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A PROOF OF THE GREEN–OSHER INEQUALITY

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Abstract. In this paper we give a different proof of the Green-Osher inequality and show that equality holds if and only if γ is a circle when $F(x)$ is a strictly convex function on $(0,+\infty)$.

1. Introduction

The classical isoperimetric inequality states that for a simple closed curve γ in the Euclidean plane $\mathbb{R}^2$ of length L and enclosing a region of area A, one gets

$$L^2 - 4\pi A \geqslant 0,$$

and with equality if and only if γ is a circle. This fact was known to the ancient Greeks, a mathematical proof was only given, however, in the 19th century by Steiner [22] and Edler [4]. Since then there have been many proofs, sharpened forms, generalizations, and applications of it, see e.g., Bonnesen-Fenchel [2], Dai-Xu-Zhou [3], Lawlor [12], Lax [13], Martini-Mustafaev [14], Mitrinović-Pečarić-Volenec [15], Ou-Pan [16], Pan-Sun-Wang [17], etc., and the literature therein. Usually, variants of the classical isoperimetric inequality do not involve integrals of curvature of the plane curve. In the 1980's, however, Gage [5] has shown an "isoperimetric inequality" which involves the integration of the squared curvature of the curve, that is,

$$\int_\gamma \kappa^2 \mathrm{d}s \geqslant \frac{\pi L}{A}, \tag{1}$$

where κ is the curvature of γ. Inequality (1) plays an essential role in the study of curve evolution problems in the plane (see Gage [6], Gage [7], Gage-Hamilton [9], Jiang-Pan [11], etc.).

In [10], Green and Osher derived the *Green-Osher inequality* via an elegant symmetric technique. Inspired by the work of Gage [8] and Peri-Wills-Zucco [20], we present a new proof of the Green-Osher inequality through the minimal annulus of the curve γ and conclude that equality holds if and only if γ is a circle. That is, we obtain the following theorem:

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THEOREM 1. *Let γ be a closed, strictly convex C^2 curve with radius of curvature $\rho(\theta)$. If $F(x)$ is a strictly convex function on $(0,+\infty)$, then*

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))\mathrm{d}\theta \geqslant \frac{1}{2}(F(-t_1)+F(-t_2)), \tag{2}$$

where t_1 and t_2 are the two roots of the Steiner polynomial of γ. Equality in (2) holds if and only if γ is a circle.

This paper is organized as follows. In Section 2 some preliminary lemmata and propositions are obtained. Moreover, the proof of the Theorem 1 is concluded in Section 3.

2. Some basic lemmata and propositions

Let γ be a closed, strictly convex C^2 curve with support function $p(\theta)$, K the domain enclosed by γ and B the unit disk. The *length* of γ is

$$L = \int_{S^1} p(\theta)+p''(\theta)\mathrm{d}\theta = \int_{S^1} p(\theta)\mathrm{d}\theta.$$

The *curvature* of γ is given by

$$\kappa(\theta) = \frac{1}{p(\theta)+p''(\theta)}$$

and the *radius of curvature* of γ is

$$\rho(\theta) = \frac{1}{\kappa(\theta)} = p(\theta)+p''(\theta).$$

The *Minkowski sum* of two convex domains K_1 and K_2 is expressed by

$$K_1+K_2 = \{x_1+x_2 \mid x_1 \in K_1, x_2 \in K_2\}.$$

Denote by A the area of K. The *outer parallel body* of a convex domain K is the Minkowski sum of K and a ball tB with radius t. If $A(t)$ denotes its area, then

$$A(t) = A+Lt+\pi t^2, \quad t \geqslant 0,$$

which is the well known *Steiner polynomial* of γ. According to the classical isoperimetric inequality, the equation $A(t)=0$ has two negative real roots. Denote by $t_1 \geqslant t_2$ the two zeros of the Steiner polynomial of γ, that is,

$$t_1 = -\frac{L}{2\pi}+\frac{u}{2\pi}, \qquad t_2 = -\frac{L}{2\pi}-\frac{u}{2\pi},$$

where $u=\sqrt{L^2-4\pi A}\geqslant 0$.

