Mathematical
Inequalities
& Applications
Volume 19, Number 2 (2016), 523–534 doi:10.7153/mia-19-39

UNCERTAINTY PRINCIPLES FOR ORTHONORMAL SEQUENCES RELATED TO LAGUERRE HYPERGROUP

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we deal with Laguerre hypergroup $\mathbb{K}=[0,+\infty)\times\mathbb{R}$. We prove an analogous of a time-frequency localization theorem for orthonormal sequences in $L^2(\mathbb{K})$. As consequence we obtain an analogous of Shapiro's Umbrella theorem. Also, we provide a mean dispersion inequality. Finally, we get a strong version of the uncertainty inequality for orthonormal bases of $L^2(\mathbb{K})$.

1. Introduction

The uncertainty principle states that a nonzero function and its Fourier transform cannot both be sharply localized. A mathematical formulation of this physical ideas is firstly developed by Heisenberg [9] in 1927.

$$\left(\int_{\mathbb{R}} x^2|f(x)|^2dx\right)\left(\int_{\mathbb{R}} \xi^2|\hat{f}(\xi)|^2d\xi\right) \geqslant \frac{1}{4}\left(\int_{\mathbb{R}} |f(x)|^2dx\right)^2.$$

It is known that Heisenberg inequality may be also written in the form

$$\|xf\|_2^2+\|\xi\hat{f}(\xi)\|_2^2 \geqslant \frac{1}{2}\|f\|_2^2$$

where $f\in L^2(\mathbb{R})$. Since that other formulations of this principle have been given in several surveys. We refer the reader to [8, 2, 14]. A first generalization of this principle is to consider a generalized Fourier transform $\mathscr{T}$ like Hankel, Dunkl, Chebli-Trimeche transforms, etc... For Laguerre transform, A. Rahmouni proved Heisenberg-Pauli-Weyl inequality

$$\|\,|(x,t)|^{\beta}f\|_{2,m_\alpha}.\|\,|(\lambda,m)|^{\frac{\beta}{2}}\mathscr{F}_Lf\|_{2,\gamma_\alpha} \geqslant C\|f\|_{2,m_\alpha}^2. \tag{1}$$

A second generalization consists on seeing the uncertainty principle as a statement on the degradation of localization when one considers successive elements of an orthonormal basis. For some of works related to uncertainty inequalities for orthonormal bases,

Mathematics subject classification (2010): 42B10, 42C25.
Keywords and phrases: Uncertainty inequality, Laguerre hypergroup, time-frequency localization theorem, mean dispersion inequality.
This research is supported by LR11ES11 Analyse Mathématiques et Applications, Tunisia.

Paper MIA-19-39

one can cite [13, 18, 10, 11, 16]. In particular, Shapiro in [18] proves that we can not find an infinite orthonormal sequence $\{e_k\}$ such that

$$|e_k(x)| \leqslant |\varphi(x)| \quad \text{and} \quad |\hat{e}_k(x)| \leqslant |\psi(x)|$$

where φ and $\psi \in L^2(\mathbb{R})$. This result is known as Shapiro's Umbrella theorem. Furthermore, Shapiro provide a mean dispersion principle which states that if the elements of an orthonormal sequence and their Fourier transforms have uniformly bounded means and dispersions then the sequence is finite.

A quantitative version of Shapiro's results was established by Jaming and Powel in [11] using Prolate Spheroidal wave functions. In this paper, we are motivated by the work of Malinnikova [12] which gives a time frequency localization theorem that implies Shapiro's Umbrella theorem, a mean dispersion inequality and a number of uncertainty inequalities invoking orthonormal bases for $L^2(\mathbb{R}^d)$. We try to find analogous results by considering orthonormal bases of $L^2(\mathbb{K})$.

The outline of the content of the paper is as follows.

In section 2 we deal with Laguerre hypergroup $\mathbb{K}$ and the Fourier Laguerre transform $\mathscr{F}_L$. We give a few results about this transformation which can be useful later.

