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SOME SHARP INEQUALITIES FOR MULTIDIMENSIONAL INTEGRAL OPERATORS WITH HOMOGENEOUS KERNEL: AN OVERVIEW AND NEW RESULTS

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Abstract. One goal of this paper is to point out the fact that a big number of inequalities proved from time to time in journal publications, both one-dimensional and multi-dimensional, are particular cases of some general results for integral operators with homogeneous kernels, including in particular, the statements on sharp constants.

Some new multidimensional Hardy-Hilbert type inequalities are derived. Moreover, a new multidimensional Pólya-Knopp inequality is proved and some examples of applications are derived from this result. The constants in all inequalities are sharp.

1. Introduction

Let $p \geqslant 1$ and denote by p' the conjugate parameter defined by $1/p+1/p'=1$ ($p'=\infty$ when $p=1$). We also assume that f and g denote arbitrary measurable functions on $\mathbb{R}^n, n\in\mathbb{Z}_+$. All constants below and in the all inequalities in this paper are sharp. In order to fix ideas we first consider the classical case $n=1$.

Hilbert's inequality. The inequality

$$\int_0^\infty \left|\int_0^\infty \frac{1}{x+y}f(x)g(y)\right| dxdy \leqslant \frac{\pi}{\sin\frac{\pi}{p}}\left(\int_0^\infty |f(x)|^p dx\right)^{\frac{1}{p}}\left(\int_0^\infty |g(y)|^{p'}dy\right)^{\frac{1}{p'}},\ p>1 \tag{1.1}$$

is called Hilbert's inequality. It can equivalently be rewritten in the form

$$\int_0^\infty \left|\int_0^\infty \frac{1}{x+y}f(y)dy\right|^p dx \leqslant \left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p \int_0^\infty |f(x)|^p dx. \tag{1.2}$$

Hardy's inequality. The first weighted form of Hardy's inequality can be written in the following way:

$$\int_0^\infty \left|x^{\alpha-1}\int_0^x \frac{f(y)}{y^\alpha}dy\right|^p dx \leqslant \left(\frac{p}{p-\alpha-1}\right)^p \int_0^\infty |f(x)|^p dx, \tag{1.3}$$

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where $p \geqslant 1$, $\alpha < p-1$. The (equivalent) dual form of (1.3) reads:

$$\int_0^\infty \left| x^{\alpha-1} \int_x^\infty \frac{f(y)}{y^\alpha} dy \right|^p dx \leqslant \left(\frac{p}{1+\alpha-p} \right)^p \int_0^\infty |f(x)|^p dx, \tag{1.4}$$

where $p \geqslant 1$, $\alpha > p-1$.

REMARK 1. For some history about these inequalities we refer to [5], [6], [8], [9], [17] and [26]. For the extensive development of these inequalities to what today is called Hardy type inequalities, we refer to the monographs [14], [16] and [18] and the references therein.

Hardy-Hilbert type inequalities for homogeneous kernels. The inequalities (1.2)-(1.4) can all be written in the form

$$\int_0^\infty \left| \int_0^\infty k(x,y) f(y) dy \right|^p dx \leqslant C^p \int_0^\infty |f(x)|^p dx, \quad p \geqslant 1, \tag{1.5}$$

with different kernels which are homogeneous of degree -1. It is also well known that the inequality (1.5) can equivalently be written in the form

$$\int_0^\infty \left| \int_0^\infty k(x,y) f(y) g(x) dy \right| dx \leqslant C \left(\int_0^\infty |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_0^\infty |g(y)|^{p'} dy \right)^{\frac{1}{p'}}. \tag{1.6}$$

REMARK 2. There is a great number of papers devoted to the proofs of (1.5) and (1.6) for concrete kernels other than classical Hilbert kernel $k(x,y) = \frac{1}{x+y}$. One weakness with many of these results is that the authors do not refer to the fact that already in 1999 (see [4] and [10]) it was given necessary and sufficient conditions for (1.5)–(1.6) to hold and with sharp constant and general kernel of degree -1. Instead of trying to refer to all such papers we just refer to the paper [28] where at least a sufficient condition has been derived for a general homogeneous kernel in the one-dimensional case. We also refer to the monograph [15], where several results concerning Hardy-Hilbert type inequalities in the last ten years can be found.

One main aim of this paper is to present and discuss this result (see Theorem 1) in this general frame. In particular, we derive sharp constants in some Hardy- Hilbert type inequalities in a n-dimensional setting. In this connection we refer to the paper [20], where also such results were obtained even with sharp constants. See also the monograph [15]. Moreover, we derive a new multidimensional Pólya-Knopp (geometric mean) type inequality and also present some applications of this result.