Let

$$r_{in} = \max\{r>0 \mid x+rB \subseteq K, \exists x \in K\}$$

and

$$r_{out} = \min\{r > 0 \mid x + rB \supseteq K, \exists x \in K\}$$

be the *inradius* and *circumradius* of γ, respectively. For $x \in K$, set

$$r_{in}(x) = \max\{r \geqslant 0 \mid x + rB \subseteq K\}$$

and

$$r_{out}(x) = \min\{r > 0 \mid x + rB \supseteq K\}.$$

The annulus of center x is defined by

$$A_x = \left\{y \in \mathbb{R}^2 \mid y \in x + r_{out}(x)B \text{ and } y \notin \operatorname{int}(x + r_{in}(x)B)\right\}.$$

When A_x contains γ and $r_{out}(x) - r_{in}(x)$ attains its minimum, the corresponding annulus is called the *minimal annulus* of γ. It is well known that there exists a unique minimal annulus of a strictly convex curve γ (cf. [1, 19, 20]). The center of the minimal annulus is denoted by x_0. Denote by ρ_{max} and ρ_{min} the maximum and minimum values of $\rho(\theta)$, respectively. If γ is not a circle, then

$$-\rho_{max} < t_2 < -r_{out} < -\frac{L}{2\pi} < -r_{in} < t_1 < -\rho_{min}. \tag{3}$$

The proof of (3) can be found in [10], and it is obvious that the above quantities are all equal when γ is a circle.

DEFINITION 1. ([10]) Consider

$$\sup\left\{\int_I \rho(\theta)\mathrm{d}\theta \mid I \subset S^1, \int_I \mathrm{d}\theta = \pi\right\}.$$

Let I_1 denote the smallest subset of S^1 with measure π and realizing the above supremum, and let I_2 be its complement. There exists an $a \in \mathbb{R}^+$ such that

$$I_1 \subseteq \{\theta \mid \rho(\theta) \geqslant a\}, \quad I_2 \subseteq \{\theta \mid \rho(\theta) \leqslant a\}.$$

Set

$$\rho_1 = \frac{1}{\pi}\int_{I_1} \rho(\theta)\mathrm{d}\theta, \quad \rho_2 = \frac{1}{\pi}\int_{I_2} \rho(\theta)\mathrm{d}\theta,$$

yielding

$$\rho_1 + \rho_2 = \frac{L}{\pi}, \quad \rho_1 \geqslant \rho_2.$$

Hence, there is a $b \geqslant 0$ such that

$$\rho_1 = \frac{L}{2\pi} + b, \quad \rho_2 = \frac{L}{2\pi} - b. \tag{4}$$

To obtain a stronger version of the Bonnesen inequality in Proposition 1, we need two significant lemmata. The first one is from Santaló [21, p.119] (see also [23]), the second one is a direct consequence of Gage [8, Pro.1.6 and Thm.1.8] (see also [20, Lemmata 1 and 2]). Define $E(s,k)$ as the area of $\{x \mid \#\{C(x,s) \bigcap \gamma\} = k, k \in \mathbb{N}\}$, where $C(x,s)$ is a circle with center x and radius s. Notice that if k is odd, then the circle must be tangent to γ at a point and $E(s,k) = 0$.

LEMMA 1. ([21]) *If γ is a simple closed planar curve with perimeter L and enclosed area A, then*

$$4Ls = 2E(s,2) + 4E(s,4) + 6E(s,6) + \cdots \tag{5}$$

and if $s \in [r_{in}, r_{out}]$, then

$$A + Ls + \pi s^2 = E(s,2) + E(s,4) + E(s,6) + \cdots. \tag{6}$$

LEMMA 2. ([8]) *Let γ be a convex planar curve and x_0 the center of its minimal annulus. If $s,t \in \gamma \cap C(x_0, r_{in}(x_0))$ and $S,T \in \gamma \cap C(x_0, r_{out}(x_0))$ and the line segments st and ST satisfy $st \cap ST \neq \emptyset$, then there is a line l with the following properties:*

(i) $l \cap K$ is a line segment with x_0 as its midpoint, where K is the domain enclosed by γ;

(ii) the points s and t lie on different sides of l, and so do S and T.

PROPOSITION 1. *If γ is a strictly convex and non-circular planar curve, x_0 is the center of the minimal annulus of γ, then*

$$A - Ls + \pi s^2 < 0, \quad s \in [r_{in}(x_0), r_{out}(x_0)].$$

Proof. From (5) and (6), we get

$$Ls - A - \pi s^2 \geqslant E(s,4), \quad s \in [r_{in}, r_{out}].$$

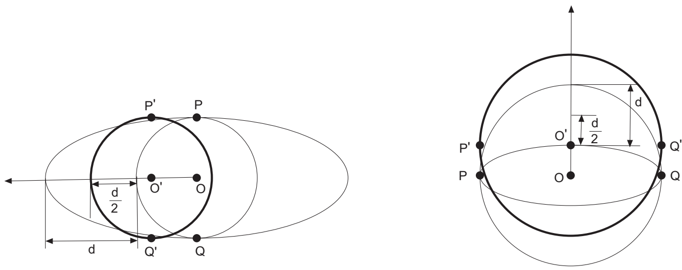
For a non-circular curve γ, we have to prove that $E(r_{in},4) > 0$ and $E(r_{out},4) > 0$, that is, $A - Lr_{in} + \pi r_{in}^2 < 0$ and $A - Lr_{out} + \pi r_{out}^2 < 0$.