In section 3, first, we provide an analogous of time-frequency theorem invoking a series of orthonormal sequences in $L^2(\mathbb{K})$. As application we get an analogous version of Shapiro's Umbrella theorem. Second, in Theorem 4, we prove a Mean Dispersion localization for orthonormal sequence in $L^2(\mathbb{K})$:

$$\forall p,q>0, \quad \sum_{n=1}^{N}\left(\| \, |(x,t)|^p \phi_n\|_{2,m_\alpha}^2 + \| \, |(\lambda,m)|^q \mathscr{F}_L\phi_n\|_{2,\gamma_\alpha}^2\right) \geqslant C N^{1+\frac{2p}{3\alpha+6}}$$

Moreover

$$\sum_n \left(\| \, |(x,t)|^p \phi_n\|_{2,m_\alpha} + \| \, |(\lambda,m)|^q \mathscr{F}_L\phi_n\|_{2,\gamma_\alpha}\right)^{-\frac{1}{2}(3\alpha+6)-\varepsilon} < +\infty.$$

Finally, we find in Theorem 6 the uncertainty principle: for $\{e_n\}_{n=1}^{\infty}$ an orthonormal basis in $L^2(\mathbb{K})$,

$$\sup_n \| \, |(x,t)|^p e_n\|_{2,m_\alpha} \| \, |(\lambda,m)|^{\frac{p}{2}} \mathscr{F}_L e_n\|_{2,\gamma_\alpha} = +\infty.$$

This result is stronger for bases than Heisenberg-Pauli-Weyl inequality (1).

2. Preliminaries

In this paper we consider the Laguerre hypergroup $\mathbb{K} = [0,+\infty)\times\mathbb{R}$ which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group. For more details one can see [1, 7, 15, 19]. We consider the following system of partial differential operators:

For all $(x,t) \in \mathbb{K}$ and $\alpha \geqslant 0$,

$$\begin{cases} D_1 = \dfrac{\partial}{\partial t} \\ D_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{2\alpha+1}{x}\dfrac{\partial}{\partial x} + x^2\dfrac{\partial^2}{\partial t^2} \end{cases}$$

For $\alpha = n-1$, $n \in \mathbb{N} \setminus \{0\}$, the operator D_2 is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$.

For $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the initial problem

$$\begin{cases} D_1 u = i\lambda\, u, \\ D_2\, u = -4|\lambda| \left(m + \dfrac{\alpha+1}{2}\right) u \\ u(0,0) = 1,\ \dfrac{\partial u}{\partial x}(0,t) = 0 \qquad \text{for all } t \in \mathbb{R}, \end{cases}$$

has a unique solution $\varphi_{\lambda,m}$ given by

$$\forall (x,t) \in \mathbb{K}, \qquad \varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathscr{L}_m^{(\alpha)}(|\lambda| x^2)$$

where $\mathscr{L}_m^{(\alpha)}$ is the Laguerre function defined on $\mathbb{R}_+$ by

$$\mathscr{L}_m^{(\alpha)}(x) = e^{-\frac{x}{2}}\frac{L_m^\alpha(x)}{L_m^\alpha(0)}$$

and L_m^α is the Laguerre polynomial of degree m and order α.

The Fourier Laguerre transform of a suitable function is given by

$$\mathscr{F}_L f(\lambda,m) = \int_{\mathbb{K}} f(x,t)\, \varphi_{-\lambda,m}(x,t) dm_\alpha(x,t)$$

where

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}}{\pi\Gamma(\alpha+1)} dx\, dt.$$

From [15], it is well known that Fourier Laguerre transform can be inverted to

$$\mathscr{F}_L^{-1} f(x,t) = \int_{\mathbb{R}\times\mathbb{N}} f(\lambda,m)\, \varphi_{\lambda,m}(x,t) d\gamma_\alpha(\lambda,m)$$

where

$$d\gamma_\alpha(\lambda,m) = L_m^\alpha(0)\delta_m \otimes |\lambda|^{\alpha+1} d\lambda.$$

For all $(x,t) \in \mathbb{K}$, we denote $|(x,t)|_{\mathbb{K}} = |(x,t)| = (x^4+4t^2)^{\frac{1}{4}}$ the homogeneous norm on $\mathbb{K}$. Let $B_r(x,t)$ be the ball centred at (x,t) of radius r. i.e

$$B_r(x,t) = \{(y,s) \in \mathbb{K}; |(y-x, s-t)|_{\mathbb{K}} < r\}$$

It was seen in [7] that for all $(x,t) \in \mathbb{K}$, $m_\alpha(B_r(x,t)) = \Omega_2 r^{2\alpha+4}$, where

$$\Omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+1)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}.$$

We denote $L^2(\mathbb{K}) = L^2(\mathbb{K}, dm_\alpha)$ the space of mesurable functions $f : \mathbb{K} \longrightarrow \mathbb{C}$ such that

$$\|f\|_{2,m_\alpha} = \left(\int_{\mathbb{K}} |f(x,t)|^2 dm_\alpha(x,t) \right)^{\frac{1}{2}} < +\infty.$$

