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ON THE COMPACTNESS OF THE STEVIĆ–SHARMA OPERATOR ON THE LOGARITHMIC BLOCH SPACES

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Abstract. Let $H(\mathbb{D})$ denote the space of all analytic functions on the unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$, $\psi_1, \psi_2 \in H(\mathbb{D})$, and φ be an analytic self-map of $\mathbb{D}$. In this paper, we characterize the compactness of the Stević-Sharma operator on the logarithmic Bloch spaces.

1. Introduction

We begin with a brief review of relevant concepts and results in one complex variable. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$, $H(\mathbb{D})$ the class of all analytic functions on the unit disc.

Next we are ready to introduce the needed spaces. The logarithmic Bloch space is defined as follows [55]:

$$\mathscr{B}_{\log} = \left\{ f \in H(\mathbb{D}) : \|f\| = \sup_{z\in\mathbb{D}} (1-|z|^2)\left(\log \frac{2}{1-|z|}\right)|f'(z)| < \infty \right\}.$$

The space $\mathscr{B}_{\log}$ is a Banach space under the norm $\|f\|_{\mathscr{B}_{\log}} = |f(0)| + \|f\|$. Let $\mathscr{B}_{\log,0}$ denote the subspace of $\mathscr{B}_{\log}$ consisting of those $f \in \mathscr{B}_{\log}$ such that

$$\lim_{|z|\to 1} (1-|z|^2)\left(\log \frac{2}{1-|z|}\right)|f'(z)| = 0.$$

S. Ye in [52] proved that $\mathscr{B}_{\log,0}$ is a closed subspace of $\mathscr{B}_{\log}$. It is obvious that there are unbounded $\mathscr{B}_{\log}$ functions. For example, consider the function $f(z) = \log\log \frac{e}{1-z}$. It is easily proved that for $0 < \alpha < 1$, $\mathscr{B}^{\alpha} \subsetneqq \mathscr{B}_{\log} \subsetneqq \mathscr{B}$, where $\mathscr{B}^{\alpha}$ is the α-Bloch space. For some recent papers on logarithmic Bloch-type spaces and operators on them see, for example, [1, 3, 4, 9, 14, 27, 30, 33, 34, 39, 41, 42, 46, 48, 53].

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Let $\psi \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, then define the composition, multiplication and differentiation operators on $H(\mathbb{D})$ as follows:

$$(C_\varphi f)(z) = (f \circ \varphi)(z) = f(\varphi(z)), z \in \mathbb{D};$$
$$(M_\psi f)(z) = \psi(z) f(z), z \in \mathbb{D};$$
$$Df(z) = f'(z), z \in \mathbb{D}.$$

The differentiation operator is a typical example of an unbounded linear operator on many analytic function spaces. After studying above mentioned operators, some experts started studying their products. At the beginning a special interest attracted products formed by composition and differentiation operators (see, for example, [5, 11, 12, 13, 25, 32, 35, 44]). To unify the study of these product-type operators, operators in [37, 43, 63], with the study of weighted composition operators on spaces of analytic functions, S. Stević and A. Sharma introduced a new operator as follows.

For $\psi_1, \psi_2 \in H(\mathbb{D})$ and φ denotes an analytic self-map, let

$$T_{\psi_1,\psi_2,\varphi} f(z) = \psi_1(z) f(\varphi(z)) + \psi_2(z) f'(\varphi(z)),\ f \in H(\mathbb{D}).$$

The operator $T_{\psi_1,\psi_2,\varphi}$ was studied by S. Stević and co-workers for the first time in [49, 50] and later studied, for example, in [20, 57, 59]. This operator is related to the various products of multiplication, composition, and differentiation operators. It is clear that all products of composition, multiplication and differentiation operators in the following six ways can be obtained from the operator $T_{\psi_1,\psi_2,\varphi}$ by choosing different ψ_1, ψ_2. More specifically we have

$$M_\psi C_\varphi D = T_{0,\psi,\varphi};\ \ M_\psi D C_\varphi = T_{0,\psi\varphi',\varphi};\ C_\varphi M_\psi D = T_{0,\psi\circ\varphi,\varphi};$$
$$D M_\psi C_\varphi = T_{\psi',\psi\varphi,\varphi}; C_\varphi D M_\psi = T_{\psi'\circ\varphi,\psi\varphi,\varphi};\ D C_\varphi M_\psi = T_{\psi'\circ\varphi\varphi',(\psi\circ\varphi)\varphi',\varphi}.$$

Product-type operators on some spaces of analytic functions on the unit disk have been the object of study in several recent years (see, for example, [6, 7, 8, 10, 15, 16, 17, 18, 19, 26, 37, 54, 56, 62, 63] and also related references therein). Quite recently, Y. Yu and Y. Liu in [58] have considered the boundedness of the product-type operator $T_{\psi_1,\psi_2,\varphi}$ on logarithmic Bloch spaces. The study of product-type operators has attracted considerable interest recently. For some other original sources related to the operators on the unit disk or the unit ball in $\mathbb{C}^n$, see, for example, [28, 29, 31, 36, 38, 40, 45, 47, 51] as well as the following recent papers [21], [22], [60] and [61].

