|u(x)-u(y)|}{|x-y|}\right) \frac{\tau(|x-y|)}{|x-y|^{n-2}} d\sigma(x) d\sigma(y),$$

$$\mathscr{L}(s) := \int_{\Omega} \Psi(s|\nabla\widetilde{u}(x)|)\, \rho(x) dx.$$

Inequalities $\Psi(\sum_{i=1}^k a_i) \leqslant \frac{1}{k}\sum_{i=1}^k \Psi(k a_i)$ will be used frequently with possibly different k. They imply

$$\int_{\Omega} \Psi\left(\frac{|\widetilde{u}(x)|}{\lambda}\right) \rho(x)\, dx \prec \sum_{r=1}^m \int_{U_r^+} \Psi\left(\frac{m|\widetilde{u}_r(x)|}{\lambda}\right) \rho(x)\, dx \overset{(6.1)}{\prec} \int_{\partial\Omega} \Psi\left(\frac{m|u(x)|}{\lambda}\right) d\sigma(x) = \mathscr{I}\left(\frac{m}{\lambda}\right),$$

which gives (3.3). We also have

$$\mathscr{L}\left(\frac{1}{\lambda}\right) \prec \sum_{r=1}^{m}\int_{U_r^+}\Psi\left(\frac{m|\nabla(\kappa(x)\cdot\widetilde{u}_r(x))|}{\lambda}\right)\rho(x)dx \leqslant \sum_{r=1}^{m}(\mathscr{A}_r+\mathscr{B}_r),\ \text{where}$$

$$\mathscr{A}_r := \int_{U_r^+}\Psi\left(\frac{2m\|\nabla\kappa\|_\infty|\widetilde{u}_r(x)|}{\lambda}\right)\rho(x)dx \overset{(6.1)}{\prec} \int_{\partial\Omega}\Psi\left(\frac{2m\|\nabla\kappa\|_\infty|u(x)|}{\lambda}\right)d\sigma(x)$$

$$= \mathscr{I}\left(\frac{c_1}{\lambda}\right),\ \text{where } c_1 = 2m\|\nabla\kappa\|_\infty,$$

$$\mathscr{B}_r := \int_{U_r^+}\Psi\left(\frac{2m|\kappa||\nabla\widetilde{u}_r(x)|}{\lambda}\right)\rho(x)dx$$

$$\overset{(6.2)}{\prec} \int_{\Lambda_r}\int_{\Lambda_r}\Psi\left(\frac{2mB_2}{\lambda}\frac{|u_r(x)-u_r(y)|}{|x-y|}\right)\frac{\tau(|x-y|)}{|x-y|^{n-2}}d\sigma(x)d\sigma(y)$$

$$+\int_{\Lambda_r}\Psi\left(\frac{2mB_3}{\lambda}|u_r(x)|\right)d\sigma(x) =: \mathscr{C}_r+\mathscr{D}_r.$$

Clearly,

$$\mathscr{D}_r \prec \int_{\partial\Omega}\Psi\left(\frac{2mB_3}{\lambda}|u(x)|\right)d\sigma(x) = \mathscr{I}\left(\frac{c_2}{\lambda}\right),\ \text{where } c_2 = 2mB_3.$$

To estimate $\mathscr{C}_r$, we note that when $x,y\in\Lambda_r$ we have

$$\frac{|u_r(x)-u_r(y)|}{|x-y|} = \frac{|u(x)\phi_r(x)-u(y)\phi_r(y)|}{|x-y|}$$
$$\leqslant |u(x)|\frac{|\phi_r(x)-\phi_r(y)|}{|x-y|}+|\phi_r(y)|\frac{|u(x)-u(y)|}{|x-y|}$$
$$\leqslant |u(x)|\cdot c_3+\frac{|u(x)-u(y)|}{|x-y|},$$

where $c_3 = \max\{\|\nabla\phi_1\|_\infty,\|\nabla\phi_2\|_\infty,\cdots,\|\nabla\phi_m\|_\infty\}$. Therefore $\mathscr{C}_r \prec \mathscr{E}_r+\mathscr{F}_r$, where

$$\mathscr{E}_r := \int_{\Lambda_r}\int_{\Lambda_r}\Psi\left(\frac{c_4|u(x)|}{\lambda}\right)\frac{\tau(|x-y|)}{|x-y|^{n-2}}d\sigma(x)d\sigma(y),\ \text{where } c_4 = 4mB_2c_3,$$