Now we consider the quasinorm defined on $\mathbb{R} \times \mathbb{N}$ by

$$\mathcal{N}(\lambda,m) = |(\lambda,m)| = 4|\lambda|(m + \frac{\alpha+1}{2}).$$

We denote

$$\mathbb{B}_r(0,0) = \{(\lambda,m) \in \mathbb{R} \times \mathbb{N};\ |(\lambda,m)| = \mathcal{N}(\lambda,m) < r\}$$

and we have from [17]

$$\gamma_\alpha(\mathbb{B}_r(0,0)) = \frac{2\,r^{\alpha+2}}{\alpha+2} \sum_{m \geqslant 0} \frac{L_m^\alpha(0)}{(4m+2\alpha+2)^{\alpha+2}}.$$

We introduce $L^2(\mathbb{R} \times \mathbb{N})$ the space of measurable function $g : \mathbb{R} \times \mathbb{N} \longrightarrow \mathbb{C}$ which verifies

$$\|g\|_{2,\gamma_\alpha} = \left(\int_{\mathbb{R} \times \mathbb{N}} |g(\lambda,m)|^2 d\gamma_\alpha(\lambda,m) \right)^{\frac{1}{2}} < +\infty.$$

Nessibi and Trimèche proved in [15] the following Plancherel formula

$$\|\mathscr{F}_L f\|_{2,\gamma_\alpha} = \|f\|_{2,m_\alpha}.$$

We introduce the dilated of $(x,t) \in \mathbb{K}$ by $\delta_r(x,t) = (rx, r^2 t)$ and the dilated of $(\lambda,m) \in \mathbb{R} \times \mathbb{N}$ by $\delta'_r(\lambda,m) = (r^2\lambda, m)$. For $f_r(x,t) = r^{-(2\alpha+4)} f(\delta_{\frac{1}{r}}(x,t))$ we have

$$\int_{\mathbb{K}} f_r(x,t) dm_\alpha(x,t) = \int_{\mathbb{K}} f(x,t) dm_\alpha(x,t).$$

As in [6], we define $D_r f = r^{-(\alpha+2)} f(\delta_{\frac{1}{r}}(x,t))$. By a change of variables, we get

$$\mathscr{F}_L D_r f = \hat{D}_{\frac{1}{r}} \mathscr{F}_L f \tag{2}$$

where

$$\hat{D}_r f(\lambda,m) = r^{-(\alpha+2)} f(\delta'_{\frac{1}{r}}(\lambda,m)).$$

3. Main results

3.1. Time frequency localization related to Laguerre hypergroup

THEOREM 1. *Let* $\{\phi_n\}_{n=1}^N$ *be an orthonormal system in* $L^2(\mathbb{K})$*,* T *be a measurable subset of* $\mathbb{K}$ *and* W *be a measurable subset of* $\mathbb{R}\times\mathbb{N}$*. Assume that*

$$\int_T |\phi_n|^2 dm_\alpha = 1-a_n^2 \quad \text{and} \quad \int_W |\mathscr{F}_L\phi_n|^2 d\gamma_\alpha = 1-b_n^2.$$

Then

$$\sum_{n=1}^N (1-\frac{3}{2}a_n-\frac{3}{2}b_n) \leqslant m_\alpha(T)\,\gamma_\alpha(W).$$

Proof. Let $T\subset\mathbb{K}$ and $W\subset\mathbb{R}\times\mathbb{N}$ be two measurable subsets such that $m_\alpha(T)<+\infty$ and $\gamma_\alpha(W)<+\infty$. We denote χ_T and χ_W the characteristic functions of T and W. If $f\in L^2(\mathbb{K})$, we consider similarly to [3, 6] P_T and P_W the operators given by

$$P_T f = \chi_T f \qquad \text{and} \qquad P_W f = \mathscr{F}_L^{-1}(\chi_W \mathscr{F}_L f).$$

We have

$$P_W\,P_T f(x,t) = \int_{\mathbb{R}\times\mathbb{N}} \chi_W(\lambda,m)\mathscr{F}_L P_T f(\lambda,m)\varphi_{\lambda,m}(x,t)\,d\gamma_\alpha(\lambda,m).$$