Inspired by the above results, the purpose of the paper is to study the compactness of the operator $T_{\psi_1,\psi_2,\varphi}$ on the logarithmic Bloch spaces $\mathscr{B}_{\log}$ ($\mathscr{B}_{\log,0}$). These results can be seen as extensions of our earlier results on this operator (see [58]), where Y. Yu and Y. Liu investigated the boundedness. Throughout the paper, the letter C denotes a positive constant which may vary at each occurrence but it is independent of the essential variables.

The paper is organized as follows. Section 2 contains lemmas needed to prove the main results. Section 3 considers the compactness of the operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log}(\mathscr{B}_{\log,0}) \to \mathscr{B}_{\log}$ and the compactness of the operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log}(\mathscr{B}_{\log,0}) \to \mathscr{B}_{\log,0}$.

2. Auxiliary results

For a better understanding, in this section we list up the following auxiliary results that are needed to prove our main results. The following lemma is folklore.

LEMMA 1. *Let $f \in \mathscr{B}_{\log}$. Then there is a positive constant C independent of f such that*

$$|f(z)| \leqslant C\left(2+\log\log\frac{2}{1-|z|}\right)\|f\|_{\mathscr{B}_{\log}}$$

and

$$|f(z)| \leqslant 2C\left(\log\log\frac{2}{1-|z|}\right)\|f\|_{\mathscr{B}_{\log}},\ |z| \geqslant 1-1/e^{e^2}.$$

The following lemma in [26, Lemma 1] plays an important role in characterizing the boundedness and the compactness of the operators under consideration in this paper.

LEMMA 2. *Let $f \in \mathscr{B}_{\log}$. Then there is a positive constant C independent of f such that*

$$|f^{(n)}(z)| \leqslant \frac{C\|f\|_{\mathscr{B}_{\log}}}{(1-|z|^2)^n \log\frac{2}{1-|z|}},$$

for every $z \in \mathbb{D}$, and all positive integer $n = 1,2,\cdots$.

The following three lemmas (see, for example, [58, Theorem 3.1, Theorem 3.2, Theorem 3.3]) are provided only for the convenience of the reader.

LEMMA 3. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$ and φ denote an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log} \to \mathscr{B}_{\log}$ *is bounded;*

(b) $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log,0} \to \mathscr{B}_{\log}$ *is bounded;*

(c) $\psi_1 \in \mathscr{B}_{\log}$,

$$M_1 := \sup_{z\in\mathbb{D}} (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)|\log\log\frac{2}{1-|\varphi(z)|} < \infty, \tag{1}$$

$$M_2 := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}} < \infty, \tag{2}$$

and

$$M_3 := \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2\log\frac{2}{1-|\varphi(z)|}} < \infty. \tag{3}$$

LEMMA 4. *Let* $\psi_1, \psi_2 \in H(\mathbb{D})$ *and* φ *denote an analytic self-map of* $\mathbb{D}$. *Then* $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log,0} \to \mathscr{B}_{\log,0}$ *is a bounded operator if and only if* $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log,0} \to \mathscr{B}_{\log}$ *is a bounded operator,*

$$\lim_{|z|\to 1}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)| = 0, \tag{4}$$

$$\lim_{|z|\to 1}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z)+\psi_2'(z)| = 0, \tag{5}$$

and

$$\lim_{|z|\to 1}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)| = 0. \tag{6}$$

LEMMA 5. *Let* $\psi_1, \psi_2 \in H(\mathbb{D})$ *and* φ *denote an analytic self-map of* $\mathbb{D}$. *If* $\psi_1 \in \mathscr{B}_{\log,0}$,

$$\lim_{|z|\to 1}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)|\log\log\frac{2}{1-|\varphi(z)|} = 0,$$

$$\lim_{|z|\to 1}\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}} = 0,$$

and

$$\lim_{|z|\to 1}\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2\log\frac{2}{1-|\varphi(z)|}} = 0,$$

then $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log} \to \mathscr{B}_{\log,0}$ *is bounded.*

The following criterion for the compactness follows by standard arguments (see, for example, the proofs of the corresponding lemmas in [2, Proposition 3.11]). The details will not be pursued here.

LEMMA 6. *Let* $\psi_1, \psi_2 \in H(\mathbb{D})$ *and* φ *denote an analytic self-map of* $\mathbb{D}$. *Then* $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log}(\mathscr{B}_{\log,0}) \to \mathscr{B}_{\log}$ *is compact if and only if* $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log}(\mathscr{B}_{\log,0}) \to \mathscr{B}_{\log}$ *is bounded and for any bounded sequence* $\{f_n\}$ *in* $\mathscr{B}_{\log}(\mathscr{B}_{\log,0})$ *which converges to zero uniformly on compact subsets of* $\mathbb{D}$ *as* $n \to \infty$, *we have* $\|T_{\psi_1,\psi_2,\varphi} f_n\|_{\mathscr{B}_{\log}} \to 0$ *as* $n \to \infty$.

The following lemma can be proved similar to Lemma 1 in [23] (see, also [24]). The details are omitted.

LEMMA 7. *A closed set* K *in* $\mathscr{B}_{\log,0}$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1}\sup_{f\in K}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|f'(z)| = 0.$$

3. The compactness of the operator $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log}(\mathscr{B}_{\log,0})\to\mathscr{B}_{\log}(\mathscr{B}_{\log,0})$

First we characterize the compactness of the operator $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log}(\mathscr{B}_{\log,0})\to\mathscr{B}_{\log}$.

