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HAUSDORFF OPERATORS ON THE WEIGHTED HERZ–TYPE HARDY SPACES

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Abstract. In this paper, we study the high-dimensional Hausdorff operators on the weighted Herz-type Hardy spaces and obtain some substantial extensions from the previous results in [3]. Particularly, for the Hausdorff operators, we establish their sharp boundedness on the power weighted Herz-type Hardy spaces. Our results reveal that the Housdorff operators have better performance in the Herz-type Hardy spaces $H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ $(h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w))$ than their performance in the Hardy spaces $H^p(\mathbf{R}^n;w)$ $(h^p(\mathbf{R}^n;w))$ when $0<p<1$.

1. Introduction

Fix an integrable function ϕ on the half-line $(0,\infty)$. The classical Hausdorff operator h_ϕ with the kernel function ϕ is defined in the integral form by

$$h_\phi(f)(x) = \int_0^\infty \frac{\phi(t)}{t} f\left(\frac{x}{t}\right) dt,$$

for all Schwartz functions f. By a change of variables, one easily sees that for $x>0$,

$$h_\phi(f)(x) = \int_0^\infty \frac{\phi(x/t)}{t} f(t)\, dt.$$

Thus, many classical operators can be derived from h_ϕ if we choose suitable kernel functions ϕ (see [4, 5, 8, 29]). These operators include the Hardy operator, the adjoint Hardy operator [7, 10, 11] and the Cesàro operator [19, 38]. The Hardy-Littlewood-Pólya operator and the Riemann-Liouville fractional integral can also be derived from the Hausdorff operator.

For $n \geqslant 2$, a natural n-dimensional extension of h_ϕ is the operator

$$H_\Phi(f)(x) = \int_{\mathbf{R}^n} \frac{\Phi(y)}{|y|^n} f\left(\frac{x}{|y|}\right) dy,$$

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where Φ is a fixed integrable function. Same as h_ϕ, in the last 15 years, the operator H_Φ and its varieties received extensive studies. By the Minkowski inequality, it is easy to see that H_Φ is bounded on the Lebesgue space $L^p(\mathbf{R}^n)$ for all $1 \leqslant p \leqslant \infty$, provided that Φ satisfies the size condition

$$\int_{\mathbf{R}^n} |\Phi(y)|\,|y|^{-n+n/p}\,dy < \infty.$$

The same argument can show that H_Φ is also bounded on the real Hardy space $H^1(\mathbf{R}^n)$ if we assume that Φ is a Lebesgue integrable function on $\mathbf{R}^n$. The modern systematic study of Hausdorff operators was started by Liflyand and Móricz in [26]. For more involved situations about the Hausdorff operator on the Hardy spaces and other spaces, the reader can see [6, 12, 13, 21, 22, 23, 24, 27, 28, 39, 41, 43], among many others. However, except in the case $n = 1$, we can not find any paper in literature about the research of the Hausdorff operator H_Φ on the space $H^p(\mathbf{R}^n)$ when $n \geqslant 2$ and $0 < p < 1$. Even in the case $n = 1$, Liflyand and Miyachi [25] proved that there is a bounded function ϕ, whose support is contained in $[a,b] \subset (0,\ \infty)$ such that h_ϕ is not bounded on $H^p(\mathbf{R})$ for any $0 < p < 1$. Thus, finding a good substitute of H^p in which the Hausdorff operator is bounded becomes an interesting research topic.

In this paper, we will study a more general operator

$$H_{\Phi,A}(f)(x) = \int_{\mathbf{R}^n} \frac{\Phi(y)}{|y|^n} f(A(y)x)dy,$$

on the weighted Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ and on the weighted local Herz-type Hardy space $h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ for $0 < p < \infty$. Here, in the definition of $H_{\Phi,A}$, $A(y)$ is an $n \times n$ matrix satisfying $\det A(y) \neq 0$ almost everywhere in the support of Φ.

The operator $H_{\Phi,A}$ was initially studied by Lerner and Liflyand in [21] to study its boundedness on the Hardy space $H^1(\mathbf{R}^n)$. It is easy to see that the Hausdorff operator H_Φ is a special case of $H_{\Phi,A}$ if one chooses a special matrix $A(y)$. In [21], Lerner and Liflyand obtained the following theorem.

THEOREM A.

$$\|H_{\Phi,A}(f)\|_{H^1(\mathbf{R}^n)} \preceq \left(\int_{\mathbf{R}^n} \frac{|\Phi(y)|}{|y|^n} \left\|A^{-1}(y)\right\|^n dy\right) \|f\|_{H^1(\mathbf{R}^n)},$$

where $\left\|A^{-1}(y)\right\|$ *is an operator norm of* $A^{-1}(y)$ *to be specified later on.*

Besides the Lebesgue and Hardy spaces, the Herz-type spaces are other important function spaces raised in the research work from harmonic analysis and its related topics. In 1964, Beurling [2] first introduced some fundamental form of Herz spaces in order to study some convolution algebras. Four years later, Herz [18] gave versions of the spaces defined in a slightly different but more convenience setting. Since then, the theory of Herz spaces has been significantly developed, and these spaces have turned out to be quite useful in analysis. For example, they were used by Baernstein and Sawyer [1] to characterize the multipliers on the standard Hardy spaces, and used by Lu and Yang [35] in the study on certain partial differential equations.

On the other hand, the theory of Hardy spaces associated with Herz spaces has been developed in [15, 31]. These new Hardy spaces can be regarded as the local version at the origin of the classical Hardy spaces $H^p(\mathbf{R}^n)$ and are good substitutes for $H^p(\mathbf{R}^n)$ when we study the boundedness of non-translation invariant operators (see [32]). For the weighted case, Lu and Yang in 1995 introduced the weighted Herz-type Hardy spaces $H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ and established their atomic decompositions, Fan and Yang in 1997 introduced weighted local Herz-type Hardy spaces $h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ and established their atomic decompositions and Lee in 2006 found the molecular characterizations of weighted Herz-type Hardy spaces. We will recall the definitions of $H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ and $h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ and list their atomic characterizations in the second section. For proofs of the atomic decompositions and further details about these spaces, the reader can see the book [37] and the papers [9, 20, 30, 33, 34, 36].

Now, we find that the Herz-type Hardy space is not only a good substitute for $H^p(\mathbf{R}^n)$ when one studies the boundedness of non-translation invariant operators, but it also plays a quite different role from the Hardy space $H^p(\mathbf{R}^n)$, when $0<p<1$, in the study of the boundedness of the Hausdorff operator. In a recent paper, the authors obtained the following result:

THEOREM B. ([3]) *Let* $0<p\leqslant 1<q<\infty$, *and* $n(1-1/q)\leqslant\alpha<\infty$. *We have*

$$\|H_{\Phi,A}(f)\|_{H\dot{K}_q^{\alpha,1}(\mathbf{R}^n)}$$

$$\preceq \int_{\mathbf{R}^n}\frac{|\Phi(y)|}{|y|^n}\left\|A^{-1}(y)\right\|^{\alpha}\ \left|\det A^{-1}(y)\right|^{1/q}dy\ \|f\|_{H\dot{K}_q^{\alpha,1}(\mathbf{R}^n)},$$

and for $0<p<1$,

$$\|H_{\Phi,A}(f)\|_{H\dot{K}_q^{\alpha,p}(\mathbf{R}^n)}$$

$$\preceq \left(\int_{\mathbf{R}^n}\frac{|\Phi(y)|}{|y|^n}\left\|A^{-1}(y)\right\|^{\alpha}\ \left|\det A^{-1}(y)\right|^{1/q}\left(1+\left|\log\left\|A^{-1}(y)\right\|\right|\right)^{\sigma}dy\right)\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbf{R}^n)}$$

with $\sigma>\frac{1-p}{p}$.

The theorem reveals that, unlike its performance in the Hardy space H^p, the Hausdorff operator $H_{\Phi,A}$ is indeed bounded on the spaces $H\dot{K}_q^{\alpha,p}$ for all $0<p<1$, under only some size condition assumption on the kernel function Φ.

The main purpose of this paper is to further extend Theorem B to the weighted Herz type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$, as well as the weighted local Herz type Hardy space $h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$. Also, in the case $p=1$ and $w(x)=|x|^{\beta}$, $-n<\beta\leqslant 0$, we obtain the sharp size condition on Φ for the boundedness of $H_{\Phi,A}$ on the space $H\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|\cdot|^{\beta})$, under some reasonable assumption on the matrix A.