Using Fubini's theorem, we get

$$P_W\,P_T f(x,t) = \int_{\mathbb{K}} f(x',t')\mathscr{N}(x',t',x,t)\,dm_\alpha(x',t')$$

where

$$\mathscr{N}(x',t',x,t) = \chi_T(x',t')\,\mathscr{F}_L^{-1} g_{(x',t')}(x,t)$$

and

$$g_{(x',t')}(\lambda,m) = \chi_W(\lambda,m)\,\varphi_{-\lambda,m}(x',t').$$

This shows that $P_W P_T$ is an integral operator with kernel $\mathscr{N}$ verifying

$$\|\mathscr{N}\|^2_{L^2(\mathbb{K})\otimes L^2(\mathbb{K})} = \int_{\mathbb{K}} |\chi_T(x',t')|^2 \left(\int_{\mathbb{K}} |\mathscr{F}_L^{-1} g_{(x',t')}(x,t)|^2\,dm_\alpha(x,t)\right) dm_\alpha(x',t').$$

From Plancherel formula, we have

$$\begin{aligned}\|\mathscr{N}\|^2_{L^2(\mathbb{K})\otimes L^2(\mathbb{K})} &= \int_{\mathbb{K}} |\chi_T(x',t')| \left(\int_{\mathbb{R}\times\mathbb{N}} |\chi_W(\lambda,m)\varphi_{-\lambda,m}(x',t')|^2\,d\gamma_\alpha(\lambda,m)\right) dm_\alpha(x',t') \\ &\leqslant m_\alpha(T)\,\gamma_\alpha(W).\end{aligned}$$

Therefore the Hilbert-Schmidt norm of $P_W P_T$

$$\|P_W P_T\|^2_{HS} = \|\mathscr{N}\|^2_{L^2(\mathbb{K})\otimes L^2(\mathbb{K})} \leqslant m_\alpha(T)\,\gamma_\alpha(W). \tag{3}$$

Now, we consider the corresponding self-adjoint operator

$$Q = (P_W P_T)^* P_W P_T = P_T P_W P_T.$$

Since

$$tr(Q) = \|P_W P_T\|_{HS}^2 \leqslant m_\alpha(T)\,\gamma_\alpha(W)$$

then

$$\sum_{n=1}^{N} \langle Q\phi_n, \phi_n\rangle \leqslant tr(Q) \leqslant m_\alpha(T)\,\gamma_\alpha(W).$$

On the other hand

$$\langle Q\phi_n, \phi_n\rangle = \langle P_W P_T \phi_n, P_T \phi_n\rangle.$$

Denote $T^c = \mathbb{K}\setminus T$ and $W^c = \mathbb{R}\times\mathbb{N}\setminus W$, we can see that

$$\langle Q\phi_n, \phi_n\rangle = \langle \phi_n, \phi_n\rangle - \langle \phi_n, P_{T^c}\phi_n\rangle - \langle P_T\phi_n, P_{W^c}\phi_n\rangle - \langle P_W P_T\phi_n, P_{T^c}\phi_n\rangle.$$

Applying Cauchy-Schwartz inequality, we obtain

$$\langle Q\phi_n, \phi_n\rangle \geqslant 1 - 2a_n - b_n$$

which implies

$$\sum_{n=1}^{N} 1 - 2a_n - b_n \leqslant m_\alpha(T)\,\gamma_\alpha(W). \tag{4}$$

Furthermore, if we consider $\tilde{Q} = (P_T P_W)^* P_T P_W$ then we get similary

$$\sum_{n=1}^{N} 1 - a_n - 2b_n \leqslant m_\alpha(T)\,\gamma_\alpha(W) \tag{5}$$

Relations (4) and (5) allows us to conclude the desired time-frequency localization inequality. □

The following corollary is an immediate consequence of Theorem 1.

COROLLARY 1. *Let $\{\phi_n\}_{n=1}^N$ be an orthonormal system in $L^2(\mathbb{K})$, such that ϕ_n is ε-concentrated on the ball $B_{r_0}(0,0)$ of $\mathbb{K}$ and $\mathscr{F}_L\phi_n$ is ε-concentrated on the ball $\mathbb{B}_{\rho_0}(0,0)$ of $\mathbb{R}\times\mathbb{N}$, for each $n = 1,...,N$, i.e*

$$\int_{|(x,t)|<r_0} |\phi_n|^2 dm_\alpha \geqslant 1-\varepsilon^2, \qquad \int_{|(\lambda,m)|<\rho_0} |\mathscr{F}_L\phi_n|^2 d\gamma_\alpha \geqslant 1-\varepsilon^2.$$

Then

$$N \leqslant \frac{c(\alpha)}{1-3\varepsilon}\, r_0^{2\alpha+4}\,\rho_0^{\alpha+2}$$

where

$$c(\alpha) = \frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}(\alpha+2)^2\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)} \sum_{m\geqslant 0} \frac{L_m^\alpha(0)}{(4m+2\alpha+2)^{\alpha+2}}.$$