THEOREM 1. *Let $\psi_1,\psi_2\in H(\mathbb{D})$ and φ denote an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log}\to\mathscr{B}_{\log}$ *is compact;*
(b) $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log,0}\to\mathscr{B}_{\log}$ *is compact;*
(c) $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log}\to\mathscr{B}_{\log}$ *is bounded,*

$$\lim_{|\varphi(z)|\to 1}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)|\log\log\frac{2}{1-|\varphi(z)|}=0, \tag{7}$$

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}}=0, \tag{8}$$

and

$$\lim_{|\varphi(z)|\to 1}\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2\log\frac{2}{1-|\varphi(z)|}}=0. \tag{9}$$

Proof. $(c)\Rightarrow(a)$. Suppose that $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log}\to\mathscr{B}_{\log}$ is bounded, and (7), (8), and (9) hold. To prove that $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log}\to\mathscr{B}_{\log}$ is compact, for any bounded sequence $\{f_k\}$ in $\mathscr{B}_{\log}$ with $f_k\to 0$ uniformly on compact subsets of $\mathbb{D}$, let $\|f_k\|_{\mathscr{B}_{\log}}\leqslant 1$, it suffices, in view of Lemma 6, to show that

$$\|T_{\psi_1,\psi_2,\varphi}f_k\|_{\mathscr{B}_{\log}}\to 0 \text{ as } k\to\infty.$$

By (7), (8), and (9), we have for any $\varepsilon>0$, there exists $\rho\in(1/e^{e^2},1)$ such that

$$(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)\log\log\frac{2}{1-|\varphi(z)|}|\psi_1'(z)|<\varepsilon, \tag{10}$$

$$\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}}<\varepsilon, \tag{11}$$

and

$$\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2\log\frac{2}{1-|\varphi(z)|}}<\varepsilon, \tag{12}$$

for $\rho < |\varphi(z)| < 1$. Since the operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log,0} \to \mathscr{B}_{\log}$ is bounded, by Lemma 3 one has $\psi_1 \in \mathscr{B}_{\log}$, (1), (2), and (3) hold. Since $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$, Cauchy's estimate shows that f_k' and f_k'' converge to 0 uniformly on compact subsets of $\mathbb{D}$, there exists a $K_0 \in \mathbb{N}$ such that $k > K_0$ implies that

$$\begin{aligned}
&|(T_{\psi_1,\psi_2,\varphi} f_k)(0)| + \sup_{|\varphi(z)|\leqslant\rho} (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|(T_{\psi_1,\psi_2,\varphi} f_k)'(z)| \\
&\leqslant |\psi_1(0)f_k(\varphi(0)) + \psi_2(0)f_k'(\varphi(0))| \\
&\quad + \sup_{|\varphi(z)|\leqslant\rho} (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)||f_k(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|\leqslant\rho} (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z) + \psi_2'(z)||f_k'(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|\leqslant\rho} (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)||f_k''(\varphi(z))| \\
&\leqslant |\psi_1(0)||f_k(\varphi(0))| + |\psi_2(0)f_k'(\varphi(0))| + \|\psi_1\|_{\mathscr{B}_{\log}} \sup_{|\varphi(z)|\leqslant\rho} |f_k(\varphi(z))| \\
&+ M_2\left(1-\rho^2\right)\log\frac{2}{1-\rho}\sup_{|\varphi(z)|\leqslant\rho}|f_k'(\varphi(z))| + M_3\left(1-\rho^2\right)^2\log\frac{2}{1-\rho}\sup_{|\varphi(z)|\leqslant\rho}|f_k''(\varphi(z))| \\
&< C\varepsilon.
\end{aligned} \tag{13}$$

When $k > K_0$, from (10), (11), (12), (13), Lemmas 1, and 2, one has

$$\begin{aligned}
&\|T_{\psi_1,\psi_2,\varphi} f_k\|_{\mathscr{B}_{\log}} \\
&= |\left(T_{\psi_1,\psi_2,\varphi} f_k\right)(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)\left|\left(T_{\psi_1,\psi_2,\varphi} f_k\right)'(z)\right| \\
&\leqslant \left(|\left(T_{\psi_1,\psi_2,\varphi} f_k\right)(0)| + \sup_{|\varphi(z)|\leqslant\rho}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)\left|\left(T_{\psi_1,\psi_2,\varphi} f_k\right)'(z)\right|\right) \\
&\quad + \sup_{\rho<|\varphi(z)|<1}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)\left|\left(T_{\psi_1,\psi_2,\varphi} f_k\right)'(z)\right| \\
&< C\varepsilon + 2C\sup_{\rho<|\varphi(z)|<1}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)\log\log\frac{2}{1-|\varphi(z)|}|\psi_1'(z)|\|f_k\|_{\mathscr{B}_{\log}} \\
&\quad + C\sup_{\rho<|\varphi(z)|<1}\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}}\|f_k\|_{\mathscr{B}_{\log}} \\
&\quad + C\sup_{\rho<|\varphi(z)|<1}\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2\log\frac{2}{1-|\varphi(z)|}}\|f_k\|_{\mathscr{B}_{\log}} \\
&< 5C\varepsilon,
\end{aligned}$$

it follows that the operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log} \to \mathscr{B}_{\log}$ is compact.