For an invertible matrix $B=(b_{ij})_{n\times n}$, set

$$\|B\|=\left(\sum_{i,j=1}^{n}|b_{ij}|^2\right)^{1/2}.$$

It is easy to see that $\|B\|$ is the norm of B and

$$\|B\|^{-n} \leqslant |\det\left(B^{-1}\right)| \leqslant \|B^{-1}\|^{n}. \tag{1}$$

Here and throughout this paper, we use the notation $A \preceq B$ to denote that there is a constant $C > 0$ independent of all essential values and variables such that $A \leqslant CB$. We use the notation $A \simeq B$, if there exist positive constants C and c, independent of all essential values and variables, such that $cB \leqslant A \leqslant CB$. Also, we use A_p to denote the set of all A_p weights whose definitions can be found in the next section.

Now we are in a position to state our results. Our first result is about the boundedness of $H_{\Phi,A}$ on the Herz type Hardy space.

THEOREM 1. *Let* $1 < q_i < \infty$, $n(1-1/q_i) \leqslant \alpha_i < \infty$, $i = 1, 2$ *and* $1/q_1 + \alpha_1/n = 1/q_2 + \alpha_2/n$. *Suppose that* $w \in A_1$ *with the critical index* r_w *for the reverse Hölder condition and* $q_1 > q_2 r_w/(r_w - 1)$.

(*i*) *If* $1 \leqslant p < \infty$, *then we have*

$$\|H_{\Phi,A}f\|_{H\dot{K}_{q_2}^{\alpha_2,p}(\mathbf{R}^n;w)} \preceq C_1 \|f\|_{H\dot{K}_{q_1}^{\alpha_1,p}(\mathbf{R}^n;w)},$$

where

$$\begin{aligned} C_1 = &\int_{\|A^{-1}(y)\|\geqslant 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} dy \\ &+ \int_{\|A^{-1}(y)\|<1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_2 q_2/q_1} dy. \end{aligned}$$

(*ii*) *If* $0 < p < 1$, *then we have, for any* $\sigma > (1-p)/p$,

$$\|H_{\Phi,A}f\|_{H\dot{K}_{q_2}^{\alpha_2,p}(\mathbf{R}^n;w)} \preceq C_2 \|f\|_{H\dot{K}_{q_1}^{\alpha_1,p}(\mathbf{R}^n;w)},$$

where

$$\begin{aligned} C_2 = &\int_{\|A^{-1}(y)\|\geqslant 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} \left(1 + \log \|A^{-1}(y)\|\right)^{\sigma} dy \\ &+ \int_{\|A^{-1}(y)\|<1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_2 q_2/q_1} \left(1 - \log \|A^{-1}(y)\|\right)^{\sigma} dy. \end{aligned}$$

Our second result is about the boundedness of $H_{\Phi,A}$ on the local Herz type Hardy space.

THEOREM 2. *Let* $1 < q_i < \infty$, $n(1-1/q_i) \leqslant \alpha_i < \infty$, $i = 1, 2$ *and* $1/q_1 + \alpha_1/n = 1/q_2 + \alpha_2/n$. *Suppose that* $w \in A_1$ *with the critical index* r_w *for the reverse Hölder condition and* $q_1 > q_2 r_w/(r_w - 1)$.

(*i*) *If* $1 \leqslant p < \infty$, *then we have*

$$\|H_{\Phi,A}f\|_{h\dot{K}_{q_2}^{\alpha_2,p}(\mathbf{R}^n;w)} \preceq C_3 \|f\|_{h\dot{K}_{q_1}^{\alpha_1,p}(\mathbf{R}^n;w)},$$

where

$$C_3 = \int_{\|A^{-1}(y)\| \geqslant 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} dy + \int_{\|A^{-1}(y)\| < 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} dy.$$

(*ii*) *If* $0 < p < 1$*, then we have, for any* $\sigma > (1-p)/p$*,*

$$\|H_{\Phi,A} f\|_{h\dot{K}_{q_2}^{\alpha_2,p}(\mathbf{R}^n;w)} \preceq C_4 \|f\|_{h\dot{K}_{q_1}^{\alpha_1,p}(\mathbf{R}^n;w)},$$

where

$$C_4 = \int_{\|A^{-1}(y)\| \geqslant 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} \left(1 + \log \|A^{-1}(y)\|\right)^{\sigma} dy + \int_{\|A^{-1}(y)\| < 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \max\left\{1, \|A^{-1}(y)\|^{\alpha_2 q_2/q_1} \left|\log \|A^{-1}(y)\|\right|^{\sigma}\right\} dy.$$

When the weight is reduced to the power function, we have the following results.

THEOREM 3. *Let* $1 < q < \infty$*,* $n(1-1/q) \leqslant \alpha < \infty$ *and* $-n < \beta \leqslant 0$*.*
(*i*) *If* $1 \leqslant p < \infty$*, then we have*

$$\|H_{\Phi,A} f\|_{H\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n;|\cdot|^{\beta})} \preceq C_5 \|f\|_{H\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n;|\cdot|^{\beta})},$$

where

$$C_5 = \int_{\boldsymbol{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} \|A^{-1}(y)\|^{\alpha(1+\beta/n)} dy.$$

(*ii*) *If* $0 < p < 1$*, then we have, for any* $\sigma > (1-p)/p$*,*

$$\|H_{\Phi,A} f\|_{H\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n;|\cdot|^{\beta})} \preceq C_6 \|f\|_{H\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n;|\cdot|^{\beta})},$$

where

$$C_6 = \int_{\boldsymbol{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} \|A^{-1}(y)\|^{\alpha(1+\beta/n)} \left(1 + \left|\log \|A^{-1}(y)\|\right|\right)^{\sigma} dy.$$

THEOREM 4. *Let* $1 < q < \infty$*,* $n(1-1/q) \leqslant \alpha < \infty$ *and* $-n < \beta \leqslant 0$*.*
(*i*) *If* $1 \leqslant p < \infty$*, then we have*

$$\|H_{\Phi,A} f\|_{h\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n;|\cdot|^{\beta})} \preceq C_7 \|f\|_{h\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n;|\cdot|^{\beta})},$$

where

$$C_7 = \int_{\|A^{-1}(y)\| < 1} \frac{|\Phi(y)|}{|y|^n} \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} dy + \int_{\|A^{-1}(y)\| \geqslant 1} \frac{|\Phi(y)|}{|y|^n} \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} \|A^{-1}(y)\|^{\alpha(1+\beta/n)} dy.$$

(*ii*) *If* $0 < p < 1$, *then we have, for any* $\sigma > (1-p)/p$,

$$\|H_{\Phi,A}f\|_{h\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n;|\cdot|^\beta)} \preceq C_8\|f\|_{h\dot{K}_q^{\alpha,p}(\boldsymbol{R}^n;|\cdot|^\beta)},$$

where

$$C_8 = \int_{\|A^{-1}(y)\|\geqslant 1} \frac{|\Phi(y)|}{|y|^n}\left(\|A(y)\|^{-\beta}|\det A^{-1}(y)|\right)^{1/q}\|A^{-1}(y)\|^{\alpha(1+\beta/n)}\left(1+\left|\log\|A^{-1}(y)\|\right|\right)^{\sigma}dy$$
$$+\int_{\|A^{-1}(y)\|<1}\frac{|\Phi(y)|}{|y|^n}\left(\|A(y)\|^{-\beta}|\det A^{-1}(y)|\right)^{1/q}\max\left\{1,\|A^{-1}(y)\|^{\alpha(1+\beta/n)}\left|\log\|A^{-1}(y)\|\right|^{\sigma}\right\}dy.$$

Especially in the case $p = 1$, if $\|A^{-1}(y)\|$ and $\|A(y)\|^{-1}$ are comparable, we can obtain the following sharp result.