Another immediate application of Theorem 1 is the so called Shapiro's Umbrella theorem

THEOREM 2. *Let ϕ and ψ be nonnegative functions in $L^2(\mathbb{K})$ and $L^2(\mathbb{R}\times\mathbb{N})$ and let $\{e_n\}_{n=1}^N$ be an orthonormal sequence that satisfies*

$$|e_n| \leqslant \phi \quad \text{and} \quad |\mathscr{F}_L e_n| \leqslant \psi.$$

Then, for all $\varepsilon \in (0,\frac{1}{3})$, there exists $R_{\varepsilon,\phi} > 0$ and $R_{\varepsilon,\psi} > 0$ such that

$$N \leqslant C R_{\varepsilon,\phi}^{2\alpha+4} R_{\varepsilon,\psi}^{\alpha+2} \tag{6}$$

where C depends only on α and ε.

Proof. Let $\varepsilon > 0$. Since ϕ belongs to $L^2(\mathbb{K})$ and ψ belongs to $L^2(\mathbb{R}\times\mathbb{N})$ then there exist $R_{\varepsilon,\phi} > 0$ and $R_{\varepsilon,\psi} > 0$ such that

$$\int_{B^c_{R_{\varepsilon,\phi}}(0,0)} |\phi|^2\, dm_\alpha \leqslant \varepsilon^2 \quad \text{and} \quad \int_{\mathbb{B}^c_{R_{\varepsilon,\psi}}(0,0)} |\psi|^2\, d\gamma_\alpha \leqslant \varepsilon^2.$$

Therefore

$$\int_{B_{R_{\varepsilon,\phi}}(0,0)} |e_n|^2\, dm_\alpha \geqslant 1-\varepsilon^2 \quad \text{and} \quad \int_{\mathbb{B}_{R_{\varepsilon,\psi}}(0,0)} |\mathscr{F}_L e_n|^2\, d\gamma_\alpha \geqslant 1-\varepsilon^2.$$

Relation (6) yields from Corollary 1. $\square$

This theorem is a quantitative version of Shapiro's Umbrella theorem: it proves that the sequence $\{e_n\}$ is finite but also it gives a quantitative estimation of the number of the sequence elements.

3.2. Mean Dispersion inequality related to Laguerre hypergroup

To prove an analogous of Mean Dispersion inequality related to Laguerre hypergroup, we need to introduce the following notations. For $p,q > 0$ and $f \in L^2(\mathbb{K})$, we denote

$$\tau_p^p(f) = \int_{\mathbb{K}} |(x,t)|^{2p} |f(x,t)|^2\, dm_\alpha(x,t)$$

and

$$\nu_q^q(\mathscr{F}_L f) = \int_{\mathbb{R}\times\mathbb{N}} |(\lambda,m)|^{2q} |\mathscr{F}_L f(\lambda,m)|^2\, d\gamma_\alpha(\lambda,m).$$

Then we have similarly as Theorem 2

THEOREM 3. *Let p, q be positive numbers and $\{\phi_n\}_{n=1}^N$ be an orthonormal system in $L^2(\mathbb{K})$ that satisfies*

$$\tau_p(\phi_n) \leqslant J \quad \text{and} \quad \nu_q(\mathscr{F}_L \phi_n) \leqslant K.$$

Then

$$N \leqslant c(\alpha,p,q) J^{\alpha+2} K^{\frac{1}{2}(\alpha+2)}.$$

Proof. Let $\varepsilon \in (0, \frac{1}{3})$. Clearly, each ϕ_n is ε-concentrated on the ball $\{|(x,t)|^{2p} \leqslant \varepsilon^{-2} J^p\}$ and each $\mathscr{F}_L\phi_n$ is ε-concentrated on the ball $\{|(\lambda,m)|^{2q} \leqslant \varepsilon^{-2} K^q\}$. We deduce the wanted result from Corollary 1. □