$(a) \Rightarrow (b)$. It is obvious.

$(b) \Rightarrow (c)$. It is clear that the compactness of $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log,0} \to \mathscr{B}_{\log}$ implies the boundedness of $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log,0} \to \mathscr{B}_{\log}$. By Lemma 3, $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log} \to \mathscr{B}_{\log}$ is bounded. If $\|\varphi\|_\infty < 1$, it is clear that the limit in (7), (8), and (9) is vacuously equal to zero. Hence, assume that $\|\varphi\|_\infty = 1$, let $\{z_k\}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. We can use the test functions

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-z\overline{\varphi(z_k)})\log\frac{2}{1-|\varphi(z_k)|}} - \frac{2(1-|\varphi(z_k)|^2)^2}{\left(1-z\overline{\varphi(z_k)}\right)^2\log\frac{2}{1-|\varphi(z_k)|}} + \frac{(1-|\varphi(z_k)|^2)^3}{\left(1-z\overline{\varphi(z_k)}\right)^3\log\frac{2}{1-|\varphi(z_k)|}}.$$

By (3.14) and (3.15) in [58] we have $\sup\limits_{k\in\mathbb{N}}\|f_k\|_{\mathscr{B}_{\log}} \leqslant C$ and $f_k \in \mathscr{B}_{\log,0}$. For $|z|<1$, we have

$$\begin{aligned}
|f_k(z)| &\leqslant \left|\frac{1-|\varphi(z_k)|^2}{\left(1-z\overline{\varphi(z_k)}\right)\log\frac{2}{1-|\varphi(z_k)|}}\right| + \left|\frac{2(1-|\varphi(z_k)|^2)^2}{\left(1-z\overline{\varphi(z_k)}\right)^2\log\frac{2}{1-|\varphi(z_k)|}}\right| \\
&\quad + \left|\frac{(1-|\varphi(z_k)|^2)^3}{\left(1-z\overline{\varphi(z_k)}\right)^3\log\frac{2}{1-|\varphi(z_k)|}}\right| \\
&\leqslant \frac{(1+|\varphi(z_k)|)(1-|\varphi(z_k)|)}{(1-|\varphi(z_k)|)\log\frac{2}{1-|\varphi(z_k)|}} + \frac{2(1+|\varphi(z_k)|)^2(1-|\varphi(z_k)|)^2}{(1-|\varphi(z_k)|)^2\log\frac{2}{1-|\varphi(z_k)|}} \\
&\quad + \frac{(1+|\varphi(z_k)|)^3(1-|\varphi(z_k)|)^3}{(1-|\varphi(z_k)|)^3\log\frac{2}{1-|\varphi(z_k)|}} \\
&\leqslant \frac{2}{\log\frac{2}{1-|\varphi(z_k)|}} + \frac{8}{\log\frac{2}{1-|\varphi(z_k)|}} + \frac{8}{\log\frac{2}{1-|\varphi(z_k)|}} \\
&= \frac{18}{\log\frac{2}{1-|\varphi(z_k)|}} \to 0 \text{ as } k\to\infty.
\end{aligned}$$

We see that f_k converges to 0 uniformly on $\mathbb{D}$, hence f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then f_k is a bounded sequence in $\mathscr{B}_{\log,0}$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$. By Lemma 6 we obtain

$$\lim_{k\to\infty}\|T_{\psi_1,\psi_2,\varphi}f_k\|_{\mathscr{B}_{\log}} = 0.$$

Using (3.11), (3.12), and (3.13) in [58], we have

$$f_k(\varphi(z_k)) = f_k'(\varphi(z_k)) = 0, \quad f_k''(\varphi(z_k)) = \frac{2\left(\overline{\varphi(z_k)}\right)^2}{\left(1-|\varphi(z_k)|^2\right)^2\log\frac{2}{1-|\varphi(z_k)|}}.$$

From the compactness of the operator $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log,0}\to\mathscr{B}_{\log}$ we get

$$\begin{aligned}&\frac{2(1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_2(z_k)\varphi'(z_k)|\left|\overline{\varphi(z_k)}\right|^2}{\left(1-|\varphi(z_k)|^2\right)^2\log\frac{2}{1-|\varphi(z_k)|}}\\&\leqslant\|T_{\psi_1,\psi_2,\varphi}f_k\|_{\mathscr{B}_{\log}}\to 0\ \text{ as } k\to\infty.\end{aligned}\tag{14}$$

By (14) and $|\varphi(z_k)|\to 1$ we have

$$\lim_{k\to\infty}\frac{(1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_2(z_k)\varphi'(z_k)|}{\left(1-|\varphi(z_k)|^2\right)^2\log\frac{2}{1-|\varphi(z_k)|}}=0.$$

This proves (9).