THEOREM 5. *Let* $1 < q < \infty$, $n(1-1/q) \leqslant \alpha < \infty$, $-n < \beta \leqslant 0$ *and* Φ *be a nonnegative function. Suppose that all entries of the same row of* $A(y)$ *are nonnegative uniformly on* $y \in \mathrm{supp}(\Phi)$ *or non positive uniformly on* $y \in \mathrm{supp}(\Phi)$ *and there is a constant* C *independent of* y *such that* $\|A^{-1}(y)\| \leqslant C\|A(y)\|^{-1}$ *for all* $y \in \mathrm{supp}(\Phi)$. *Then* $H_{\Phi,A}$ *is bounded on* $H\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|\cdot|^\beta)$ *if and only if*

$$\int_{\mathbf{R}^n}\frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^{\alpha(1+\beta/n)+(n+\beta)/q}dy < \infty.$$

For the local space $h\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|\cdot|^\beta)$, we are able to obtain the following necessity for the boundedness of $H_{\Phi,A}$ on $h\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|\cdot|^\beta)$.

THEOREM 6. *Let* $1 < q < \infty$, $n(1-1/q) \leqslant \alpha < \infty$, $-n < \beta \leqslant 0$ *and* Φ *be a nonnegative function. Suppose that there is a constant* C *independent of* y *such that* $\|A^{-1}(y)\| \leqslant C\|A(y)\|^{-1}$ *for all* $y \in \mathrm{supp}(\Phi)$.

(*i*) *If* $H_{\Phi,A}$ *is bounded on* $h\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|\cdot|^\beta)$ *with* $\alpha(1+\beta/n)+(n+\beta)/q \neq n$, *then we have*

$$\int_{\|A^{-1}(y)\|\geqslant 1}\frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^{\alpha(1+\beta/n)+(n+\beta)/q}dy + \int_{\|A^{-1}(y)\|<1}\frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^n dy < \infty.$$

(*ii*) *If* $H_{\Phi,A}$ *is bounded on* $h\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|\cdot|^\beta)$ *with* $\alpha(1+\beta/n)+(n+\beta)/q = n$, *then we have*

$$\int_{\|A^{-1}(y)\|\geqslant 1}\frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^n dy + \int_{\|A^{-1}(y)\|<1}\frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^n\left(1-\log\|A^{-1}(y)\|\right)dy < \infty.$$

Finally in this section, we want to make a few remarks about our main theorems.

REMARK 1. Suppose $A(y) = \mathrm{diag}[1/\lambda_1(y), \dots, 1/\lambda_n(y)]$ with $\lambda_i(y) \neq 0$, for $i = 1, \dots, n$. Denote

$$M(y) = \max\{|\lambda_1(y)|, \dots, |\lambda_n(y)|\},\ m(y) = \min\{|\lambda_1(y)|, \dots, |\lambda_n(y)|\}.$$

If there is a constant $C \geqslant 1$ independent of y such that $M(y) \leqslant Cm(y)$, then it is easy to check that $A(y)$ satisfies the assumptions of Theorem 6. Furthermore, if we assume that, for all $i = 1, \dots, n$, $\lambda_i(y) > 0$ uniformly on $y \in \mathbf{R}^n$ or $\lambda_i(y) < 0$ uniformly on $y \in \mathbf{R}^n$, then $A(y)$ also satisfies the conditions of Theorem 5.

REMARK 2. Let α, β and q be as in Theorem 6. It is clear that $\|A^{-1}(y)\|^n \geqslant \|A^{-1}(y)\|^{(n+\beta)/q}$ if $\|A^{-1}(y)\| < 1$. Therefore, comparing with Theorem 4 (i) and Theorem 6, we raise an open question: Is the assumption in Theorem 4 sharp at the critical index $p = 1$?

In the second section, we will introduce some necessary notation and definitions, as well as some known results to be used later in the paper. We will prove the main theorems in Section 3.

2. Notation and Definitions

We start this section by recalling some standard definitions and notation. The theory of A_p weight was first introduced by Muckenhoupt in study of weighted L^p boundedness of Hardy-Littlewood maximal functions in [40]. A weight is a nonnegative, locally integrable function on $\mathbf{R}^n$.

DEFINITION 1. Let $1 < p < \infty$. We say that a weight $w \in A_p$ if there exists a constant C such that for all balls B,

$$\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-1/(p-1)}dx\right)^{p-1} \leqslant C.$$

We say that a weight $w \in A_1$ if there is a constant C such that for all balls B,

$$\frac{1}{|B|}\int_B w(x)\,dx \leqslant C \operatorname*{essinf}_{x\in B} w(x).$$

We define $A_\infty = \cup_{1\leqslant p<\infty} A_p$.

It is known that $A_p \subset A_r$ for all $r > p$, and that if $w \in A_p$ with $1 < p < \infty$ then $w \in A_q$ for some $1 < q < p$. Therefore, we may use $q_w := inf\{q > 1 : w \in A_q\}$ to denote the *critical index* of w. Obviously, if $w \in A_q$, $q > 1$, then we have $1 \leqslant q_w < q$. A close relation to A_∞ is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant C such that

$$\left(\frac{1}{|B|}\int_B w^r(x)dx\right)^{1/r} \leqslant \frac{C}{|B|}\int_B w(x)dx$$

for all balls $B \subset \mathbf{R}^n$, we then say that w satisfies the reverse Hölder condition of order r and write $w \in RH_r$. It is well known that $w \in A_\infty$ if and only if there exists some $r > 1$ such that $w \in RH_r$. Moreover, if $w \in RH_r$, $r > 1$, then $w \in RH_{r+\varepsilon}$ for some $\varepsilon > 0$. We thus write $r_w \equiv \sup\{r > 1 : w \in RH_r\}$ to denote the *critical index of w for the reverse Hölder condition.*

An important example of A_p weight is the power function $|x|^\alpha$. It is known that $|x|^\alpha$ is an A_1 weight if and only if $-n < \alpha \leqslant 0$. Also, $|x|^\alpha \in \cap_{(n+\alpha)/n<p<\infty} A_p$ if $0 < \alpha < \infty$, where $(n+\alpha)/n$ is called the critical index of $|x|^\alpha$.

We denote by $B(x,R)$ the Euclidean ball centered at x with radius R. For any $w \in A_\infty$ and any Lebesgue measurable set E, write $w(E) = \int_E w(x)dx$ and the Lebesgue measure of E by $|E|$. We have the following standard characterization of A_p weights (see [17] or [42]).

PROPOSITION 1. *Let $w \in A_p \cap RH_r$, $p \geqslant 1$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \left(\frac{|E|}{|B|}\right)^p \leqslant \frac{w(E)}{w(B)} \leqslant C_2 \left(\frac{|E|}{|B|}\right)^{(r-1)/r}$$

for any measurable subset E of a ball B. Especially, for any $\lambda > 1$,

$$w(B(x_0,\lambda R)) \leqslant C\lambda^{np} w(B(x_0,R)).$$

PROPOSITION 2. *Let f be a nonnegative locally integrable function. If $w \in A_p$, $p \geqslant 1$, then*

$$\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} f(x)\,dx \leqslant C\left(\frac{1}{w(B(x_0,R))}\int_{B(x_0,R)} f^p(x)\,w(x)\,dx\right)^{1/p}.$$

Given a weight function w on $\mathbf{R}^n$, as usual we denote by $L^p_w(\mathbf{R}^n)$ the weighted Lebesgue space of all functions satisfying

$$\|f\|_{L^p_w} = \left(\int_{\mathbf{R}^n} |f(x)|^p w(x)dx\right)^{1/p} < \infty.$$

We denote $L^\infty_w = L^\infty$ and $\|f\|_{L^\infty_w} = \|f\|_{L^\infty}$ for $p = \infty$.

Let $B_k = \{x \in \mathbf{R}^n : |x| \leqslant 2^k\}, D_k = B_k \setminus B_{k-1}$ and χ_{D_k} be the characteristic function of D_k for $k \in \mathbf{Z}$.