REMARK 3. In the one dimensional case the inequality

$$\int_0^\infty \exp\left(\int_0^\infty \ln|f(y)| dy \right) dx \leqslant e \int_0^\infty |f(x)| dx \tag{1.7}$$

is just a limit case as $p \to \infty$ of the classical Hardy inequality ((1.3) with $\alpha = 0$)

$$\int_0^\infty \left(\frac{1}{x}\int_0^x |f(y)|dy\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(x)|^p dx,$$

In fact, just replace $|f(x)|$ by $|f(x)|^{1/p}$ and use the fact that (the scale of power-means) $\left(\frac{1}{x}\int_0^x |f(y)|^{1/p}dy\right)^p$ converges to the geometric mean $exp\left(\frac{1}{x}\int_0^x \ln|f(y)|dy\right)$ and $\left(\frac{p}{p-1}\right)^p \to e$ as $p \to \infty$.

Sometimes (1.7) is called Knopp's inequality with reference to the paper [13] but Hardy himself in his 1925 paper paper [5] informed that Pólya pointed out this argument to him so we prefer to call the inequality (1.7) for Pólya-Knopp's inequality.

The paper is organized as follows: The announced characterization of the general sharp inequality involving homogeneous kernels is presented in Section 2 (see Theorem 1). Both well-known and new multidimensional Hardy-Hilbert type inequalities are pointed out and/or complemented. The announced new multidimensional Pólya-Knopp inequality is proved in Section 3 (see Theorem 2). As consequences of this result also some new multidimensional Pólya-Knopp inequalities generated by the Riesz potential and multidimensional Hardy inequality are proved (see Theorems 3 and 4, respectively). Finally, a proof of Theorem 1 can be found in Section 4 (Appendix).

2. Inequalities for multidimensional integral operators with homogeneous kernel

We consider the inequality

$$\left(\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n} k(x,y)f(y)\,dy\right|^p dx\right)^{\frac{1}{p}} \leqslant C_{k,p}\left(\int_{\mathbb{R}^n}|f(x)|^p dx\right)^{\frac{1}{p}}, \quad 1 \leqslant p \leqslant \infty, \tag{2.1}$$

for multidimensional integral operators

$$\mathbf{K}f(x) := \int_{\mathbb{R}^n} k(x,y)f(y)\,dy \tag{2.2}$$

with a kernel $k(x,y)$.

2.1. Sharp constant for homogeneous and rotation invariant kernels

The sharp constant $C_{k,p}$ in (2.1) depending on p and the kernel k is known to be explicitly calculated for general non-negative kernels $k(x,y)$ in the case of the homogeneous and rotation invariant kernel.

In this Subsection we present this general statement (Theorem 1) and in the next subsections we give both well-known and new examples of its applications and consequences.

We assume the following:

1^0. the kernel $k(x,y)$ is *homogeneous of degree* $-n$, i.e.

$$k(tx,ty) = t^{-n}k(x,y), \qquad t>0, \quad x,y \in \mathbb{R}^n, \tag{2.3}$$

2^0. it is *invariant with respect to rotations*, i.e.

$$k[\omega(x),\omega(y)] = k(x,y), \qquad x,y \in \mathbb{R}^n \tag{2.4}$$

for all rotation $\omega(x)$ in $\mathbb{R}^n$. Let

$$\varkappa_p = \int_{\mathbb{R}^n} |k(\sigma,y)| \frac{dy}{|y|^{\frac{n}{p}}}, \quad \sigma \in \mathbb{S}^{n-1}, \tag{2.5}$$

where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. In the sequel

$$|\mathbb{S}^{n-1}| = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

denotes its surface measure. Because of the invariance condition (2.4), the integral in (2.5) does not depend on the choice of $\sigma \in \mathbb{S}^{n-1}$ (see details in the book [12, Section 5]), so one may choose $\sigma = e_1 = (1,0,...,0)$ in (2.5). We also pronounce that the condition (2.3) is crucial in the class of homogeneous kernels.In fact by using a standard dilation argument, we see that (2.1) can not hold for homogeneity of $k \neq -n$.

The statements of the following theorem are known. We briefly dwell on its history. The sufficiency part of it without the sharp constant was first given in [19] (for $n=1$ see also [7, p. 229]). A simpler proof was given in [21]. The necessity of the condition $\varkappa_p < \infty$ for the boundedness and the sharpness of the constant in such a general case was proved in [10]. A complete proof of Theorem 1 in its final form was presented in [11], see also its presentation in the book [12, page 70, Theorem 6.4].