$$\mathscr{F}_r := \int_{\Lambda_r}\int_{\Lambda_r}\Psi\left(\frac{c_5}{\lambda}\frac{|u(x)-u(y)|}{|x-y|}\right)\frac{\tau(|x-y|)}{|x-y|^{n-2}}d\sigma(x)d\sigma(y)$$

$$\prec \int_{\partial\Omega}\int_{\partial\Omega}\Psi\left(\frac{c_5}{\lambda}\frac{|u(x)-u(y)|}{|x-y|}\right)\frac{\tau(|x-y|)}{|x-y|^{n-2}}d\sigma(x)d\sigma(y) = \mathscr{J}\left(\frac{c_5}{\lambda}\right),$$

where $c_5 = 4mB_2$. For further estimation of $\mathscr{E}_r$ we show that

$$\sup_{r=1,\dots,m}\sup_{x\in\Lambda_r}\int_{\Lambda_r}\frac{\tau(|x-y|)}{|x-y|^{n-2}}d\sigma(y) \leqslant Const. \tag{6.3}$$

This gives

$$\mathscr{E}_r \prec \int_{\partial\Omega}\Psi\left(\frac{c_4|u(x)|}{\lambda}\right)d\sigma(x) = \mathscr{I}\left(\frac{c_4}{\lambda}\right).$$

Combining all the estimates we get

$$\begin{aligned}
\mathscr{L}\left(\frac{1}{\lambda}\right) &\prec \sum_{r=1}^{m}(\mathscr{A}_r+\mathscr{B}_r) \prec \mathscr{I}\left(\frac{c_1}{\lambda}\right)+\sum_{r=1}^{m}(\mathscr{C}_r+\mathscr{D}_r)\\
&\prec \mathscr{I}\left(\frac{c_1}{\lambda}\right)+\mathscr{I}\left(\frac{c_2}{\lambda}\right)+\sum_{r=1}^{m}(\mathscr{E}_r+\mathscr{F}_r)\\
&\prec \mathscr{I}\left(\frac{c_1}{\lambda}\right)+\mathscr{I}\left(\frac{c_2}{\lambda}\right)+\mathscr{I}\left(\frac{c_4}{\lambda}\right)+\mathscr{J}\left(\frac{c_5}{\lambda}\right) \prec \mathscr{I}\left(\frac{D}{\lambda}\right)+\mathscr{J}\left(\frac{B}{\lambda}\right),
\end{aligned}$$

where $D := \max\{c_1,c_2,c_4\}$, $B = c_5$. This would imply (3.4), wile to obtain norm inequalities (3.2) and (3.6) we have to apply Lemma 2.7. The linearity of the mapping follows from the construction.

We are left with the proof of (6.3). For this, we note that $\Lambda_r = \tilde{A}_r^{-1}(S_r)$, where $S_r = \{(x',\alpha_r(x')), x' \in aQ'\}$ is the graph of Lipschitz function (Definition 2.1). Therefore

$$G_r(x) := \int_{\Lambda_r}\frac{\tau(|x-y|)}{|x-y|^{n-2}}d\sigma(y) = \int_{S_r}\frac{\tau(|x-y|)}{|x-y|^{n-2}}d\sigma(y)$$

as $\tilde{A}_r$ changes neither Hausdorff measure nor distances. Using the bi-Lipschitz mapping $l_r(x) := \tilde{\alpha}_r \circ T^{-1} \circ i$ where $\tilde{\alpha}_r$ is as Assumption B instead of $\tilde{\alpha}$, $i : Q' \mapsto Q' \times \{0\}$ is an embedding $x' \mapsto (x',0)$, we recognize that

$$G_r(x) = \int_{l_r(Q')}\frac{\tau(|x-y|)}{|x-y|^{n-2}}d\sigma(y) = \int_{Q'}\frac{\tau(|l_r(x')-l_r(y')|)}{|l_r(x')-l_r(y')|^{n-2}}|Dl_r(y')|d(y').$$

Because of bi-Lipschitzity of l_r we have $|Dl_r(y')| \prec 1$ and there exist constants $e_1,e_2 > 0$ such that $e_1|x'-y'| \leqslant |l_r(x')-l_r(y')| \leqslant e_2|x'-y'|$. Therefore

$$\tau(|l_r(x')-l_r(y')|) \prec \tau_{e_1,e_2}(|x'-y'|) \overset{\textit{Lemma } 5.1}{\prec} \tau(|x'-y'|)$$

and

$$\frac{\tau(|l_r(x')-l_r(y')|)}{|l_r(x')-l_r(y')|^{n-2}}|Dl_r(y')| \prec \frac{\tau(|x'-y'|)}{|x'-y'|^{n-2}}.$$