Since γ is strictly convex, the largest inscribed circle is unique (denote it by C), and it is clear that there are at least two tangent points of γ and C, say P and Q, so that $\angle POQ \leqslant \pi$ (cf. [10, Lemma 1.11]). In the direction of the bisector of $\angle POQ$, if there is a distance d between γ and C, then C can be moved a distance, say $d/2$, in the direction as shown in Figure 1a so that γ intersects C at 4 points, and by continuity, $E(r_{in},4) > 0$. If there is always a tangent point in the direction of all angle bisectors, thus $d \equiv 0$, then γ must be a circle, which shows that we can use the above progress when γ is non-circular. Similarly, we can get $E(r_{out},4) > 0$ (see Figure 1b).

If γ is strictly convex and centrally symmetric, then $r_{in} = r_{in}(x_0)$ and $r_{out} = r_{out}(x_0)$. Thus, $A - Lr_{in}(x_0) + \pi r_{in}^2(x_0) = A - Lr_{in} + \pi r_{in}^2 < 0$ and $A - Lr_{out}(x_0) + \pi r_{out}^2(x_0) = A - Lr_{out} + \pi r_{out}^2 < 0$.

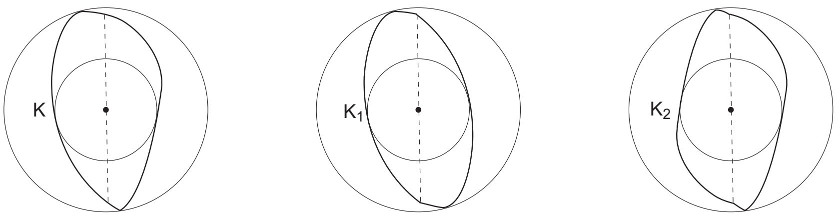
If γ is strictly convex but not centrally symmetric, we need some symmetrization procedures as in Figure 2.

According to Lemma 2, we can cut the domain enclosed by γ into two parts by a chord with midpoint x_0 as in Figure 2a. Let L_i and A_i denote the arc length and the area of each part, respectively. By applying a symmetry with respect to x_0 to the two parts, we get two centrally symmetric domains, K_1 and K_2, as in Figure 2b and Figure 2c. Meanwhile, $r_{in}(x_0)$ and $r_{out}(x_0)$ are equal in the three figures.



(a) r_{in} (b) r_{out}

Figure 1: *Strictly convex and non-circular*



(a) Cut (b) Left part (c) Right part

Figure 2: *Symmetrization*

Since K_1 is convex, we get

$$2A_1 - 2L_1 r_{in}(x_0) + \pi r_{in}^2(x_0) < 0. \tag{7}$$

For a non-convex domain K_2, denote the convex hull of K_2 in Figure 2c by K_2'. The set K_2' has perimeter $\tilde{L}_2 \leqslant 2L_2$ and area $\tilde{A}_2 \geqslant 2A_2$; so

$$2A_2 - 2L_2 r_{in}(x_0) + \pi r_{in}^2(x_0) \leqslant \tilde{A}_2 - \tilde{L}_2 r_{in}(x_0) + \pi r_{in}^2(x_0) < 0. \tag{8}$$

By (7) and (8), we have

$$A - L r_{in}(x_0) + \pi r_{in}(x_0)^2 < 0.$$

By a similar argument we get

$$A - L r_{out}(x_0) + \pi r_{out}(x_0)^2 < 0.$$

Thus, for a strictly convex and non-circular planar curve γ, we have $A - L r_{in}(x_0) + \pi r_{in}(x_0)^2 < 0$ and $A - L r_{out}(x_0) + \pi r_{out}(x_0)^2 < 0$. Furthermore, from a property of quadratic function, it follows that $A - Ls + \pi s^2 < 0$, $r_{in}(x_0) \leqslant s \leqslant r_{out}(x_0)$. $\square$