THEOREM 1. *Let* $1 \leqslant p \leqslant \infty$ *and the kernel* $k(x,y)$ *satisfy the assumptions* (2.3)*–*(2.4)*. If*

$$\varkappa_p < \infty,$$

then the inequality (2.1) *holds with* $C(k,p) = \varkappa_p$*. If* $k(x,y) \geqslant 0$*, then the condition* $\varkappa_p < \infty$ *is also necessary for* (2.1) *to hold and* $\varkappa_p$ *is the best constant.*

Since the proof of Theorem 1 was published in sources more related to operator theory than to inequalities, we find it convenient to reproduce that proof here for the integral inequalities' audience; it is given in Appendix.

REMARK 4. There are also known statements of a similar nature for operators with a kernel homogeneous of degree $-n$, with rotation invariance condition replaced by a more general assumption. We do not touch this case but refer the reader to [11] and [12].

The following multidimensional Hilbert type inequality is obtained from Theorem 1 by just calculating the integral in (2.5).

EXAMPLE 1. (Hilbert type inequality) Let $\lambda > 0, \alpha > 0$ and $1 \leqslant p < \infty$. Then

$$\int\limits_{\mathbb{R}^n} \left| |x|^{\beta+\lambda\alpha-n} \int\limits_{\mathbb{R}^n} \frac{f(y)\,dy}{|y|^\beta \left(|x|^\lambda + |y|^\lambda\right)^\alpha} \right|^p dx \leqslant \varkappa^p \int\limits_{\mathbb{R}^n} |f(x)|^p dx \tag{2.6}$$

holds if and only if $\beta < \frac{n}{p'}$ and $\alpha\lambda > \frac{n}{p'} - \beta$ and $\varkappa_{p,\beta} = \int\limits_{\mathbb{R}^n} \frac{|y|^{-\beta-\frac{n}{p}} dy}{\left(1+|y|^\lambda\right)^\alpha} = \frac{|\mathbb{S}^{n-1}|}{\lambda} \int\limits_0^\infty \frac{\rho^{\frac{1}{\lambda}\left(\frac{n}{p'}-\beta\right)-1}}{(1+\rho)^\alpha}$ $= \frac{|\mathbb{S}^{n-1}|}{\lambda} B\left(\frac{1}{\lambda}\left(\frac{n}{p'}-\beta\right), \alpha - \frac{1}{\lambda}\left(\frac{n}{p'}-\beta\right)\right)$ is the sharp constant.

REMARK 5. A similar inequality with “more anisotropic” kernel in which $|x|, |y|$ are replaced by $\|x\|_\beta, \|y\|_\beta$ where $\|x\|_\beta := \left(|x_1|^\beta + \cdots |x_n|^\beta\right)^{\frac{1}{\beta}}$, formally is not contained in Theorem 1 since $\|x\|_\beta$ is not rotation invariant (see Remark 4 though), but so modified inequality follows from the inequality (2.6) by the evident change of variables and the function; it was considered (with $\alpha = 1$) over $\mathbb{R}^n_+$ in [28, formula (3.3)].

2.2. Some known examples of applications of Theorem 1

Let now

$$I^\alpha f(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)\,dy}{|x-y|^{n-\alpha}}, \quad 0 < \alpha < n, \tag{2.7}$$

be the Riesz potential operator with the normalizing constant

$$\gamma_n(\alpha) = \frac{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}. \tag{2.8}$$

EXAMPLE 2. (Stein-Weiss inequality) The best constant C for the inequality (see [24])

$$\int_{\mathbb{R}^n} |x|^\mu \, |I^\alpha f(x)|^p \; dx \leqslant C^p \int_{\mathbb{R}^n} |x|^\gamma |f(x)|^p \; dx, \tag{2.9}$$

valid if and only if $1 \leqslant p < \infty$, $\alpha p - n < \gamma < n(p-1)$, $\mu = \gamma - \alpha p$, was obtained in [22] by means of Theorem 1:

$$C = 2^{-\alpha} \frac{\Gamma\left(\frac{n(p-1)-\gamma}{2p}\right)\Gamma\left(\frac{n+\gamma-\alpha p}{2p}\right)}{\Gamma\left(\frac{n+\gamma}{2p}\right)\Gamma\left(\frac{n(p-1)+\alpha p-\gamma}{2p}\right)}. \tag{2.10}$$

(In the case when $\frac{\alpha}{2}$ is an integer, the sharp constant was calculated in [2] by other means, for non-integer $\frac{\alpha}{2}$ but $\mu = 0$ we refer to [27] and for general μ but $p = 2$ to [4]).

In the next section we give some other consequences of Theorem 1 to some classical inequalities, where the best constants seem never to have been observed, up to our knowledge.