DEFINITION 2. Suppose $\alpha \in \mathbf{R}$, $0 < p$, $q < \infty$. Let w be a weight on $\mathbf{R}^n$. The homogeneous weighted Herz space $\dot{K}^{\alpha,p}_q(\mathbf{R}^n;w)$ is defined by

$$\dot{K}^{\alpha,p}_q(\mathbf{R}^n;w) = \left\{f \in L^q_{loc}(\mathbf{R}^n \setminus \{0\};w) : \|f\|_{\dot{K}^{\alpha,p}_q(\mathbf{R}^n;w)} < \infty\right\},$$

where

$$\|f\|_{\dot{K}^{\alpha,p}_q(\mathbf{R}^n;w)} = \left\{\sum_{k=-\infty}^{+\infty} w(B_k)^{\alpha p/n} \|f\chi_{D_k}\|^p_{L^q(\mathbf{R}^n,w)}\right\}^{1/p}.$$

Let φ be a function in $S(\mathbf{R}^n)$ (the class of Schwartz functions) satisfying

$$\int_{\mathbf{R}^n} \varphi(x)\,dx \neq 0.$$

Set

$$f^{++}(x) = \sup_{0<s} |\varphi_s * f(x)|, \; f^{+}(x) = \sup_{0<s<1} |\varphi_s * f(x)|,$$

where $\varphi_s(x) = \frac{1}{s^n}\varphi\left(\frac{x}{s}\right)$.

DEFINITION 3. Let $\alpha \in \mathbf{R}$, $0 < p < \infty$, $1 < q < \infty$ and $w \in A_1$. The homogeneous weighted Herz-type Hardy space $H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ and the homogeneous weighted local Herz-type Hardy space $h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ are defined, respectively, by

$$H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w) = \left\{f \in S'(\mathbf{R}^n) : f^{++} \in \dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)\right\},$$

$$h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w) = \left\{f \in S'(\mathbf{R}^n) : f^{+} \in \dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)\right\},$$

where

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)} = \|f^{++}\|_{\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)}, \|f\|_{h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)} = \|f^{+}\|_{\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)}.$$

It is known that the definitions of weighted (local) Herz-type Hardy spaces are flexible on the choice of the function φ. If $w=1$, we denote these spaces by $H\dot{K}_q^{\alpha,p}(\mathbf{R}^n)$ and $h\dot{K}_q^{\alpha,p}(\mathbf{R}^n)$, respectively. Obviously, $\dot{K}_p^{0,p}(\mathbf{R}^n) = L^p(\mathbf{R}^n)$, $H\dot{K}_p^{0,p}(\mathbf{R}^n) = H^p(\mathbf{R}^n)$ and $h\dot{K}_p^{0,p}(\mathbf{R}^n) = h^p(\mathbf{R}^n)$ for all $0 < p < \infty$, where $H^p(\mathbf{R}^n)$ and $h^p(\mathbf{R}^n)$ are the real Hardy space and the real local Hardy space (see [16]), respectively.

When $\alpha \geqslant n(1-1/q)$, the Herz-type Hardy space is an ideal space in place of the Herz space. If $0<p<\infty$, $1<q<\infty$, and $w \in A_1$, Lu and Yang [34] showed that

$$H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w) \cap L_{loc}^q(\mathbf{R}^n \setminus \{0\};w) = \dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$$

for $0 < \alpha < n(1-1/q)$, while

$$H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w) \cap L_{loc}^q(\mathbf{R}^n \setminus \{0\};w) \subset \dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$$

for $n(1-1/q) \leqslant \alpha < \infty$, and the inclusion is proper.

DEFINITION 4. Suppose $0<p<\infty$, $\gamma < 1 < q < \infty$, $n(1-1/q) \leqslant \alpha < \infty$, and $\tau \geqslant [\alpha + n(1/q-1)]$, where $[s]$ denotes the maximal integer less than s.

(i) A function $a(x)$ on $\mathbf{R}^n$ is said to be a central $(\alpha,q;w)$-atom if it satisfies

$$\operatorname{supp}(a) \subset B(0,r) = \{x \in \mathbf{R}^n : |x| < r\}, \tag{2}$$

$$\|a\|_{L^q(\mathbf{R}^n,w)} \leqslant w(B(0,r))^{-\alpha/n}, \tag{3}$$

$$\int_{\mathbf{R}^n} a(x)x^{\beta}\,dx = 0 \tag{4}$$

for all multi-indices β with $|\beta| \leqslant \tau$.

(ii) A function $a(x)$ on $\mathbf{R}^n$ is said to be a central $(\alpha,q;w)$- block if it satisfies (2) and (3).

Similar to the real Hardy spaces, the space $H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ and $h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ have atomic decompositions.

PROPOSITION 3. *Let $0<p<\infty$, $1<q<\infty$, $n(1-1/q)\leqslant\alpha<\infty$ and $w\in A_1$.*
(i) $f\in H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ if and only if

$$f=\sum_{k=-\infty}^{+\infty}\lambda_k a_k$$

in Schwartz's distributional sense, where $\sum_{k=-\infty}^{+\infty}|\lambda_k|^p<\infty$, and each a_k is a central $(\alpha,q;w)$-atom with support in B_k. Moreover,

$$\|f\|_{H\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)}\approx inf\left\{\left(\sum_{k=-\infty}^{+\infty}|\lambda_k|^p\right)\right\}^{1/p},$$

where the infimum is taken over all decompositions of f as above.
(ii) $f\in h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)$ if and only if

$$f=\sum_{k=-\infty}^{+\infty}\lambda_k a_k$$

in Schwartz's distributional sense, where $\sum_{k=-\infty}^{+\infty}|\lambda_k|^p<\infty$ and each a_k $(k\leqslant 0)$ is a central $(\alpha,q;w)$-atom and each a_k $(k>0)$ is a central $(\alpha,q;w)$-block with support in B_k. Moreover,

$$\|f\|_{h\dot{K}_q^{\alpha,p}(\mathbf{R}^n;w)}\approx inf\left\{\left(\sum_{k=-\infty}^{+\infty}|\lambda_k|^p\right)\right\}^{1/p},$$

where the infimum is taken over all decompositions of f as above.

3. Proof of the Theorems

3.1. Proof of Theorem 1

Firstly, we prove (i). According to part (i) of Proposition 3, any $f\in H\dot{K}_{q_1}^{\alpha_1,p}(\mathbf{R}^n;w)$ has an atomic decomposition

$$f=\sum_{k=-\infty}^{\infty}\lambda_k a_k,$$

where $\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p}\preceq\|f\|_{H\dot{K}_{q_1}^{\alpha_1,p}(\mathbf{R}^n;w)}$ and each a_k is a central $(\alpha_1,q_1;w)$-atom with support in B_k. Now

$$H_{\Phi,A}(f)=\sum_{k=-\infty}^{\infty}\lambda_k H_{\Phi,A}(a_k).$$

To prove the theorem, it suffices to show that

$$H_{\Phi,A}(a_k) = \sum_{j=-\infty}^{\infty} \widetilde{\lambda}_{kj} H_{\Phi,A}\left(a_{kj}\right),$$

where each $H_{\Phi,A}\left(a_{kj}\right)$ is a central $(\alpha_2, q_2; w)$-atom and

$$\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}|^p \preceq 1$$

uniformly on $k \in \mathbf{Z}$.

We rewrite

$$\begin{aligned} H_{\Phi,A}(a_k)(x) &= \sum_{j=-\infty}^{\infty} \int_{2^j \leqslant \|A^{-1}(y)\| < 2^{j+1}} \frac{\Phi(y)}{|y|^n} a_k(A(y)x)dy \\ &:= \sum_{j=-\infty}^{\infty} b_{kj}(x). \end{aligned} \tag{5}$$

It is easy to check that each b_{kj} satisfies the same cancellation condition as a_k. Without loss of generality, we may assume that supp $(a_k) \subset B(0,\rho)$. Hence we have supp $(a_k(A(y)\cdot)) \subset B(0, \|A^{-1}(y)\|\rho)$ and

$$\text{supp}(b_{kj}) \subset B(0, 2^{j+1}\rho).$$

Also, the size of b_{kj} is

$$\|b_{kj}\|_{L^{q_2}(\mathbf{R}^n;w)} \preceq \int_{2^j \leqslant \|A^{-1}(y)\| < 2^{j+1}} \frac{|\Phi(y)|}{|y|^n} \|a_k(A(y)\cdot)\|_{L^{q_2}(\mathbf{R}^n;w)} dy. \tag{6}$$