THEOREM 4. *Let p and q be positive and let $\{\phi_n\}_n$ be an orthonormal sequence in $L^2(\mathbb{K})$. Then*

$$\sum_{n=1}^{N} \left(\| \, |(x,t)|^p \phi_n \|_{2,m_\alpha}^2 + \| \, |(\lambda,m)|^q \mathscr{F}_L\phi_n \|_{2,\gamma_\alpha}^2 \right) \geqslant C N^{1+\frac{2p}{3\alpha+6}} \tag{7}$$

where C depends on α, p and q. Furthermore, for all $\varepsilon > 0$, we have

$$\sum_{n} \left(\| \, |(x,t)|^p \phi_n \|_{2,m_\alpha} + \| \, |(\lambda,m)|^q \mathscr{F}_L\phi_n \|_{2,\gamma_\alpha} \right)^{-\frac{1}{2}(3\alpha+6)-\varepsilon} < +\infty. \tag{8}$$

Proof. Let $\{\phi_n\}_n$ be an orthonormal sequence in $L^2(\mathbb{K})$. For all $k \in \mathbb{Z}$, we consider

$$P_k = \{n;\ \max(\tau_p(\phi_n), \nu_q(\mathscr{F}_L\phi_n)) \in [2^{k-1}, 2^k)\}.$$

Then

$$\int_{\mathbb{K}} |(x,t)|^{2p} |\phi_n(x,t)|^2 dm_\alpha \leqslant 2^{kp} \quad \text{and} \quad \int_{\mathbb{R}\times\mathbb{N}} |(\lambda,m)|^{2q} |\mathscr{F}_L\phi_n(\lambda,m)|^2 d\gamma_\alpha \leqslant 2^{kq}$$

whenever $n \in \cup_{j=-\infty}^{k} P_j$. Therefore one can see that ϕ_n is $\frac{1}{4}$-concentrated on the ball centered at $(0,0)$ and of radius $2^{\frac{k}{2}+\frac{2}{p}}$ and $\mathscr{F}_L\phi_n$ is $\frac{1}{4}$-concentrated on the ball centered at $(0,0)$ and of radius $2^{\frac{k}{2}+\frac{2}{q}}$. From Corollary 1, we have the number of elements in $\cup_{j=-\infty}^{k} P_j$ is less then $C(\alpha,p,q) 2^{\frac{1}{2}(3\alpha+6)k}$ where $C(\alpha,p,q)$ is a constant that does not depend on k. This shows that when $C(\alpha,p,q)2^{\frac{1}{2}(3\alpha+6)k} < 1$, the number of elements in $\cup_{j=-\infty}^{k} P_j$ is null. Consequently, there exists k_0 such that P_k is empty for all $k \leqslant k_0$. We remark that when $q = \frac{p}{2}$ and $p > 1$ this result can be also deduced from Heisenberg-Pauli-Weyl inequality proved in [17].

For $N > 2C(\alpha,p,q)$, let choose k such that

$$2C(\alpha,p,q) 2^{\frac{1}{2}(3\alpha+6)(k-1)} < N \leqslant 2C(\alpha,p,q) 2^{\frac{1}{2}(3\alpha+6)k}.$$

The first inequality shows that at least half of $\{1,\ldots,N\}$ does not belong to $\cup_{j=-\infty}^{k-1} P_j$. We remark that when $n \notin \cup_{j=-\infty}^{k-1} P_j$, $\tau_p^p(\phi_n) \geqslant 2^{(k-1)p}$ which implies

$$\sum_{n=1}^{N} \left(\tau_p^p(\phi_n) + \nu_q^q(\mathscr{F}_L\phi_n) \right) \geqslant \frac{N}{2}.2^{(k-1)p} \geqslant a(\alpha,p,q) N^{1+\frac{2p}{3\alpha+6}}.$$

For $N < 2C(\alpha,p,q)$, we have

$$\sum_{n=1}^{N} \left(\tau_p^p(\phi_n) + \nu_q^q(\mathscr{F}_L\phi_n) \right) \geqslant N 2^{(k_0-1)p}.$$

Since $C(\alpha,p,q)2^{\frac{1}{2}(3\alpha+6)k_0} = r_0 \in (0,1)$ then $2^{(k_0-1)p} \geqslant cte.(2C(\alpha,p,q))^{\frac{2p}{3\alpha+6}}$ and relation (7) follows.