This together with change of variables $z = x'-y'$ and using polar coordinates implies

$$G_r(x) \prec \int_{z\in x+Q'}\frac{\tau(|z|)}{|z|^{n-2}}dz \prec \int_{B(0,2\sqrt{n-1})}\frac{\tau(|z|)}{|z|^{n-2}}dz = \theta_{n-2}\int_0^{2\sqrt{n-1}}\tau(r)dr < \infty.$$

This shows (6.3) and completes the arguments. □

7. Appendix. Proofs of Lemmas 4.1 and 4.2

Proof of the Lemma 4.1. Let $\beta : aQ' \mapsto S$, $\beta(x') = (x',\alpha(x'))$. As β forms the map on S, we have

$$\int_S \Psi(s|g(x)|)\,d\sigma(x) = \int_{aQ'}\Psi\left(s|g(\beta(x'))|\right)\|D\beta(x')\|\,dx' \geqslant \int_{aQ'}\Psi\left(s|g(\beta(x'))|\right)\,dx'.$$

Last inequality holds because

$$D\beta(x') = \begin{bmatrix} 1 & 0 & \dots & 0 & \frac{\partial\alpha(x')}{\partial x_1} \\ 0 & 1 & \dots & 0 & \frac{\partial\alpha(x')}{\partial x_2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & \frac{\partial\alpha(x')}{\partial x_{n-1}} \end{bmatrix},$$

so that $\|D\beta(x')\| = \sqrt{1+|\nabla\alpha(x')|^2} \geqslant 1$. Note that

$$\int_{aQ'} \Psi\left(s|g(\beta(x'))|\right) dx' = \int_{aQ'\times\{0\}} \Psi(s|f(x)|)\, d\sigma(x).$$

This gives (4.4).

To obtain inequality (4.5), we use similar arguments dealing with product measure:

$$\begin{aligned} \mathscr{J}_2(s) &= \int_S\int_S \Psi\left(s\frac{|g(x)-g(y)|}{|x-y|}\right)\frac{\omega(|x-y|)}{|x-y|^{n-2}}d\sigma(x)d\sigma(y) \\ &\geqslant \int_{aQ'}\int_{aQ'} \Psi\left(s\frac{|g(\beta(x'))-g(\beta(y'))|}{|\beta(x')-\beta(y')|}\right)\frac{\omega(|\beta(x')-\beta(y')|)}{|\beta(x')-\beta(y')|^{n-2}}dx'dy'. \end{aligned}$$

As

$$|x'-y'| \leqslant |\beta(x')-\beta(y')| \leqslant \sqrt{1+\|\nabla\alpha\|_\infty^2}\cdot|x'-y'| = a_1|x'-y'|,$$

and $\beta(x') = \tilde{\alpha}(x',0)$, we get

$$\frac{\omega(|\beta(x')-\beta(y')|)}{|\beta(x')-\beta(y')|^{n-2}} \geqslant \frac{1}{a_1^{n-2}}\cdot\frac{\omega(|\beta(x')-\beta(y')|)}{|x'-y'|^{n-2}} = \frac{\omega(|\tilde{\alpha}(x',0)-\tilde{\alpha}(y',0)|)}{a_1^{n-2}\cdot|x'-y'|^{n-2}}.$$

From there inequality (4.5) follows.

To obtain (4.6) we compute that:

$$\begin{aligned} \mathscr{K}_1(s) = \int_{V^+} \Psi(s|\widetilde{g}(y)|)\rho(y)dy &= \int_{\widetilde{\alpha}(Q_{a,b})} \Psi\left(s|\widetilde{f}(\widetilde{\alpha}^{-1}(y))|\right)\rho(y)dy \\ &= \int_{Q_{a,b}} \Psi\left(s|\widetilde{f}(x)|\right)|\mathrm{det}D\widetilde{\alpha}(x)|\rho((\widetilde{\alpha}(x)))dx \end{aligned}$$

and it suffices to note that $|\mathrm{det}D\widetilde{\alpha}(x)| = 1$. Inequality (4.7) holds by almost the same arguments as above, but in place of $|\widetilde{g}(y)|$ we deal with

$$|\nabla\widetilde{g}(y)| = |\nabla\widetilde{f}(\widetilde{\alpha}^{-1}(y))\cdot\nabla\widetilde{\alpha}^{-1}(y))| \leqslant |\nabla\widetilde{f}(\widetilde{\alpha}^{-1}(y)|\cdot(1+a_0). \tag{7.1}$$