Since $q_1 > q_2 r_w/(r_w - 1)$, there is $1 < r < r_w$ such that $q_1 = q_2 r' = q_2 r/(r-1)$. By virtue of the reverse Hölder condition and Proposition 2, we obtain

$$\begin{aligned} &\|a_k(A(y)\cdot)\|_{L^{q_2}(\mathbf{R}^n;w)} \\ &\preceq \left(\int_{A^{-1}(y)B(0,\rho)} |a_k(A(y)x)|^{q_1} dx\right)^{1/q_1} \left(\int_{A^{-1}(y)B(0,\rho)} w^r(x)dx\right)^{1/(rq_2)} \\ &\preceq |\det A^{-1}(y)|^{1/q_1} \left(\int_{B(0,\rho)} |a_k(z)|^{q_1} dz\right)^{1/q_1} \left(\int_{B(0,\|A^{-1}(y)\|\rho)} w^r(x)dx\right)^{1/(rq_2)} \\ &\preceq |\det A^{-1}(y)|^{1/q_1} |B(0,\rho)|^{1/q_1} \left(\frac{1}{w(B(0,\rho))} \int_{B(0,\rho)} |a_k(z)|^{q_1} w(z)dz\right)^{1/q_1} \\ &\quad \times \frac{|B(0,\|A^{-1}(y)\|\rho)|^{1/(rq_2)}}{|B(0,\|A^{-1}(y)\|\rho)|^{1/q_2}} w(B(0,\|A^{-1}(y)\|\rho))^{1/q_2}. \end{aligned}$$

Hence

$$\begin{aligned}
&\|a_k(A(y)\cdot)\|_{L^{q_2}(\mathbf{R}^n;w)}\\
&\preceq |\det A^{-1}(y)|^{1/q_1}\frac{\|A^{-1}(y)\|^{n/(rq_2)}}{\|A^{-1}(y)\|^{n/q_2}}\frac{w\left(B(0,\|A^{-1}(y)\|\rho)\right)^{1/q_2}}{w(B(0,\rho))^{1/q_1}}\|a_k\|_{L^{q_1}(\mathbf{R}^n;w)}\\
&\preceq \left(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^n}\right)^{1/q_1}\frac{w\left(B(0,\|A^{-1}(y)\|\rho)\right)^{1/q_2}}{w(B(0,\rho))^{1/q_1}}w(B(0,\rho))^{-\alpha_1/n} \qquad (7)\\
&\preceq \left(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^n}\right)^{1/q_1}\left(\frac{w(B(0,2^{j+1}\rho))}{w(B(0,\rho))}\right)^{1/q_2+\alpha_2/n} w(B(0,2^{j+1}\rho))^{-\alpha_2/n}, \qquad (8)
\end{aligned}$$

where the last inequality is due to $1/q_2+\alpha_2/n=1/q_1+\alpha_1/n$.

When $j\geqslant 0$, (8) shows that

$$\begin{aligned}
\|a_k(A(y)\cdot)\|_{L^{q_2}(\mathbf{R}^n;w)} &\preceq 2^{jn(1/q_2+\alpha_2/n)}\left(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^n}\right)^{1/q_1} w(B(0,2^{j+1}\rho))^{-\alpha_2/n}\\
&\preceq |\det A^{-1}(y)|^{1/q_1}\|A^{-1}(y)\|^{\alpha_1}w(B(0,2^{j+1}\rho))^{-\alpha_2/n}. \qquad (9)
\end{aligned}$$

When $j=-1$, we have

$$\|a_k(A(y)\cdot)\|_{L^{q_2}(\mathbf{R}^n;w)}\preceq\left(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^n}\right)^{1/q_1} w(B(0,\rho))^{-\alpha_2/n}. \qquad (10)$$

When $j<-1$, by Proposition 1, (8) yields that

$$\begin{aligned}
\|a_k(A(y)\cdot)\|_{L^{q_2}(\mathbf{R}^n;w)} &\preceq \left(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^n}\right)^{1/q_1} 2^{jn(1/q_2+\alpha_2/n)/r'}w(B(0,2^{j+1}\rho))^{-\alpha_2/n}\\
&\preceq |\det A^{-1}(y)|^{1/q_1}\|A^{-1}(y)\|^{\alpha_2q_2/q_1}w(B(0,2^{j+1}\rho))^{-\alpha_2/n}. \qquad (11)
\end{aligned}$$

From (6)–(11), we have

$$\|b_{kj}\|_{L^{q_2}(\mathbf{R}^n;w)}\preceq\widetilde{\lambda}_{kj}\,w(B(0,2^{j+1}\rho))^{-\alpha_2/n},$$

where

$$\widetilde{\lambda}_{kj}=\int_{2^j\leqslant\|A^{-1}(y)\|<2^{j+1}}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|A^{-1}(y)\|^{\alpha_1}dy,\ \text{ if } \ j\geqslant 0$$

and

$$\widetilde{\lambda}_{kj}=\int_{2^j\leqslant\|A^{-1}(y)\|<2^{j+1}}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|A^{-1}(y)\|^{\alpha_2q_2/q_1}dy,\quad \text{if } \ j<0.$$

Let

$$b_{kj} = \widetilde{\lambda}_{kj} a_{kj}.$$

It is easy to check that each a_{kj} is a central $(\alpha_2, q_2; w)$-atom. Next we will show that $\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}|^p$ is uniformly bounded on $k \in \mathbf{Z}$.

When $1 \leqslant p < \infty$,

$$\begin{aligned}\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}|^p &\preceq \left(\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}| \right)^p \\ &\preceq \Bigg(\int_{\|A^{-1}(y)\| \geqslant 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} dy \\ &\quad + \int_{\|A^{-1}(y)\| < 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_2 q_2/q_1} dy \Bigg)^p. \end{aligned} \tag{12}$$

(i) of Theorem 1 now is proved.

It remains to prove part (ii). When $0 < p < 1$,

$$\begin{aligned}\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}|^p &= \sum_{j=1}^{\infty} |\widetilde{\lambda}_{kj}|^p + \sum_{j=-\infty}^{-1} |\widetilde{\lambda}_{kj}|^p + |\widetilde{\lambda}_{k0}|^p \\ &:= I + II + |\widetilde{\lambda}_{k0}|^p. \end{aligned} \tag{13}$$

Then

$$\begin{aligned} I &\preceq \left(\sum_{j=1}^{\infty} j^{\sigma} |\widetilde{\lambda}_{kj}| \right)^p \left(\sum_{j=1}^{\infty} j^{-(1-p)/(\sigma p)} \right)^{1/(1-p)} \\ &\preceq \left(\int_{\|A^{-1}(y)\| \geqslant 2} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} \left(\log \|A^{-1}(y)\| \right)^{\sigma} dy \right)^p. \end{aligned} \tag{14}$$

$$\begin{aligned} II &\preceq \left(\sum_{j=-\infty}^{-1} |j|^{\sigma} |\widetilde{\lambda}_{kj}| \right)^p \left(\sum_{j=-\infty}^{-1} |j|^{-(1-p)/(\sigma p)} \right)^{1/(1-p)} \\ &\preceq \left(\int_{\|A^{-1}(y)\| < 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_2 q_2/q_1} \left| \log \|A^{-1}(y)\| \right|^{\sigma} dy \right)^p. \end{aligned} \tag{15}$$

(13)–(15) yield that, if $0 < p < 1$,

$$\begin{aligned} &\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}|^p \\ &\preceq \Bigg(\int_{\|A^{-1}(y)\| \geqslant 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} \left(1 + \left| \log \|A^{-1}(y)\| \right| \right)^{\sigma} dy \\ &\quad + \int_{\|A^{-1}(y)\| < 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_2 q_2/q_1} \left(1 + \left| \log \|A^{-1}(y)\| \right| \right)^{\sigma} dy \Bigg)^p. \end{aligned} \tag{16}$$

The proof of theorem is completed. □

3.2. Proof of Theorem 2

Firstly, we prove (i). According to part (ii) of Proposition 3, any $f\in h\dot{K}_{q_1}^{\alpha_1,p}(\mathbf{R}^n;w)$ has an atomic decomposition

$$f=\sum\lambda_k a_k,$$

where $(\sum|\lambda_k|^p)^{1/p}\preceq\|f\|_{h\dot{K}_{q_1}^{\alpha_1,p}(\mathbf{R}^n;w)}$ and each a_k $(k\leqslant 0)$ is a central $(\alpha_1,q_1;w)$-atom and a_k $(k>0)$ is a central $(\alpha_1,q_1;w)$-block with support in B_k. By Theorem 1 (i), it suffices to show that, for any $k\geqslant 1$,

$$H_{\Phi,A}(a_k)=\sum_{j=-\infty}^{\infty}\widetilde{\lambda}_{kj}H_{\Phi,A}(a_{kj}),$$

where each $H_{\Phi,A}(a_{kj})$ is a central $(\alpha_2,q_2;w)$-block and

$$\sum_{j=-\infty}^{\infty}|\widetilde{\lambda}_{kj}|^p\preceq 1$$

uniformly on $k\in\mathbf{Z}^+$.