Let now prove (8). We have, for $n \in P_k$,

$$\tau_p(\phi_n) + \nu_q(\mathscr{F}_L\phi_n) \geqslant \max\left(\tau_p(\phi_n), \nu_q(\mathscr{F}_L\phi_n)\right) \geqslant 2^{k-1}.$$

Since $\mathbb{N} = \cup_{k\in\mathbb{Z}} P_k$ and P_k is empty for all $k \leqslant k_0$, we get

$$\sum_n \left(\tau_p(\phi_n) + \nu_q(\mathscr{F}_L\phi_n)\right)^{-\frac{1}{2}(3\alpha+6)-\varepsilon} \leqslant \sum_{k=k_0}^{+\infty}\left(\sum_{n\in P_k} 2^{(1-k)(\frac{1}{2}(3\alpha+6)+\varepsilon)}\right)$$

$$\leqslant \sum_{k=k_0}^{+\infty} C(\alpha,p,q)2^{\frac{1}{2}(3\alpha+6)k}2^{(1-k)(\frac{1}{2}(3\alpha+6)+\varepsilon)} < +\infty. \quad \square$$

3.3. Unbounded product of dispersions

THEOREM 5. *Let $T \subset \mathbb{K}$, $W \subset \mathbb{R}\times\mathbb{N}$ be a pair of measurable subsets of finite measure $m_\alpha(T)$, $\gamma_\alpha(W) < +\infty$. Then T and W are strong annihilating pair i.e there exists a constant $C(T,W)$ such that for all $f \in L^2(\mathbb{K})$,*

$$\|f\|^2_{2,m_\alpha} \leqslant C(T,W)\left(\|f\chi_{T^c}\|^2_{2,m_\alpha} + \|\mathscr{F}_L f\chi_{W^c}\|^2_{2,\gamma_\alpha}\right).$$

Proof. Let prove that if $supp(f) \subset T$, $supp(\mathscr{F}_L f) \subset W$ and $m_\alpha(T)$, $\gamma_\alpha(W)$ are finite then $f = 0$. According to [6, Corollary 3.7], this statement implies that T and W are strong annihilating pair which gives Theorem 5.

An elementary fact on Hilbert-Schmidt operators states that

$$dim(Im(P_T) \cap Im(P_W)) = \|P_T \cap P_W\|^2_{HS} \leqslant \|P_T\, P_W\|^2_{HS}.$$

From relation (3), we have

$$dim(Im(P_T) \cap Im(P_W)) < +\infty \tag{9}$$

Assume towards a contradiction that there exists $f_0 \neq 0$ where $supp(f_0) = T_0$ and $W_0 = supp(\mathscr{F}_L f_0)$ have both finite measure $0 < m_\alpha(T_0), \gamma_\alpha(W_0) < +\infty$.

Let T_1 (resp W_1) be a measurable subset of $\mathbb{K}$ (resp $\mathbb{R}\times\mathbb{N}$) of finite measure $0 < m_\alpha(T_1) < +\infty$ (resp $0 < \gamma_\alpha(W_1) < +\infty$) such that $T_0 \subset T_1$ (resp $W_0 \subset W_1$).

We have, for $r > 0$,

$$m_\alpha(T_1 \cup \delta_r T_0) = \|\chi_{\delta_r T_0} - \chi_{T_1}\|^2_{2,m_\alpha} + \langle \chi_{\delta_r T_0}, \chi_{T_1}\rangle_{m_\alpha}.$$

The function $: r \longmapsto m_\alpha(T_1 \cup \delta_r T_0)$ is continuous on $(0,+\infty)$. The same holds for $: r \longmapsto \gamma_\alpha(W_1 \cup \delta'_r W_0)$. One deduces that there exists an infinite sequence of distinct numbers $(r_i)_{i=0}^{\infty} \subset (0,+\infty)$ with $r_0 = 1$, such that, if we denote by $T = \bigcup_{i=0}^{+\infty} \delta_{r_i} T_0$ and $W = \bigcup_{i=0}^{+\infty} \delta'_{\frac{1}{r_i}} W_0$,

$$m_\alpha(T) < 2m_\alpha(T_0) \quad \text{and} \quad \gamma_\alpha(W) < 2\gamma_\alpha(W_0).$$

We next define $f_i = D_{r_i} f_0$ so that $supp(f_i) = \delta_{r_i} T_0 \subset T$. From relation (2), we have $supp(\mathscr{F}_L f_i) = \delta'_{\frac{1}{r_i}} W_0 \subset W$.