To justify (7.1) we observe that

$$\left(\nabla\widetilde{\alpha}^{-1}\right)(y',y_n) = \begin{bmatrix} 1 & 0 & \dots & 0 & -\frac{\partial\alpha(y')}{\partial y_1} \\ 0 & 1 & \dots & 0 & -\frac{\partial\alpha(y')}{\partial y_2} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & -\frac{\partial\alpha(y')}{\partial y_{n-1}} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} =: \begin{bmatrix} 1 & 0 & \dots & 0 & a_1 \\ 0 & 1 & \dots & 0 & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & a_{n-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = B.$$

Therefore

$$\begin{pmatrix} x_1 \\ \dots \\ x_{n-1} \\ x_n \end{pmatrix} B = x_1 \begin{pmatrix} 1 \\ 0 \\ \dots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \dots \\ 0 \end{pmatrix} + \dots + x_{n-1} \begin{pmatrix} 0 \\ \dots \\ 1 \\ 0 \end{pmatrix} + x_n \begin{pmatrix} a_1 \\ \dots \\ a_{n-1} \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + x_n a_1 \\ \dots \\ x_{n-1} + x_n a_{n-1} \\ x_n \end{pmatrix} = Y.$$

Consequently, for $a = (a_1, \dots, a_{n-1})$ and $x' = (x_1, \dots, x_{n-1})$ we have

$$|Y|^2 = \sum_{i=1}^{n-1} (x_i^2 + x_n^2 a_i^2 + 2x_n a_i x_i) + x_n^2 \leqslant |x'|^2 + (|x_n||a|)^2 + 2|x_n||a \cdot x'| + x_n^2$$

$$\leqslant \left(|x'|^2 + (|x_n||a|)^2 + 2|x_n||a||x'| \right) + x_n^2 \leqslant |x|^2 (1 + |a|)^2.$$

This ends the proof of the lemma. □

Proof of the Lemma 4.2 (sketched). We only explain the computation of (4.8). As $Q_{a,b} = T^{-1}(Q)$ and $|DT^{-1}| = a^{n-1}b$, we have

$$\mathscr{L}_2(s) = \int_{Q_{a,b}} \Psi\left(s|\nabla \tilde{f}(x)|\right) \rho(\tilde{\alpha}(x))dx = a^{n-1}b \int_Q \Psi\left(s|(\nabla \tilde{f})(T^{-1}y)|\right) \rho(\tilde{\alpha}(T^{-1}y))dy$$

and as $\tilde{f}(x) = \tilde{h}(T(x))$, we have

$$\nabla \tilde{f}(z) = \nabla \tilde{h}(T(z)) DT(z) = \nabla \tilde{h}(T(z)) \cdot \begin{bmatrix} a^{-1} & 0 & \dots & 0 \\ 0 & a^{-1} & \dots & 0 \\ 0 & \dots & a^{-1} & 0 \\ 0 & \dots & 0 & b^{-1} \end{bmatrix} =: Y.$$

When $\nabla \tilde{h}(T(z)) = \begin{pmatrix} \alpha_1 \\ \dots \\ \alpha_n \end{pmatrix} = w$, we have $Y =$

$$\begin{pmatrix} \alpha_1 \\ \dots \\ \alpha_n \end{pmatrix} \cdot \begin{bmatrix} a^{-1} & 0 & \dots & 0 \\ 0 & a^{-1} & \dots & 0 \\ 0 & \dots & a^{-1} & 0 \\ 0 & \dots & 0 & b^{-1} \end{bmatrix} = \alpha_1 \begin{pmatrix} a^{-1} \\ 0 \\ \dots \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ a^{-1} \\ \dots \\ 0 \end{pmatrix} + \dots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \dots \\ b^{-1} \end{pmatrix} = \begin{pmatrix} \alpha_1 a^{-1} \\ \dots \\ \alpha_{n-1} a^{-1} \\ \alpha_n b^{-1} \end{pmatrix}.$$

Therefore $|Y| \leqslant \max\{\frac{1}{a}, \frac{1}{b}\}|w|$ and so $|\nabla \tilde{f}(T^{-1}(y))| \leqslant G|\nabla \tilde{h}(T(T^{-1}(y)))| = G|\nabla \tilde{h}(y)|$. Consequently $\mathscr{L}_2(s) \leqslant a^{n-1} b \mathscr{L}_3(sG)$ and (4.8) follows. □

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