We rewrite

$$\begin{aligned}&(H_{\Phi,A}a_k)(x)\\&=\int_{\|A^{-1}(y)\|<2^{-k}}\frac{\Phi(y)}{|y|^n}a_k(A(y)x)dy+\sum_{j=-k}^{\infty}\int_{2^j\leqslant\|A^{-1}(y)\|<2^{j+1}}\frac{\Phi(y)}{|y|^n}a_k(A(y)x)dy\\&:=\widetilde{b}_k+\sum_{j=-k}^{\infty}b_{kj}(x).\end{aligned}\tag{17}$$

When $j\geqslant -k$, by the proof of Theorem 1 (i), it is easy to check that b_{kj} can be rewritten as

$$b_{kj}=\widetilde{\lambda}_{kj}a_{kj},$$

where each a_{kj} is an $(\alpha_2,q_2;w)$-block supported in $B(0,2^{j+k+1})$ with $\|a_{kj}\|_{h\dot{K}_{q_2}^{\alpha_2,p}(\mathbf{R}^n;w)}$ $\preceq 1$, and

$$\widetilde{\lambda}_{kj}=\begin{cases}\int_{2^j\leqslant\|A^{-1}(y)\|<2^{j+1}}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|A^{-1}(y)\|^{\alpha_1}dy, & j\geqslant 0,\\ \int_{2^j\leqslant\|A^{-1}(y)\|<2^{j+1}}\frac{|\Phi(y)|}{|y|^n}|\det A^{-1}(y)|^{1/q_1}\|A^{-1}(y)\|^{\alpha_2 q_2/q_1}dy, & -k\leqslant j<0.\end{cases}\tag{18}$$

Next we turn to estimate $\widetilde{b}_k$. It is easy to see that the support of $\widetilde{b}_k$ is in $B(0,1)$ and the size of it is

$$\|\widetilde{b}_k\|_{L^{q_2}(\mathbf{R}^n;w)}\preceq\int_{\|A^{-1}(y)\|<2^{-k}}\frac{|\Phi(y)|}{|y|^n}\|a_k(A(y)\cdot)\|_{L^{q_2}(\mathbf{R}^n;w)}dy.\tag{19}$$

Let r be as in the proof of Theorem 1 such that $q_1 = q_2 r'$. By (7), we have

$$\begin{aligned}
&\|a_k(A(y)\cdot)\|_{L^{q_2}(\mathbf{R}^n;w)} \\
&\preceq \left(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^n}\right)^{1/q_1} \frac{w\left(B(0,\|A^{-1}(y)\|2^k)\right)^{1/q_2} w(B(0,1))^{\alpha_2/n}}{w(B(0,2^k))^{1/q_1+\alpha_1/n}} w(B(0,1))^{-\alpha_2/n} \\
&\preceq \left(\frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^n}\right)^{1/q_1} \left(\frac{|B(0,\|A^{-1}(y)\|2^k)|}{|B(0,2^k)|}\right)^{1/(q_2 r')} w(B(0,1))^{-\alpha_2/n} \\
&\preceq |\det A^{-1}(y)|^{1/q_1} w(B(0,1))^{-\alpha_2/n},
\end{aligned} \tag{20}$$

where the second inequality is due to $k \geqslant 1$, $\|A^{-1}(y)\| < 2^{-k}$ and Proposition 1.

(19) and (20) imply that

$$\begin{aligned}
\|\widetilde{b}_k\|_{L^{q_2}(\mathbf{R}^n;w)} &\preceq \left(\int_{\|A^{-1}(y)\|<2^{-k}} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} dy\right) w(B(0,1))^{-\alpha_2/n} \\
&:= \widetilde{\lambda}_{k(-k-1)} w(B(0,1))^{-\alpha_2/n}.
\end{aligned} \tag{21}$$

Let $\widetilde{b}_k(x) = \widetilde{\lambda}_{k(-k-1)} a_{k(-k-1)}(x)$. (21) yields that $a_{k(-k-1)}$ is a central $(\alpha_2, q_2; w)$-block and

$$\|a_{k(-k-1)}\|_{h\dot{K}_{q_2}^{\alpha_2,p}(\mathbf{R}^n;w)} \preceq 1.$$

Let

$$\widetilde{\lambda}_{kj} \equiv 0,\ j = -k-2, -k-3, \ldots. \tag{22}$$

From (18), (20) and (22), we have that, for $1 \leqslant p < \infty$,

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}|^p &\preceq \left(\sum_{j=-k-1}^{\infty} |\widetilde{\lambda}_{kj}|\right)^p \\
&\preceq \left(\int_{\|A^{-1}(y)\|<2^{-k}} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} dy\right. \\
&\quad + \int_{\|A^{-1}(y)\|\geqslant 2} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} dy \\
&\quad \left. + \int_{2^{-k}\leqslant\|A^{-1}(y)\|<2} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_2 q_2/q_1} dy\right)^p \\
&\preceq \left(\int_{\|A^{-1}(y)\|<1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} dy\right. \\
&\quad \left. + \int_{\|A^{-1}(y)\|\geqslant 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} dy\right)^p,
\end{aligned} \tag{23}$$

and, for $0 < p < 1$ and $\sigma > p/(1-p)$,

$$\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}|^p \preceq \sum_{j=-k-1}^{\infty} |\widetilde{\lambda}_{kj}|^p$$

$$\preceq \Bigg(\int_{\|A^{-1}(y)\|<2^{-k}} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} dy$$

$$+ \int_{2^{-k}\leqslant\|A^{-1}(y)\|<2} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_2 q_2/q_1} \left(1+\left|\log\|A^{-1}(y)\|\right|\right)^{\sigma} dy$$

$$+ \int_{\|A^{-1}(y)\|\geqslant 2} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} \left(\log\|A^{-1}(y)\|\right)^{\sigma} dy \Bigg)^p$$

$$\preceq \Bigg(\int_{\|A^{-1}(y)\|<1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \max\left\{1, \|A^{-1}(y)\|^{\alpha_2 q_2/q_1} \left|\log\|A^{-1}(y)\|\right|^{\sigma}\right\} dy$$

$$+ \int_{\|A^{-1}(y)\|\geqslant 1} \frac{|\Phi(y)|}{|y|^n} |\det A^{-1}(y)|^{1/q_1} \|A^{-1}(y)\|^{\alpha_1} \left(1+\log\|A^{-1}(y)\|\right)^{\sigma} dy \Bigg)^p.$$

This completes the proof of Theorem 2. □

3.3. Proof of Theorem 3

Similar to the proof of Theorem 1, it suffices to show that, for every central $(\alpha, q; |\cdot|^{\beta})$-atom a_k supported in $B(0,\rho)$,

$$H_{\Phi,A}(a_k) = \sum_{j=-\infty}^{\infty} \widetilde{\lambda}_{kj} H_{\Phi,A}(a_{kj}),$$

where each $H_{\Phi,A}(a_{kj})$ is again a central $(\alpha, q; |\cdot|^{\beta})$-atom and

$$\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}|^p \preceq 1$$

uniformly on $k \in \mathbf{Z}$.