Next, let

$$\begin{aligned}g_k(z)=&\frac{3(1-|\varphi(z_k)|^2)}{\left(1-z\overline{\varphi(z_k)}\right)\log\frac{2}{1-|\varphi(z_k)|}}\\&-\frac{5(1-|\varphi(z_k)|^2)^2}{\left(1-z\overline{\varphi(z_k)}\right)^2\log\frac{2}{1-|\varphi(z_k)|}}+\frac{2(1-|\varphi(z_k)|^2)^3}{\left(1-z\overline{\varphi(z_k)}\right)^3\log\frac{2}{1-|\varphi(z_k)|}}.\end{aligned}$$

It is easy to check that g_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, $g_k\in\mathscr{B}_{\log,0}$ and $\sup_{k\in\mathbb{N}}\|g_k\|_{\mathscr{B}_{\log}}\leqslant C$. By Lemma 6 we have

$$\lim_{k\to\infty}\|T_{\psi_1,\psi_2,\varphi}g_k\|_{\mathscr{B}_{\log}}=0.$$

In addition, one has $g_k(\varphi(z_k))=g_k''(\varphi(z_k))=0$ and

$$g_k'(\varphi(z_k))=-\frac{\overline{\varphi(z_k)}}{(1-|\varphi(z_k)|^2)\log\frac{2}{1-|\varphi(z_k)|}}.$$

Using the compactness of the operator $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log,0}\to\mathscr{B}_{\log}$ we get

$$\begin{aligned}&\frac{(1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_1(z_k)\varphi'(z_k)+\psi_2'(z_k)|\left|\overline{\varphi(z_k)}\right|}{(1-|\varphi(z_k)|^2)\log\frac{2}{1-|\varphi(z_k)|}}\\&\leqslant\|T_{\psi_1,\psi_2,\varphi}g_k\|_{\mathscr{B}_{\log}}\to 0\ \text{ as } k\to\infty.\end{aligned}\tag{15}$$

By (15) and $|\varphi(z_k)|\to 1$ we have

$$\lim_{k\to\infty}\frac{(1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_1(z_k)\varphi'(z_k)+\psi_2'(z_k)|}{(1-|\varphi(z_k)|^2)\log\frac{2}{1-|\varphi(z_k)|}}=0,$$

it implies that (8) holds.

Now we prove (7). Choose

$$h_k(z) = a_k^{-1}\left(\log\log\frac{4}{1-z\overline{\varphi(z_k)}}\right)^2,$$

where $a_k = \log\log\frac{4}{1-|\varphi(z_k)|^2}$. Then

$$h_k'(z) = \frac{2\log\log\frac{4}{1-z\overline{\varphi(z_k)}}}{a_k}\frac{\overline{\varphi(z_k)}}{\left(1-z\overline{\varphi(z_k)}\right)\log\frac{4}{1-z\overline{\varphi(z_k)}}}$$

and

$$\begin{aligned} h_k''(z) = {} & \frac{2}{a_k}\frac{\left(\overline{\varphi(z_k)}\right)^2}{\left(1-z\overline{\varphi(z_k)}\right)^2\left(\log\frac{2}{1-z\overline{\varphi(z_k)}}\right)^2} \\ & + \frac{2\log\log\frac{4}{1-z\overline{\varphi(z_k)}}}{a_k}\frac{\left(\overline{\varphi(z_k)}\right)^2}{\log\frac{4}{1-z\overline{\varphi(z_k)}}\left(1-z\overline{\varphi(z_k)}\right)^2}\left(1-\frac{1}{\log\frac{4}{1-z\overline{\varphi(z_k)}}}\right). \end{aligned}$$

Thus,

$$h_k(\varphi(z_k)) = \log\log\frac{4}{1-|\varphi(z_k)|^2} \geqslant \log\log\frac{2}{1-|\varphi(z_k)|},$$

$$|h_k'(\varphi(z_k))| \leqslant \left|\frac{2\overline{\varphi(z_k)}}{(1-|\varphi(z_k)|^2)\log\frac{4}{1-|\varphi(z_k)|^2}}\right| \leqslant \frac{2}{(1-|\varphi(z_k)|^2)\log\frac{2}{1-|\varphi(z_k)|}} \tag{16}$$

and

$$\begin{aligned} & |h_k''(\varphi(z_k))| \\ \leqslant {} & \left|\frac{2}{(1-|\varphi(z_k)|^2)^2\left(\log\frac{4}{1-|\varphi(z_k)|^2}\right)^2\log\log\frac{4}{1-|\varphi(z_k)|^2}}\right| \\ & + \left|\frac{2}{(1-|\varphi(z_k)|^2)^2\log\frac{4}{1-|\varphi(z_k)|^2}}\right|\left(1-\frac{1}{\log\frac{4}{1-|\varphi(z_k)|^2}}\right) \\ \leqslant {} & \frac{2}{(1-|\varphi(z_k)|^2)^2\left(\log\frac{2}{1-|\varphi(z_k)|}\right)^2\log\log 4} + \frac{2}{(1-|\varphi(z_k)|^2)^2\log\frac{2}{1-|\varphi(z_k)|}} \\ \leqslant {} & \frac{\frac{2}{\log\log 4}+2}{(1-|\varphi(z_k)|^2)^2\log\frac{2}{1-|\varphi(z_k)|}}. \end{aligned} \tag{17}$$

By a direct calculation, we may easily prove that h_k converges to 0 uniformly on compact subsets of $\mathbb{D}$, $h_k \in \mathscr{B}_{\log,0}$ and $\sup_{k\in\mathbb{N}} \|h_k\|_{\mathscr{B}_{\log}} \leqslant C$. By Lemma 6 we get that

$$\lim_{k\to\infty} \|T_{\psi_1,\psi_2,\varphi} h_k\|_{\mathscr{B}_{\log}} = 0. \tag{18}$$