2.3. Some new consequences of Theorem 1

Our next interest is the best constant in the multidimensional Hardy inequalities with power weight:

$$\left\| |x|^{\alpha-n} \int\limits_{|y|<|x|} \frac{f(y)\,dy}{|y|^{\alpha}} \right\|_{L^p(\mathbb{R}^n)} \leqslant C_1(p,\alpha)\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \leqslant p < \infty, \ \alpha < \frac{n}{p'}, \tag{2.11}$$

$$\left\| |x|^{\beta-n} \int\limits_{|y|>|x|} \frac{f(y)\,dy}{|y|^{\beta}} \right\|_{L^p(\mathbb{R}^n)} \leqslant C_2(p,\beta)\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \leqslant p < \infty, \ \beta > \frac{n}{p'}. \tag{2.12}$$

The best constant for (2.11) was calculated in the non-weighted case $\alpha = 0$ in [1], where it was shown that

$$C_1(p,0) = |B(0,1)|p', \tag{2.13}$$

where $|B(0,1)| = \frac{|\mathbb{S}^{n-1}|}{n} = \frac{2\pi^{\frac{n}{2}}}{n\Gamma\left(\frac{n}{2}\right)}$ is the volume of the unit ball. The weighted case with general weights was studied in [3], but by using this result the sharp constant can never be obtained.

PROPOSITION 1. *The sharp constants for* (2.11) *and* (2.12) *are given by*

$$C_1(p,\alpha) = \frac{|\mathbb{S}^{n-1}|}{\frac{n}{p'} - \alpha}, \quad \text{resp.} \quad C_2(p,\beta) = \frac{|\mathbb{S}^{n-1}|}{\beta - \frac{n}{p'}}. \tag{2.14}$$

Proof. In (2.11) and (2.12) we deal with the boundedness of the integral operator with the kernels

$$k_1(x,y) = \frac{|x|^{\alpha-n}}{|y|^{\alpha}} \chi_{B(0,|x|)}(y) \quad \text{resp.} \quad k_2(x,y) = \frac{|x|^{\beta}}{|y|^{\beta+n}} \chi_{\mathbb{R}^n \setminus B(0,|x|)}(y),$$

to which Theorem 1 is applicable. It gives

$$C_1(p,\alpha) = \int\limits_{B(0,1)} \frac{dy}{|y|^{\alpha+\frac{n}{p}}} = |\mathbb{S}^{n-1}| \int\limits_0^1 \rho^{n-1-\alpha-\frac{n}{p}}\,d\rho = \frac{|\mathbb{S}^{n-1}|}{\frac{n}{p'} - \alpha}.$$

Similarly

$$C_2(p,\beta) = \int\limits_{|y|>1} \frac{dy}{|y|^{\beta+\frac{n}{p}}} = |\mathbb{S}^{n-1}| \int\limits_1^\infty \rho^{n-1-\beta-\frac{n}{p}}\,d\rho = \frac{|\mathbb{S}^{n-1}|}{\beta - \frac{n}{p'}}. \qquad \square$$

This proof shows that (2.11)–(2.12) can be complemented in various ways (and still keeping the sharp constant), i.e. the following

PROPOSITION 2. *The inequality*

$$\left\| |x|^{\alpha-n} \int\limits_{a|x|<|y|<b|x|} \frac{f(y)\,dy}{|y|^{\alpha}} \right\|_{L^p(\mathbb{R}^n)} \leqslant C(p,\alpha)\|f\|_{L^p(\mathbb{R}^n)}, \quad 0<a<b<\infty, \tag{2.15}$$

holds for all $1 \leqslant p < \infty$ *and* $\alpha \in \mathbb{R}$, *where the sharp constant is equal to*

$$C(p,\alpha) = |\mathbb{S}^{n-1}| \left| \frac{b^{\frac{n}{p'}-\alpha} - a^{\frac{n}{p'}-\alpha}}{\frac{n}{p'}-\alpha} \right|, \quad \alpha \neq \frac{n}{p'},$$

with $\frac{b^{\frac{n}{p'}-\alpha}-a^{\frac{n}{p'}-\alpha}}{\frac{n}{p'}-\alpha}$ *replaced by* $\ln\frac{b}{a}$ *when* $\alpha = \frac{n}{p'}$

REMARK 6. Note that (2.15) may be regarded as an extension and unification of (2.11) and (2.12) with the sharp constants given by (2.14). In fact, (2.11) and (2.12) follow by just using (2.15) with $a=0$, $b=1$ resp. $a=1$, $b=\infty$.