LEMMA 3. *If γ is a closed, strictly convex and non-circular curve in the plane, and x_0 is the center of the minimal annulus of γ, then*

$$-t_1 < r_{in}(x_0) < \frac{L}{2\pi} < r_{out}(x_0) < -t_2. \tag{9}$$

Proof. If γ is a strictly convex and non-circular curve, then by Proposition 1, it follows that

$$A - Ls + \pi s^2 < 0, \quad r_{in}(x_0) \leqslant s \leqslant r_{out}(x_0).$$

Thus, $-t_1 < r_{in}(x_0)$ and $r_{out}(x_0) < -t_2$. From $r_{in}(x_0)B \subset K$ and $K \subset r_{out}(x_0)B$ (the inclusion is strict, since γ is a non-circular curve), it follows that $r_{in}(x_0) < \frac{L}{2\pi}$ and $r_{out}(x_0) > \frac{L}{2\pi}$. □

COROLLARY 1. *If γ is a closed, strictly convex and non-circular C^2 curve in the plane, and x_0 is the center of the minimal annulus of γ, then*

$$\rho_{\min} < -t_1 < r_{in}(x_0) \leqslant r_{in} < \frac{L}{2\pi} < r_{out} \leqslant r_{out}(x_0) < -t_2 < \rho_{\max}. \tag{10}$$

Proof. From the definition of r_{in} and r_{out}, it is obvious that $r_{in}(x_0) \leqslant r_{in}$ and $r_{out} \leqslant r_{out}(x_0)$. Then (10) follows from (3) and (9). Moreover, if γ is centrally symmetric with respect to x_0, then $r_{in} = r_{in}(x_0)$ and $r_{out} = r_{out}(x_0)$. Hence the equalities in (10) cannot be sharpened. □

LEMMA 4. ([18]) *If γ is a closed, strictly convex and non-circular curve in the plane, then*

$$\rho_1 > \rho_2.$$

Although the next proposition has appeared in [10] and [18], in this paper, we give a simplified proof of it through the above argument.

PROPOSITION 2. *If γ is a closed, strictly convex and C^2 curve, then*

$$\rho_1 \geqslant -t_2. \tag{11}$$

Moreover, if γ is not a circle, then

$$\rho_1 > -t_2. \tag{12}$$

Proof. Let x_0 be the center of the minimal annulus of γ. From the inequalities (cf. [1])

$$A - Ls + \pi s^2 \leqslant 0, \quad r_{in}(x_0) \leqslant s \leqslant r_{out}(x_0),$$

and the classical isoperimetic inequality $L^2 - 4\pi A \geqslant 0$, it follows that

$$-t_1 \leqslant r_{in}(x_0) \leqslant r_{out}(x_0) \leqslant -t_2.$$

Choose x_0 as the origin; then $r_{in}(x_0) \leqslant p(\theta) \leqslant r_{out}(x_0)$, which implies

$$-\frac{u}{2\pi} \leqslant p(\theta) - \frac{L}{2\pi} \leqslant \frac{u}{2\pi}, \quad u = \sqrt{L^2 - 4\pi A} \geqslant 0.$$

On I_1, $\rho(\theta) - a \geqslant 0$, combine with the above inequality,

$$-\left(p(\theta) - \frac{L}{2\pi}\right)(\rho(\theta) - a) \leqslant \frac{u}{2\pi}(\rho(\theta) - a).$$

Integrating this on the interval I_1, it shows that

$$-\frac{1}{\pi}\int_{I_1}\left(p(\theta) - \frac{L}{2\pi}\right)(\rho(\theta) - a)\mathrm{d}\theta \leqslant \frac{u}{2\pi}(\rho_1 - a). \tag{13}$$

On I_2, $\rho(\theta) - a \leqslant 0$, we deduce

$$-\left(p(\theta) - \frac{L}{2\pi}\right)(\rho(\theta) - a) \leqslant -\frac{u}{2\pi}(\rho(\theta) - a).$$

Integrating this on the interval I_2, it shows that

$$-\frac{1}{\pi}\int_{I_2}\left(p(\theta) - \frac{L}{2\pi}\right)(\rho(\theta) - a)\mathrm{d}\theta \leqslant -\frac{u}{2\pi}(\rho_2 - a). \tag{14}$$

Adding (13) and (14) we get

$$-\frac{1}{\pi}\int_{S^1}\left(p(\theta) - \frac{L}{2\pi}\right)(\rho(\theta) - a)\mathrm{d}\theta \leqslant \frac{u}{2\pi}(\rho_1 - \rho_2) = \frac{bu}{\pi}.$$

The left-hand side can be simplified to

$$\frac{L^2 - 4\pi A}{2\pi^2} = \frac{u^2}{2\pi^2},$$

thus $b \geqslant \frac{u}{2\pi}$, which implies $\rho_1 \geqslant -t_2$.