Using the triangle inequality, (16), and (17) one has

$$\begin{aligned}
&\|T_{\psi_1,\psi_2,\varphi} h_k\|_{\mathscr{B}_{\log}} \\
&\geqslant \sup_{z\in\mathbb{D}} (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)\left|(T_{\psi_1,\psi_2,\varphi} h_k)'(z)\right| \\
&= \sup_{z\in\mathbb{D}} (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)h_k(\varphi(z)) \\
&\quad + \left(\psi_1(z)\varphi'(z)+\psi_2'(z)\right)h_k'(\varphi(z)) + \psi_2(z)\varphi'(z)h_k''(\varphi(z))| \\
&\geqslant (1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_1'(z_k)h_k(\varphi(z_k)) + \left(\psi_1(z_k)\varphi'(z_k)+\psi_2'(z_k)\right)h_k'(\varphi(z_k)) \\
&\quad + \psi_2(z_k)\varphi'(z_k)h_k''(\varphi(z_k))| \\
&\geqslant (1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_1'(z_k)||h_k(\varphi(z_k))| \\
&\quad - (1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)\left|\psi_1(z_k)\varphi'(z_k)+\psi_2'(z_k)\right||h_k'(\varphi(z_k))| \\
&\quad - (1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_2(z_k)\varphi'(z_k)||h_k''(\varphi(z_k))| \\
&\geqslant (1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_1'(z_k)|\log\log\frac{2}{1-|\varphi(z_k)|} \\
&\quad - 2\frac{(1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_1(z_k)\varphi'(z_k)+\psi_2'(z_k)|}{(1-|\varphi(z_k)|^2)\log\frac{2}{1-|\varphi(z_k)|}} \\
&\quad - \left(\frac{2}{\log\log 4}+2\right)\frac{(1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_2(z_k)\varphi'(z_k)|}{(1-|\varphi(z_k)|^2)^2\log\frac{2}{1-|\varphi(z_k)|}}.
\end{aligned}$$

By (8), (9), and (18) we obtain

$$\begin{aligned}
&(1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_1'(z_k)|\log\log\frac{2}{1-|\varphi(z_k)|} \\
&\leqslant \|T_{\psi_1,\psi_2,\varphi} h_k\|_{\mathscr{B}_{\log}} + 2\frac{(1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_1(z_k)\varphi'(z_k)+\psi_2'(z_k)|}{(1-|\varphi(z_k)|^2)\log\frac{2}{1-|\varphi(z_k)|}} \\
&\quad + \left(\frac{2}{\log\log 4}+2\right)\frac{(1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)|\psi_2(z_k)\varphi'(z_k)|}{(1-|\varphi(z_k)|^2)^2\log\frac{2}{1-|\varphi(z_k)|}} \\
&\longrightarrow 0 \text{ as } k\to\infty.
\end{aligned} \tag{19}$$

Using (19) we get

$$\lim_{k\to\infty}(1-|z_k|^2)\left(\log\frac{2}{1-|z_k|}\right)\log\log\frac{2}{1-|\varphi(z_k)|}|\psi_1'(z_k)|=0,$$

and consequently (7) holds, and that proves Theorem 1. □

Let $\psi\in H(\mathbb{D})$ and φ denote an analytic self-map of $\mathbb{D}$, define a linear operator $W_{\psi,\varphi}$ as follows:

$$W_{\psi,\varphi}f=M_\psi C_\varphi f,\ f\in H(\mathbb{D}).$$

The operator $W_{\psi,\varphi}$ is called the weighted composition operator. From Theorem 1 we can get the characterization of the compactness of the weighted composition operator $uC_\varphi=W_{u,\varphi}$ on $\mathscr{B}_{\log}$.

COROLLARY 1. ([3, Theorem 3.4], [53, Theorem 4.1]) *Let $u\in H(\mathbb{D})$ and φ denote an analytic self-map of $\mathbb{D}$. Then the weighted composition operator $uC_\varphi:\mathscr{B}_{\log}\to\mathscr{B}_{\log}$ is compact if and only if the weighted composition operator $uC_\varphi:\mathscr{B}_{\log}\to\mathscr{B}_{\log}$ is bounded,*

$$\lim_{|\varphi(z)|\to1}|u'(z)|\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)\log\log\frac{2}{1-|\varphi(z)|}=0,$$

and

$$\lim_{|\varphi(z)|\to1}\frac{|u(z)\varphi'(z)|\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}}=0.$$

From Theorem 1 we also can get the characterization of the compactness of the operator $DW_{\psi,\varphi}$ or $W_{\psi,\varphi}D$ on $\mathscr{B}_{\log}$, which to the best of our knowledge, have not appeared in the literature. The boundedness and compactness of the operator $DW_{\psi,\varphi}$ or $W_{\psi,\varphi}D$ on H^∞ were investigated in [10]. The boundedness and compactness of the operator $DW_{\psi,\varphi}$ from weighted Bergman space to weighted Zygmund space was investigated in [7]. The boundedness and compactness of the operator $DW_{\psi,\varphi}$ from the weighted Bergman-Orlicz space to the Bers type space, weighted Bloch space and weighted Zygmund space was investigated in [8]. The boundedness and compactness of the operator DC_φ or $C_\varphi D$ on the Hardy space H^2 were investigated in [25].