We restrict ourselves by one example more:

PROPOSITION 3. *The inequality*

$$\left\| |x|^{\alpha+\gamma-n} \int\limits_{|y|<|x|} \frac{f(y)\,dy}{|y|^{\alpha}(|x|-|y|)^{\gamma}} \right\|_{L^p(\mathbb{R}^n)} \leqslant C(p,\alpha,\gamma)\|f\|_{L^p(\mathbb{R}^n)}, \quad 1 \leqslant p < \infty, \tag{2.16}$$

holds, where $\alpha < \frac{n}{p'}$ *and* $\gamma < 1$ *and the sharp constant is equal to* $C(p,\alpha,\gamma) = |\mathbb{S}^{n-1}| B\left(\frac{n}{p'}-\alpha, 1-\gamma\right)$.

The proofs of Propositions 2 and 3 are similar to that of Proposition 1, so we omit the details.

3. Some new Pólya-Knopp type inequalities with sharp constants

Besides the constant $\varkappa_p$, we also introduce the limit constants

$$\varkappa_\infty = \int\limits_{\mathbb{R}^n} k(e_1,y)dy$$

and

$$\varkappa^* = n\frac{\int_{\mathbb{R}^n} k(e_1,y)\ln\frac{1}{|y|}\,dy}{\int_{\mathbb{R}^n} k(e_1,y)dy}$$

assuming that $k(x,y) \geqslant 0$ and which may be zero only on a set of measure zero. For simplicity in this Section we assume that all considered functions are non-negative. Our main theorem reads:

THEOREM 2. *Let* $f(x) \geqslant 0$, *let* $\varkappa_\infty < \infty$ *and* $\varkappa_p < \infty$ *for some* $p > 1$. *If* $\varkappa^* < \infty$, *then*

$$\int_{\mathbb{R}^n} \exp\left(\frac{1}{\varkappa_\infty}\int_{\mathbb{R}^n} k(x,y)\ln f(y)\,dy\right)dx \leqslant e^{\varkappa^*}\int_{\mathbb{R}^n} f(x)dx \tag{3.1}$$

and the constant $e^{\varkappa^*}$ *is sharp.*

Proof. First we observe that

$$\varkappa_\infty < \infty \text{ and } \varkappa_p < \infty \text{ for some } p \implies \varkappa_q < \infty \text{ for all } q > p,$$

because

$$\varkappa_q \leqslant \int_{|y|<1} k(e_1,y)|y|^{-n/p}dy + \int_{|y|>1} k(e_1,y)dy \leqslant \varkappa_p + \varkappa_\infty.$$

Therefore, we can apply Theorem 1 for all sufficiently large p.

We rewrite the inequality (2.1) guaranteed by that theorem as

$$\left\|\frac{1}{\varkappa_\infty}\int_{\mathbb{R}^n} k(x,y)f(y)\,dy\right\|_{L^p(\mathbb{R}^n)} \leqslant \frac{\varkappa_p}{\varkappa_\infty}\|f\|_{L^p(\mathbb{R}^n)}, \tag{3.2}$$

Here we replace $f(x)$ by $f^\lambda(x)$, and also p by $\frac{1}{\lambda}$, where λ is an arbitrarily small positive number, and make use of the relation

$$\|f^\lambda\|_p = \|f\|_{\lambda p}^\lambda. \tag{3.3}$$

We get

$$\left\|\left(\frac{1}{\varkappa_\infty}\int_{\mathbb{R}^n} k(x,y)f^\lambda(y)\,dy\right)^{\frac{1}{\lambda}}\right\|_{L^1(\mathbb{R}^n)} \leqslant \left(\frac{\varkappa_{n\lambda}}{\varkappa_\infty}\right)^{\frac{1}{\lambda}}\|f\|_{L^1(\mathbb{R}^n)}, \tag{3.4}$$

Denote for brevity

$$g_\lambda(x) = \frac{1}{\varkappa_\infty}\int_{\mathbb{R}^n} k(x,y)f^\lambda(y)\,dy.$$

Since $\lim_{\lambda\to 0} g_\lambda(x) = 1$, for almost all x we have

$$\lim_{\lambda\to 0}(g_\lambda(x))^{\frac{1}{\lambda}} = \lim_{\lambda\to 0} e^{\frac{\ln g_\lambda(x)}{\lambda}} = e^{\lim\limits_{\lambda\to 0}\frac{d}{d\lambda}\ln g_\lambda(x)}.$$

Hence

$$\lim_{\lambda\to 0}(g_\lambda(x))^{\frac{1}{\lambda}} = \exp\left(\frac{1}{\varkappa_\infty}\int_{\mathbb{R}^n} k(x,y)\ln f(y)\,dy\right). \tag{3.5}$$