Let b_{kj} be as in (5). It is easy to see that it satisfies the same cancellation condition as a_k, $\mathrm{supp}(b_{kj}) \subset B(0, 2^{j+1}\rho)$ and the size of b_{kj} satisfies

$$\|b_{kj}\|_{L^q(\mathbf{R}^n;|\cdot|^{\beta})} \preceq \int_{2^j\leqslant\|A^{-1}(y)\|<2^{j+1}} \frac{|\Phi(y)|}{|y|^n} \|a_k(A(y)\cdot)\|_{L^q(\mathbf{R}^n;|\cdot|^{\beta})} dy. \tag{24}$$

Since

$$\|a_k(A(y)\cdot)\|_{L^q(\mathbf{R}^n;|\cdot|^{\beta})}$$

$$\preceq \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} \|a_k(\cdot)\|_{L^q(\mathbf{R}^n;|\cdot|^{\beta})}$$

$$\preceq \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} \rho^{-\alpha(n+\beta)/n} \tag{25}$$

$$\preceq \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} \|A^{-1}(y)\|^{\alpha(1+\beta/n)} \left(\int_{B(0,2^{j+1}\rho)} |x|^{\beta} dx\right)^{-\alpha/n}, \tag{26}$$

the last inequality is due to $2^j \leqslant \|A^{-1}(y)\| < 2^{j+1}$.

By (24) and (26), we have

$$\begin{aligned}
&\|b_{kj}\|_{L^q(\mathbf{R}^n;|\cdot|^\beta)} \\
&\preceq \left(\int_{B(0,2^{j+1}\rho)} |x|^\beta dx\right)^{-\alpha/n} \\
&\quad \times \left(\int_{2^j \leqslant \|A^{-1}(y)\| < 2^{j+1}} \frac{|\Phi(y)|}{|y|^n} \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} \|A^{-1}(y)\|^{\alpha(1+\beta/n)} dy\right) \\
&:= \widetilde{\lambda}_{kj} \left(\int_{B(0,2^{j+1}\rho)} |x|^\beta dx\right)^{-\alpha/n}. \qquad (27)
\end{aligned}$$

Let

$$b_{kj} = \widetilde{\lambda}_{kj} a_{kj}.$$

It is easy to check that each a_{kj} is a central $(\alpha, q; |\cdot|^\beta)$-atom. By a similar discussion as in (12) and (16) respectively, we have

$$\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}|^p \preceq \left(\int_{\mathbf{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} \|A^{-1}(y)\|^{\alpha(1+\beta/n)} dy\right)^p, \quad (28)$$

if $1 \leqslant p < \infty$ and

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} |\widetilde{\lambda}_{kj}|^p \qquad (29) \\
&\preceq \left(\int_{\mathbf{R}^n} \frac{|\Phi(y)|}{|y|^n} \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} \|A^{-1}(y)\|^{\alpha(1+\beta/n)} \left(1 + \left|\log \|A^{-1}(y)\|\right|\right)^\sigma dy\right)^p,
\end{aligned}$$

if $0 < p < 1$ and $(1-p)/p < \sigma$.

The proof of Theorem 3 is completed. □

3.4. Proof of Theorem 4

Since the proof is similar to the proofs of Theorem 2 and 3, we just sketch the outline of the proof. Let b_{kj} and $\widetilde{b}_k$ be as in (17). According to (26), we have, for $k \geqslant 1$ and $j \geqslant -k$,

$$\begin{aligned}
&\|b_{kj}(\cdot)\|_{L^q(\mathbf{R}^n;|\cdot|^\beta)} \\
&\preceq \left(\int_{B(0,2^{j+1}\rho)} |x|^\beta dx\right)^{-\alpha/n} \\
&\quad \times \left(\int_{2^j \leqslant \|A^{-1}(y)\| < 2^{j+1}} \frac{|\Phi(y)|}{|y|^n} \left(\|A(y)\|^{-\beta} |\det A^{-1}(y)|\right)^{1/q} \|A^{-1}(y)\|^{\alpha(1+\beta/n)} dy\right).
\end{aligned}$$

By (25), we have

$$\|\widetilde{b}_k(\cdot)\|_{L^q(\mathbf{R}^n;|\cdot|^\beta)} \preceq 2^{-k\alpha(n+\beta)/n}\int_{\|A^{-1}(y)\|<2^{-k}} \frac{|\Phi(y)|}{|y|^n}\left(\|A(y)\|^{-\beta}|\det A^{-1}(y)|\right)^{1/q} dy$$
$$\preceq \left(\int_{\|A^{-1}(y)\|<1} \frac{|\Phi(y)|}{|y|^n}\left(\|A(y)\|^{-\beta}|\det A^{-1}(y)|\right)^{1/q}\right)\left(\int_{B(0,1)}|x|^\beta dx\right)^{-\alpha/n}.$$

This completes the proof of the theorem. □

3.5. Proof of Theorem 5

If $\|A^{-1}(y)\| \preceq \|A(y)\|^{-1}$, then (1) gives that

$$\|A^{-1}(y)\|^n \simeq \|A(y)\|^{-n} \simeq |\det A^{-1}(y)|. \tag{30}$$

The "*if*" part (i) of Theorem 3 is easily obtained from Theorem 1. Next we will show the "*only if*" part. For simplicity, we show the case $n=2$, since the proof of case $n \geqslant 3$ is only notation difference and it does not require new idea.

Set $A(y) = (c_{ij}(y))_{2\times 2}$. Since for any given $i \in \{1,2\}$, $c_{i,1}(y)c_{i,2}(y) \geqslant 0$ for all $y \in \mathbf{R}^2$, without loss of generality, we assume $c_{1,1}(y)$, $c_{1,2}(y) \geqslant 0$ and $c_{2,1}(y)$, $c_{2,2}(y) \leqslant 0$. Then, for all vectors $x=(x_1,x_2)$ with $x_i>0$, $i=1,\ 2$, we have

$$A(y)x \in \{(z_1,z_2)\in \mathbf{R}^2 \mid z_1 \geqslant 0, z_2 \leqslant 0\}.$$

Let a be a function with support on $B(0,1)$ and satisfying

$$a(x_1,x_2) = \begin{cases} 1, & x_1x_2>0,\\ 0, & x_1x_2=0,\\ -1, & x_1x_2<0.\end{cases}$$

It is easy to check that a is a central $(\alpha,q;|\cdot|^\beta)$-atom and a satisfies

$$\|a(\cdot)\|_{H\dot{K}_q^{\alpha,1}(\mathbf{R}^2;|\cdot|^\beta)} \preceq 1.$$

Suppose

$$\int_{\mathbf{R}^2} \frac{\Phi(y)}{|y|^2}\|A^{-1}(y)\|^{\alpha(1+\beta/2)+(2+\beta)/q}dy = \infty.$$

If $H_{\Phi,A}$ is bounded on $H\dot{K}_q^{\alpha,1}(\mathbf{R}^2;|\cdot|^\beta)$, we have

$$\|H_{\Phi,A}(a)\|_{H\dot{K}_q^{\alpha,1}(\mathbf{R}^2;|\cdot|^\beta)} \preceq \|a\|_{H\dot{K}_q^{\alpha,1}(\mathbf{R}^2;|\cdot|^\beta)} \preceq 1.$$

On the other hand,

$$\|H_{\Phi,A}(a)\|_{H\dot{K}_q^{\alpha,1}(\mathbf{R}^2;|\cdot|^\beta)} \succeq \|H_{\Phi,A}(a)\|_{\dot{K}_q^{\alpha,1}(\mathbf{R}^2;|\cdot|^\beta)}$$
$$\succeq \sum_{k=-\infty}^{+\infty}\left(2^{k\alpha(2+\beta)/2}\|(H_{\Phi,A}a)\chi_{C_k}\|_{L^q(\mathbf{R}^2;|\cdot|^\beta)}\right). \tag{31}$$

Since

$$\|g\chi_{C_k}\|_{L^1(\mathbf{R}^2;|\cdot|^\beta)} \preceq \left(\int_{C_k} |g(x)|^q |x|^\beta dx\right)^{1/q} \left(\int_{C_k} |x|^\beta dx\right)^{1/q'}$$
$$\preceq 2^{k(2+\beta)/q'} \|g\chi_{C_k}\|_{L^q(\mathbf{R}^2;|\cdot|^\beta)}, \qquad (32)$$