COROLLARY 2. *Let $\psi\in H(\mathbb{D})$ and φ denote an analytic self-map of $\mathbb{D}$. Then the weighted composition followed by differentiation $DW_{\psi,\varphi}:\mathscr{B}_{\log}\to\mathscr{B}_{\log}$ is compact if and only if the operator $DW_{\psi,\varphi}:\mathscr{B}_{\log}\to\mathscr{B}_{\log}$ is bounded,*

$$\lim_{|\varphi(z)|\to1}\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)|\psi''(z)|\log\log\frac{2}{1-|\varphi(z)|}=0,$$

$$\lim_{|\varphi(z)|\to1}\frac{\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)|(\psi(z)+\psi'(z))\varphi'(z)+\psi'(z)\varphi(z)|}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}}=0,$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi(z)\varphi(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2 \log\frac{2}{1-|\varphi(z)|}} = 0.$$

COROLLARY 3. *Let $\psi \in H(\mathbb{D})$ and φ denote an analytic self-map of $\mathbb{D}$. Then the differentiation followed by weighted composition $W_{\psi,\varphi}D : \mathscr{B}_{\log} \to \mathscr{B}_{\log}$ is compact if and only if the operator $W_{\psi,\varphi}D : \mathscr{B}_{\log} \to \mathscr{B}_{\log}$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi'(z)|}{\left(1-|\varphi(z)|^2\right) \log\frac{2}{1-|\varphi(z)|}} = 0,$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi(z)\varphi(z)|}{\left(1-|\varphi(z)|^2\right)^2 \log\frac{2}{1-|\varphi(z)|}} = 0.$$

Next we consider the compactness of the operator $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log} \to \mathscr{B}_{\log,0}$. The compactness of operators of which the range is in $\mathscr{B}_{\log,0}$ has a close relation with Lemma 7.

THEOREM 2. *Let $\psi_1, \psi_2 \in H(\mathbb{D})$ and φ denote an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log} \to \mathscr{B}_{\log,0}$ *is compact;*
(b) $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log,0} \to \mathscr{B}_{\log,0}$ *is compact;*
(c) $\psi_1 \in \mathscr{B}_{\log,0}$,

$$\lim_{|z|\to 1} (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)|\log\log\frac{2}{1-|\varphi(z)|} = 0, \tag{20}$$

$$\lim_{|z|\to 1} \frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{\left(1-|\varphi(z)|^2\right) \log\frac{2}{1-|\varphi(z)|}} = 0, \tag{21}$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2 \log\frac{2}{1-|\varphi(z)|}} = 0. \tag{22}$$

Proof. $(c) \Rightarrow (a)$. Suppose that $\psi_1 \in \mathscr{B}_{\log,0}$, (20), (21), and (22) hold. By Lemma 5, it is clear that $T_{\psi_1,\psi_2,\varphi} : \mathscr{B}_{\log} \to \mathscr{B}_{\log,0}$ is bounded. Since for every $z \in \mathbb{D}$, $f \in \mathscr{B}_{\log}$, by Lemmas 1 and 2 we have

$$
\begin{aligned}
&(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)\left|\left(T_{\psi_1,\psi_2,\varphi}f\right)'(z)\right| \\
&= (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)f(\varphi(z)) \\
&\quad + \left(\psi_1(z)\varphi'(z)+\psi_2'(z)\right)f'(\varphi(z))+\psi_2(z)\varphi'(z)f''(\varphi(z))| \\
&\leqslant (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)||f(\varphi(z))| \\
&\quad + (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z)+\psi_2'(z)||f'(\varphi(z))| \\
&\quad + (1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)f''(\varphi(z))| \\
&\leqslant C\|f\|_{\mathscr{B}_{\log}}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)|\left(2+\log\log\frac{2}{1-|\varphi(z)|}\right) \\
&\quad + C\|f\|_{\mathscr{B}_{\log}}\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z)+\psi_2'(z)|}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}} \\
&\quad + C\|f\|_{\mathscr{B}_{\log}}\frac{(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2\log\frac{2}{1-|\varphi(z)|}}.
\end{aligned}
\tag{23}
$$

Taking the supremum in inequality (23) over all $f\in\mathscr{B}_{\log}$ such that $\|f\|_{\mathscr{B}_{\log}}\leqslant 1$ and letting $|z|\to 1$, yields

$$
\lim_{|z|\to 1}\sup_{\|f\|_{\mathscr{B}_{\log}}\leqslant 1}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)\left|\left(T_{\psi_1,\psi_2,\varphi}f\right)'(z)\right| = 0.
$$

Therefore, by Lemma 7, we have $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log}\to\mathscr{B}_{\log,0}$ is compact.

$(a)\Rightarrow(b)$. This implication is clear.