Now assume that we have a vanishing linear combinations of dilates of f_0

$$\sum_{finite} \alpha_i f_i(x,t) = 0.$$

We denote $\beta_i = \alpha_i r_i^{-(\alpha+2)}$ and $g(\frac{x}{r_i}) = f_0(\frac{x}{r_i}, 0)$ then

$$\sum_{finite} \beta_i g(\frac{x}{r_i}) = 0.$$

Applying the euclidien Fourier we get

$$\sum_{finite} \beta_i r_i \mathscr{F} g(r_i x) = 0.$$

Since $g \in L^1(\mathbb{R})$ then $\mathscr{F} g \in C_0$. Invoking [5, lemma 2.1], one can see that $\mathscr{F} g$ has linearly independent dilates. Therefore $\beta_i = 0$ so that $\alpha_i = 0$, which proves that f_i are linearly independent. Consequently, $dim(Im(P_T) \cap Im(P_W)) = +\infty$ which contradicts relation (9). $\square$

LEMMA 1. *Let b and c be positive numbers, there exists a nonzero function f in $L^2(\mathbb{K})$ such that $f(x,t) = 0$ when $|(x,t)| \leqslant b$, and $\mathscr{F}_L f(\lambda,m) = 0$ when $|(\lambda,m)| \leqslant c$.*

Proof. We consider the space E_c of $f \in L^2(\mathbb{K})$ such that $\mathscr{F}_L f(\lambda,m) = 0$ when $|(\lambda,m)| \geqslant c$. From Theorem 5, there exists C such that

$$\|f\|_{2,m_\alpha} \leqslant C \, \|f \chi_{\{|(x,t)|>b\}}\|_{2,m_\alpha}.$$

The last inequality implies that the traces of functions from the space E_c on $\{|(x,t)| > b\}$ form a closed subspace F_c in $L^2(\{|(x,t)| > b\})$ which is obviously not the whole space. Thus there exists $f \in F_c^{\perp} \subset L^2(\{|(x,t)| > b\})$ such that

$$\langle f, g \rangle = \int_{|(x,t)|>b} f(x,t) \overline{g(x,t)} \, dm_\alpha(x,t) = 0 \tag{10}$$

for any $g \in E_c$. We extend f by zero on the ball $B_b(0,0)$ in order to get the required function. In fact, since $\mathscr{F}_L^* = \mathscr{F}_L^{-1}$ then relation (10) implies that $\mathscr{F}_L f = 0$ on $\mathbb{B}_c(0,0)$. $\square$

THEOREM 6. *Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis in $L^2(\mathbb{K})$ and p be positive, then*

$$\sup_n \| \, |(x,t)|^p e_n\|_{2,m_\alpha} \| \, |(\lambda,m)|^{\frac{p}{2}} \mathscr{F}_L e_n\|_{2,\gamma_\alpha} = +\infty.$$

Proof. Assume that there exists an orthonormal basis such that

$$\| \, |(x,t)|^p e_n\|_{2,m_\alpha} \| \, |(\lambda,m)|^{\frac{p}{2}} \mathscr{F}_L e_n\|_{2,\gamma_\alpha} \leqslant C^{2p}.$$

We introduce

$$A_k = \{e_n \, ; \, \tau_p(e_n) \in (2^{-k}C, 2^{-k+1}C]\}$$

where k is integer. For $e_n \in A_k$ we have $\| \, |(\lambda,m)|^{\frac{p}{2}} \mathscr{F}_L e_n\|_{2,\gamma_\alpha} \leqslant C^p 2^{kp}$. Then e_n is $\frac{1}{4}-$concentrated on the ball $B_{C^{\frac{1}{2}} 2^{-\frac{k}{2}+\frac{1}{2}+\frac{2}{p}}}(0,0)$ and $\mathscr{F}_L e_n$ is $\frac{1}{4}-$concentrated on the ball $\mathbb{B}_{C2^{k+\frac{4}{p}}}(0,0)$. From Corollary 1, we have the number of elements in A_k is bounded by a constant that does not depend on k. Lemma 1 allows us to consider $f \in L^2(\mathbb{K})$, $\|f\|_{2,m_\alpha} = 1$, that vanishes on $B_R(0,0)$ and its Fourier-Laguerre transform vanishes on $\mathbb{B}_R(0,0)$. If $k \geqslant 0$ and $e_n \in A_k$, we get

$$|\langle f, e_n \rangle|^2 \leqslant R^{-2p} \tau_p^p(e_n) \leqslant (2C)^p R^{-2p} 2^{-kp}.$$

If $k < 0$ and $e_n \in A_k$ similarly we have

$$|\langle f, e_n \rangle|^2 = |\langle \mathscr{F}_L f, \mathscr{F}_L e_n \rangle|^2 \leqslant (C)^p R^{-p} 2^{kp}.$$

Since $\sum_n |\langle f, e_n \rangle|^2 = 1$ and $\cup A_k = \{e_n\}_{n=1}^{\infty}$ then choosing R large enough we get a contradiction. □

Acknowledgement. The author is thankful to the referee for his careful reading and valuable comments.

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(Received May 27, 2015)

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