Similarly

$$\lim_{\lambda\to 0}\left(\frac{\varkappa_{n\lambda}}{\varkappa_\infty}\right)^{\frac{1}{\lambda}} = e^{\frac{n}{\varkappa_\infty}\int_{\mathbb{R}^n} k(e_1,y)\ln\frac{1}{|y|}dy} \tag{3.6}$$

and from (3.4) we arrive at (3.1). □

REMARK 7. The relation (3.5) also follows from the well-known fact that the scale of powermeans

$$P_\lambda = \left(\frac{1}{\varkappa_\infty}\int\limits_{\mathbb{R}^n} k(x,y)g^\lambda(y)\,dy\right)^{1/\lambda}$$

decreases to the geometric mean

$$\exp\left(\frac{1}{\varkappa_\infty}\int\limits_{\mathbb{R}^n} k(x,y)\ln f(y)\,dy\right)$$

as $\lambda\to 0_+$. The relation (3.6) follows from the same principle.

3.1. Pólya-Knopp type inequality generated by the Riesz potential

In the following theorem we use the notation $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ for the Euler ψ-function, and define

$$B_n(a,b) := \frac{\gamma_n(a)\gamma_n(b)}{\gamma_n(a+b)}$$

where γ_n is defined by (2.8).

THEOREM 3. *Let* $0<\alpha<\nu<n$. *Then*

$$\int\limits_{\mathbb{R}^n}\exp\left(\frac{|x|^{\nu-\alpha}}{B_n(\alpha,n-\nu)}\int\limits_{\mathbb{R}^n}\frac{\ln f(y)\,dy}{|y|^\nu|x-y|^{n-\alpha}}\right)dx \leqslant e^{\varkappa^*}\int\limits_{\mathbb{R}^n} f(x)dx, \tag{3.7}$$

with

$$\varkappa^* = \frac{n}{2}\left[\psi\left(\frac{\nu-\alpha}{2}\right)+\psi\left(\frac{n-\nu+\alpha}{2}\right)-\psi\left(\frac{\nu}{2}\right)-\psi\left(\frac{n-\nu}{2}\right)\right] \tag{3.8}$$

The constant $e^{\varkappa^*}$ in (3.7) is sharp.

Proof. We have to derive the inequality (3.7) with its sharp constant (3.8) from the general inequality (3.1) when

$$k(x,y) = \frac{1}{\gamma_n(\alpha)}\frac{|x|^{\nu-\alpha}}{|y|^\nu|x-y|^{n-\alpha}}.$$

To this end, we first calculate the constant

$$\varkappa_\infty = \int_{\mathbb{R}^n} k(e_1,y)dy = \frac{1}{\gamma_n(\alpha)}\int_{\mathbb{R}^n}\frac{dy}{|y|^\nu|e_1-y|^{n-\alpha}} = I^\alpha\left(\frac{1}{|y|^\nu}\right)(e_1),$$

i.e. we see that it is the fractional integral of a power function, calculated at the point $x = e_1$. Values of such integrals are known:

$$I^\alpha\left(\frac{1}{|y|^\nu}\right)(x) = \frac{\gamma_n(n-\nu)}{\gamma_n(n+\alpha-\nu)}|x|^{\alpha-\nu}, \quad \alpha < \nu < n,$$

see [23, formula (25.38)]. Hence

$$\varkappa_\infty = \frac{\gamma_n(n-\nu)}{\gamma_n(n+\alpha-\nu)}.$$

To calculate $\varkappa^*$, we rewrite the above obtained expression for $\varkappa_\infty$, explicitly showing the dependence on the parameter ν:

$$\varkappa_\infty = \frac{1}{\gamma_n(\alpha)}\int_{\mathbb{R}^n}\frac{dy}{|y|^\nu|e_1-y|^{n-\alpha}} = 2^\alpha\frac{\Gamma\left(\frac{n-\nu}{2}\right)\Gamma\left(\frac{\nu-\alpha}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{n-\nu+\alpha}{2}\right)}. \tag{3.9}$$

We observe that

$$\frac{\partial}{\partial\nu}\varkappa_\infty = \frac{1}{\gamma_n(\alpha)}\int_{\mathbb{R}^n}\frac{\ln\frac{1}{|y|}dy}{|y|^\nu|e_1-y|^{n-\alpha}}.$$

Differentiating the last equality in (3.9) with respect to the parameter ν and dividing by $\varkappa_\infty$, after some straightforward calculations we arrive at the expression for $\varkappa^*$ written in (3.8), which completes the proof. □