(31) and (32) imply that

$$\begin{aligned}
&\|H_{\Phi,A}(a)\|_{H\dot{K}_q^{\alpha,1}(\mathbf{R}^2;|\cdot|^\beta)} \\
\succeq &\sum_{k=-\infty}^{+\infty} \left(2^{k(2+\beta)(\alpha/2-1/q')} \|(H_{\Phi,A}a)\chi_{C_k}\|_{L^1(\mathbf{R}^2;|\cdot|^\beta)}\right) \qquad (33)\\
\succeq &\int_{x_1\geqslant 0, x_2\geqslant 0} \left(\int_{\mathbf{R}^2} \frac{\Phi(y)}{|y|^2} |a(A(y)x)| dy\right) |x|^{\beta+(2+\beta)(\alpha/2-1/q')} dx \\
\succeq &\int_{\mathbf{R}^2} \frac{\Phi(y)}{|y|^2} \left(\int_{z_1\geqslant 0, z_2\leqslant 0} |a(z)||A^{-1}(y)z|^{\beta+(2+\beta)(\alpha/2-1/q')} d(A^{-1}(y)z)\right) dy \\
\succeq &\int_{\mathbf{R}^2} \frac{\Phi(y)}{|y|^2} |\det A^{-1}(y)| \|A^{-1}(y)\|^{\beta+(2+\beta)(\alpha/2-1/q')} \left(\int_0^1 \gamma^{\beta+(2+\beta)(\alpha/2-1/q')+1} d\gamma\right) dy \\
\succeq &\int_{\mathbf{R}^2} \frac{\Phi(y)}{|y|^2} |\det A^{-1}(y)| \|A^{-1}(y)\|^{\beta+(2+\beta)(\alpha/2-1/q')} dy \\
= &\int_{\mathbf{R}^2} \frac{\Phi(y)}{|y|^2} \|A^{-1}(y)\|^{\alpha(1+\beta/2)+(2+\beta)/q} dy = \infty.
\end{aligned}$$

This leads to a contradiction. □

3.6. Proof of Theorem 6

First, we prove (i). Suppose

$$\int_{\|A^{-1}(y)\|\geqslant 1} \frac{\Phi(y)}{|y|^n} \|A^{-1}(y)\|^{\alpha(1+\beta/n)+(n+\beta)/q} dy = \infty.$$

Let $\varphi(x) = e^{-|x|^2}$ and $b(x) = \chi_{B(0,1/2)}(x)$. An easy computation shows

$$\|b(x)\|_{h\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|x|^\beta)} \preceq 1.$$

By a similar discussion of the proof of "*only if*" part of Theorem 5, we have

$$\begin{aligned}
1 \succeq &\|H_{\Phi,A}b(x)\|_{h^1_{|x|^\alpha}} \\
\succeq &\int_{\mathbf{R}^n} \frac{\Phi(y)}{|y|^n} |\det A^{-1}(y)| \|A^{-1}(y)\|^{\beta+(n+\beta)(\alpha/n-1/q')} \left(\int_{|z\leqslant 1/2|} |z|^{\beta+(n+\beta)(\alpha/n-1/q')} dz\right) dy \\
\succeq &\int_{\mathbf{R}^n} \frac{\Phi(y)}{|y|^n} \|A^{-1}(y)\|^{(n+\beta)/q+\alpha(1+\beta/n)} \left(\int_0^{1/2} \gamma^{(n+\beta)/q+\alpha(n+\beta)/n-1} d\gamma\right) dy = \infty,
\end{aligned}$$

which leads to a contradiction.

Next, suppose

$$\int_{\|A^{-1}(y)\|<1} \frac{|\Phi(y)|}{|y|^n} \|A^{-1}(y)\|^n = \infty.$$

By (33), we have

$$\begin{aligned}
&\|H_{\Phi,A}b(x)\|_{h\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|x|^\beta)} \\
&= \sum_{k=-\infty}^{+\infty} \left(|\cdot|^\beta (B_k)^{\alpha/n} \|(H_{\Phi,A}b)^+ \chi_{C_k}\|_{L^q(\mathbf{R}^n;|\cdot|^\beta)}\right) \\
&\succeq \sum_{k=-\infty}^{0} \left(2^{k(n+\beta)(\alpha/n-1/q')} \int_{2^{k-1}\leqslant|x|<2^k} \sup_{0<s<1} |(\varphi_s * (H_{\Phi,A}b))(x)|\, |x|^\beta dx\right) \\
&\succeq \int_{|x|<1} \left|\left(\varphi_{|x|} * (H_{\Phi,A}b)\right)(x)\right| |x|^{\beta+(n+\beta)(\alpha/n-1/q')} dx.
\end{aligned}$$

Here

$$\left|\left(\varphi_{|x|} * (H_{\Phi,A}b)\right)(x)\right| \succeq \int_{\mathbf{R}^n} \frac{\Phi(y)}{|y|^n} |\det A^{-1}(y)| \left(\int_{|u|<1/4} \frac{1}{|x|^n} \varphi\left(\frac{|x-A^{-1}(y)u|}{|x|}\right) du\right) dy.$$

Therefore, we obtain

$$\begin{aligned}
&\|H_{\Phi,A}b(x)\|_{h\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|x|^\beta)} \\
&\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|^n} |\det A^{-1}(y)| \\
&\quad \times \int_{|u|<1/8} \left(\int_{|x|<1/2} \frac{1}{|x|^n} \varphi\left(\frac{|x-A^{-1}(y)u|}{|x|}\right) |x|^{\beta+(n+\beta)(\alpha/n-1/q')} dx\right) du dy \\
&\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|^n} \|A^{-1}(y)\|^{(n+\beta)/q+\alpha(1+\beta/n)} \\
&\quad \times \int_{|u|<1/8} \left(\int_{|z|<1/(2\|A^{-1}(y)\|)} \frac{|\det A^{-1}(y)|}{\|A^{-1}(y)\|^n |z|^n}\right. \\
&\quad \left.\times \varphi\left(\frac{|A^{-1}(y)(z-u)|}{|A^{-1}(y)z|}\right) |z|^{\beta+(n+\beta)(\alpha/n-1/q')} dz\right) du dy \\
&\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|^n} \|A^{-1}(y)\|^{(n+\beta)/q+\alpha(1+\beta/n)} \\
&\quad \times \int_{|u|<1/8} \left(\int_{1/8<|z|<1/(2\|A^{-1}(y)\|)} \varphi\left(\frac{|A^{-1}(y)(z-u)|}{|A^{-1}(y)z|}\right) |z|^{\beta+(n+\beta)(\alpha/n-1/q')-n} dz\right) du dy.
\end{aligned}$$

For any $|u| < 1/8$ and $1/8 < |z| < 1/(2\|A^{-1}(y)\|)$, we have

$$\frac{|A^{-1}(y)(z-u)|}{|A^{-1}(y)z|} \leqslant \frac{\|A^{-1}(y)\| |z-u|}{\|A(y)\|^{-1} |z|} \leqslant C \frac{|z|+|u|}{|z|} \leqslant 2C.$$

This implies that

$$\begin{aligned}
&\|H_{\Phi,A}b(x)\|_{h\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|x|^\beta)}\\
&\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^{(n+\beta)/q+\alpha(1+\beta/n)}\\
&\quad\times \int_{|u|<1/8}\left(\int_{1/8<|z|<1/(2\|A^{-1}(y)\|)} |z|^{\beta+(n+\beta)(\alpha/n-1/q')-n}dz\right) du dy \qquad (34)\\
&\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^{(n+\beta)/q+\alpha(1+\beta/n)}\\
&\quad\times \left(\int_{1/(4\|A^{-1}(y)\|)}^{1/(2\|A^{-1}(y)\|)} \rho^{\beta+(n+\beta)(\alpha/n-1/q')-1}d\rho\right) dy\\
&\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^{(n+\beta)/q+\alpha(1+\beta/n)}\frac{1}{\|A^{-1}(y)\|^{\beta+(n+\beta)(\alpha/n-1/q')}}dy\\
&\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^n dy = \infty.
\end{aligned}$$

We now complete the proof of part (i) of Theorem 6.

It remains to prove (ii). From (34), if $\alpha(1+\beta/n)+(n+\beta)/q = n$, we have

$$\begin{aligned}
\|H_{\Phi,A}b(x)\|_{h\dot{K}_q^{\alpha,1}(\mathbf{R}^n;|x|^\beta)} &\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^n \left(\int_{1/8}^{1/(2\|A^{-1}(y)\|)} \rho^{-1}d\rho\right) dy\\
&\succeq \int_{\|A^{-1}(y)\|<1} \frac{\Phi(y)}{|y|^n}\|A^{-1}(y)\|^n(1-\log\|A^{-1}(y)\|)dy = \infty.
\end{aligned}$$

The proof of the theorem is completed. □

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