If γ is not a circle, by Lemma 3

$$-t_1 < r_{in}(x_0) < r_{out}(x_0) < -t_2,$$

and from the fact that $r_{in}(x_0) \leqslant p(\theta) \leqslant r_{out}(x_0)$, it follows that

$$-\frac{u}{2\pi} < p(\theta) - \frac{L}{2\pi} < \frac{u}{2\pi}, \quad u = \sqrt{L^2 - 4\pi A} > 0.$$

For I_1 and I_2, $\rho(\theta) \equiv a$ holds on at most one interval, unless γ is a circle. Assume that $\rho(\theta) > a$ on a interval I_1' of I_1. Thus, on this subinterval I_1', $\rho(\theta) - a > 0$,

$$-\left(p(\theta) - \frac{L}{2\pi}\right)(\rho(\theta) - a) < \frac{u}{2\pi}(\rho(\theta) - a).$$

Integrating this on the interval I_1, it shows that

$$-\frac{1}{\pi}\int_{I_1}\left(p(\theta)-\frac{L}{2\pi}\right)(\rho(\theta)-a)\mathrm{d}\theta<\frac{u}{2\pi}(\rho_1-a).$$

Together with (14), it yields

$$-\frac{1}{\pi}\int_{S^1}\left(p(\theta)-\frac{L}{2\pi}\right)(\rho(\theta)-a)\mathrm{d}\theta<\frac{u}{2\pi}(\rho_1-\rho_2)=\frac{bu}{\pi}.$$

With a similar argument, $b>\frac{u}{2\pi}>0$, which implies $\rho_1>-t_2$. □

3. The proof of Theorem 1

To make this paper complete, we restate the proof which appeared in [10]. Compared with the work of Green and Osher, we mainly deal with the equality case through Proposition 2.

The proof of Theorem 1. Applying Jensen's theorem to I_i $(i=1,2)$ we have

$$\frac{1}{\pi}\int_{I_i}F(\rho(\theta))\mathrm{d}\theta\geqslant F(\rho_i).$$

So,

$$\frac{1}{2\pi}\int_{S^1}F(\rho(\theta))\mathrm{d}\theta\geqslant\frac{1}{2}(F(\rho_1)+F(\rho_2)).$$

By (4), $\rho_1=\frac{L}{2\pi}+b$, $\rho_2=\frac{L}{2\pi}-b$, where $b\geqslant 0$. It follows from (11) that $b\geqslant\frac{u}{2\pi}\geqslant 0$. Again by the (strict) convexity of function $F(x)$ (see [10, Lemma 2.9]),

$$\begin{aligned}F(\rho_1)+F(\rho_2)&=F\left(\frac{L}{2\pi}+b\right)+F\left(\frac{L}{2\pi}-b\right)\\&\geqslant F\left(\frac{L}{2\pi}+\frac{u}{2\pi}\right)+F\left(\frac{L}{2\pi}-\frac{u}{2\pi}\right)=F(-t_1)+F(-t_2).\end{aligned}$$

Hence,

$$\frac{1}{2\pi}\int_{S^1}F(\rho(\theta))\mathrm{d}\theta\geqslant\frac{1}{2}(F(\rho_1)+F(\rho_2))\geqslant\frac{1}{2}(F(-t_1)+F(-t_2)).$$

On one hand, if γ is a circle, it is clear that equality holds in (2), since $-t_1=-t_2=\rho(\theta)$. On the other hand, to prove that γ is a circle when equality holds in (2), it is enough to show that

$$\frac{1}{2\pi}\int_{S^1}F(\rho(\theta))\mathrm{d}\theta>\frac{1}{2}(F(-t_1)+F(-t_2))$$

when γ is not a circle. If γ is not a circle, then $u=\sqrt{L^2-4\pi A}>0$. By Lemma 4 and (4), there exists a $b>0$, such that $\rho_1=\frac{L}{2\pi}+b$, $\rho_2=\frac{L}{2\pi}-b$. Furthermore, by (12),

$b > \frac{u}{2\pi} > 0$. Again, by strict convexity of the function $F(x)$,

$$F(\rho_1)+F(\rho_2) = F\left(\frac{L}{2\pi}+b\right)+F\left(\frac{L}{2\pi}-b\right)$$
$$> F\left(\frac{L}{2\pi}+\frac{u}{2\pi}\right)+F\left(\frac{L}{2\pi}-\frac{u}{2\pi}\right) = F(-t_1)+F(-t_2).$$

Therefore,

$$\frac{1}{2\pi}\int_{S^1} F(\rho(\theta))\mathrm{d}\theta \geqslant \frac{1}{2}(F(\rho_1)+F(\rho_2)) > \frac{1}{2}(F(-t_1)+F(-t_2)),$$

which completes the proof. □

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