COROLLARY 1. *In the case of Newtonian potential, i.e. in the case* $\alpha = 2$, *the sharp constant in the inequality* (3.7) *has the following value of* $\varkappa^*$:

$$\varkappa^* = \frac{n}{2}\frac{\nu - \frac{n+2}{2}}{(n-\nu)(\nu-2)}, \quad n \geqslant 3.$$

Proof. Use the property $\psi(z+1) = \psi(z) + \frac{1}{z}$ of the psi-function and the proof follows. □

3.2. A Pólya-Knopp type inequality generated by the multidimensional Hardy inequality

THEOREM 4. *Let* $\nu < n$. *Then*

$$\left\|\exp\left(\frac{n-\nu}{|\mathbb{S}^{n-1}|}|x|^{\nu-n}\int\limits_{|y|<|x|}\frac{\ln f(y)\,dy}{|y|^\nu}\right)\right\|_{L^1(\mathbb{R}^n)} \leqslant e^{\frac{n}{n-\nu}}\|f\|_{L^1(\mathbb{R}^n)}, \tag{3.10}$$

REMARK 8. (3.10) may be seen as a limit inequality of the Hardy inequality (2.11) with the sharp constant (2.14) in Proposition 1. A similar limit inequality of (2.12) can also be derived from Theorem 2.

REMARK 9. By letting $\nu = 0$ we obtain an n-dimensional variant of the classical Pólya-Knopp type inequality (1.7).

Proof. In this case we have $k(x,y) = \frac{|x|^{\nu-n}}{|y|^{\nu}}$ if $|y| < |x|$ and $k(x,y) = 0$ otherwise. A straightforward calculation yields that

$$\varkappa_\infty = \frac{|\mathbb{S}^{n-1}|}{n-\nu} \quad \text{and} \quad \varkappa^* = \frac{n}{n-nu}$$

and (3.1) turns into (3.10). □

4. Appendix: Proof of Theorem 1

The proof below is essentially borrowed from [11], but we include it for the readers convenience. First we present the following technical lemma:

LEMMA 1. *Under the assumptions* (2.3)-(2.4)*, the integrals*

$$\varkappa_p = \int_{\mathbb{R}^n} |k(\sigma,y)|\,|y|^{-\frac{n}{p}} dy, \ \sigma \in \mathbb{S}^{n-1}, \tag{4.1}$$

and

$$\kappa_p = \int_{\mathbb{R}^n} |k(x,\theta)|\,|x|^{-\frac{n}{p'}} dx, \ \ \theta \in \mathbb{S}^{n-1}, \tag{4.2}$$

where $0 < p \leqslant \infty$*, do not depend on* $\sigma \in \mathbb{S}^{n-1}$ *and* $\theta \in \mathbb{S}^{n-1}$*, respectively, and coincide with each other:* $\varkappa_p = \kappa_p$.

Proof. By $\omega_x(\eta)$, $\eta \in \mathbb{R}^n$, we denote any rotation in $\mathbb{R}^n$ such that $\omega_x(e_1) = \frac{x}{|x|}$, $e_1 = (1,0,\ldots,0)$. Then for $\xi = \omega_x(\eta)$ we have $|\xi| = |\eta|$ and $\xi \cdot \frac{x}{|x|} = \eta \cdot e_1 = \eta_1$.

Making the rotation change of variables $y = \omega_\sigma(z)$ in (4.1), using the rotation invariance $k(\sigma,y) \equiv k(e_1,z)$ of the kernel and the fact that $dy = dz$, we obtain that $\varkappa_p$ does not depend on σ. Similarly, the change of variables $x = \omega_\theta(z)$ in (4.2) shows that κ_p does not depend on θ.

We make use of the fact that $\varkappa_p$ does not depend on σ to rewrite $\varkappa_p$ as follows:

$$\varkappa_p = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} d\sigma \int_{\mathbb{S}^{n-1}} d\theta \int_0^\infty |k(\sigma,\rho\theta)|\rho^{n-1-\frac{n}{p}} d\rho\,.$$

By the homogeneity of the kernel and the change $\rho = \frac{1}{r}$, we get

$$\varkappa_p = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} d\sigma \int_{\mathbb{S}^{n-1}} d\theta \int_0^\infty |k(r\sigma,\theta)|r^{n-1-\frac{n}{p'}} dr\,.$$

Changing the order of integration in σ and θ, we see that the obtained inner integral (in σ and ρ) is equal to the integral defining $\varkappa_p$. Taking into account that $\varkappa_p$ does not depend on θ, we arrive at the equality $\varkappa_p = \kappa_p$. □

Proof of Theorem 1.

Sufficiency part. For the operator (2.2), by the Hölder inequality, we obtain

$$|(\mathbf{K}f)(x)| \leqslant \left\{\int_{R^n} |y|^{-\frac{n}{p}}|k(x,y)|dy\right\}^{\frac{1}{p'}} \left\{\int_{R^n} |y|^{\frac{n}{p'}}|k(x,y)|\cdot|f(y)|^p dy\right\}^{\frac{1}{p}} .$$

By the change of variables $y \to |x|y$ in the first integral, the homogeneity of the kernel $k(x,y)$ and Lemma 1, we obtain

$$|(\mathbf{K}f)(x)| \leqslant \frac{\varkappa_p^{\frac{1}{p'}}}{|x|^{\frac{n}{pp'}}} \left\{\int_{R^n} |y|^{\frac{n}{p'}}|k(x,y)|\cdot|f(y)|^p dy\right\}^{\frac{1}{p}} .$$

Then

$$\begin{aligned}\|\mathbf{K}f\|_p &\leqslant \varkappa_p^{\frac{1}{p'}} \left\{\int_{R^n} |f(y)|^p|y|^{\frac{n}{p'}} dy \int_{R^n} |k(x,y)|\cdot|x|^{-\frac{n}{p'}} dx\right\}^{\frac{1}{p}} \\ &= \varkappa_p^{\frac{1}{p'}} \left\{\int_{R^n} |f(y)|^p dy \int_{R^n} \left|k\left(x,\frac{y}{|y|}\right)\right|\cdot|x|^{-\frac{n}{p'}} dx\right\}^{\frac{1}{p}} = \varkappa_p^{\frac{1}{p'}}\kappa_p^{\frac{1}{p}}\|f\|_p.\end{aligned}$$

by the same Lemma 1, we arrive at (2.1) with $C(k,p) = \varkappa_p$.

Necessity part and the sharpness of the constant. Let now the kernel be non-negative. Suppose that (2.1) holds, i.e.the operator $\mathbf{K}$ is bounded. Then

$$\left|\int_{\mathbb{R}^n} (\mathbf{K}f)(x)\psi(x)dx\right| \leqslant \|\mathbf{K}\|\cdot\|f\|_p\|\psi\|_{p'} \tag{4.3}$$

for all $f \in L^p(\mathbb{R}^n)$ and $\psi \in L^{p'}(\mathbb{R}^n)$. We choose

$$f(x) = 0, \text{ if } |x| < 1, \quad \text{and} \quad f(x) = |x|^{-\varepsilon-\frac{n}{p}}, \text{ if } |x| \geqslant 1,$$

and $\psi(x) = [f(x)]^{p-1}$. Substituting this into (4.3), we get

$$\int_{\mathbb{S}^{n-1}} d\sigma \int_{\mathbb{R}^n} k(\sigma,y)|y|^{-\varepsilon-\frac{n}{p}} dy \int\limits_{r>\max(1,|y|^{-1})} r^{-p\varepsilon-1} dr \leqslant \|\mathbf{K}\|\cdot\|f\|_p^p . \tag{4.4}$$

Direct calculation yields

$$\|f\|_p^p = \frac{|\mathbb{S}^{n-1}|}{p\varepsilon}, \qquad \int\limits_{r>\max(1,|y|^{-1})} r^{-p\varepsilon-1} dr = \frac{1}{p\varepsilon}\left[\max(1,|y|^{-1}\right]^{-p\varepsilon},$$

so that the inequality (4.4) takes the form

$$\frac{1}{|\mathbb{S}^{n-1}|}\int_{\mathbb{S}^{n-1}} d\sigma \int_{\mathbb{R}^n} k(\sigma,y)|y|^{-\varepsilon-\frac{n}{p}}\left[\max(1,|y|^{-1}\right]^{-p\varepsilon} dy \leqslant \|\mathbf{K}\| . \tag{4.5}$$

By the rotation invariance of the kernel k, the inner integral in the left-hand side does not depend on σ, so that

$$\int_{\mathbb{R}^n} k(e_1,y)|y|^{-\varepsilon-\frac{n}{p}}\left[\max(1,|y|^{-1}\right]^{-p\varepsilon} dy \leqslant \|\mathbf{K}\|. \tag{4.6}$$

By the Fatou theorem (see e.g. [24]–[25] for Fatou theorem), we may pass to the limit as $\varepsilon \to 0$, which yields the inequality $\varkappa_p \leqslant \|\mathbf{K}\|$. Together with the inverse inequality proved in the sufficiency part, this gives the equality $\|\mathbf{K}\| = \varkappa_p$ and completes the proof. $\square$

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