$(b)\Rightarrow(c)$. Assume that $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log,0}\to\mathscr{B}_{\log,0}$ is compact. Firstly, it is obvious $\psi_1\in\mathscr{B}_{\log,0}$ and $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log,0}\to\mathscr{B}_{\log}$ is compact. By Theorem 1, ψ_1,ψ_2, and φ satisfy conditions (7), (8), and (9). It follows that for every $\varepsilon>0$, there exists $\rho\in(0,1)$ such that (10), (11), and (12) hold for $\rho<|\varphi(z)|<1$. On the other hand, since $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log,0}\to\mathscr{B}_{\log,0}$ is compact, then $T_{\psi_1,\psi_2,\varphi}:\mathscr{B}_{\log,0}\to\mathscr{B}_{\log,0}$ is bounded. By Lemma 4, ψ_1,ψ_2, and φ also satisfy conditions (4), (5), and (6). Thus for $\varepsilon>0$, there exists $\gamma\in(0,1)$ such that

$$
(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)| < \frac{\varepsilon}{\log\log\frac{2}{1-\rho}}, \tag{24}
$$

$$
(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)|\psi_1(z)\varphi'(z)+\psi_2'(z)| < \varepsilon(1-\rho^2)\log 2, \tag{25}
$$

and

$$\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)|\psi_2(z)\varphi'(z)| < \varepsilon(1-\rho^2)^2\log 2, \tag{26}$$

for $\gamma < |z| < 1$. Next, we prove that (24) and (10) imply (20). The proof of (21) and (22) is similar, hence it will be omitted.

From (10) one has, when $\gamma < |z| < 1$ and $\rho < |\varphi(z)| < 1$,

$$\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)\log\log\frac{2}{1-|\varphi(z)|}|\psi_1'(z)| < \varepsilon. \tag{27}$$

By (24) we get, when $\gamma < |z| < 1$ and $|\varphi(z)| \leqslant \rho$,

$$\begin{aligned}&\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)\log\log\frac{2}{1-|\varphi(z)|}|\psi_1'(z)|\\ &\leqslant \left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)|\psi_1'(z)|\log\log\frac{2}{1-\rho} < \varepsilon.\end{aligned} \tag{28}$$

Having in mind (27) and (28) we conclude that (20)) holds. This finishes the proof. □

Due to Theorem 2, the characterization of the compactness of the weighted composition operator uC_φ on $\mathscr{B}_{\log,0}$ is now obvious.

COROLLARY 4. ([53, Theorem 4.2]) *Let $u \in H(\mathbb{D})$ and φ denote an analytic self-map of $\mathbb{D}$. Then the weighted composition operator $uC_\varphi : \mathscr{B}_{\log,0} \to \mathscr{B}_{\log,0}$ is compact if and only if $u \in \mathscr{B}_{\log,0}$,*

$$\lim_{|z|\to 1}|u'(z)|\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)\log\log\frac{2}{1-|\varphi(z)|} = 0,$$

and

$$\lim_{|z|\to 1}\frac{|u(z)\varphi'(z)|\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}} = 0.$$

REMARK 1. In [53, Theorem 4.2], the condition $u \in \mathscr{B}_{\log,0}$ is missing. In fact, if

$$\inf_{z\in\mathbb{D}}\log\log\frac{2}{1-|\varphi(z)|} = C > 0,$$

then

$$\begin{aligned}&\lim_{|z|\to 1}|u'(z)|\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)\\ &\leqslant \frac{1}{C}\lim_{|z|\to 1}|u'(z)|\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)\log\log\frac{2}{1-|\varphi(z)|} = 0.\end{aligned}$$

If

$$\inf_{z\in\mathbb{D}} \log\log \frac{2}{1-|\varphi(z)|} = 0,$$

then

$$\lim_{|z|\to 1} |u'(z)|\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)\log\log\frac{2}{1-|\varphi(z)|} = 0$$

$\not\Rightarrow$

$$\lim_{|z|\to 1} |u'(z)|\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right) = 0.$$

Thus we think the condition $u \in \mathscr{B}_{\log,0}$ in Corollary 4 should not be deleted.

COROLLARY 5. *Let $\psi \in H(\mathbb{D})$ and φ denote an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $DW_{\psi,\varphi} : \mathscr{B}_{\log,0} \to \mathscr{B}_{\log,0}$ *is compact;*
(b) $DW_{\psi,\varphi} : \mathscr{B}_{\log} \to \mathscr{B}_{\log,0}$ *is compact;*
(c) $\psi' \in \mathscr{B}_{\log,0}$,

$$\lim_{|z|\to 1} \left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)|\psi''(z)|\log\log\frac{2}{1-|\varphi(z)|} = 0,$$

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)|(\psi(z)+\psi'(z))\varphi'(z)+\psi'(z)\varphi(z)|}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}} = 0,$$

and

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)|\psi(z)\varphi(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2\log\frac{2}{1-|\varphi(z)|}} = 0.$$

COROLLARY 6. *Let $\psi \in H(\mathbb{D})$ and φ denote an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) $W_{\psi,\varphi}D : \mathscr{B}_{\log,0} \to \mathscr{B}_{\log,0}$ *is compact;*
(b) $W_{\psi,\varphi}D : \mathscr{B}_{\log} \to \mathscr{B}_{\log,0}$ *is compact;*
(c)

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)|\psi(z)|}{\left(1-|\varphi(z)|^2\right)\log\frac{2}{1-|\varphi(z)|}} = 0$$

and

$$\lim_{|z|\to 1} \frac{\left(1-|z|^2\right)\left(\log\frac{2}{1-|z|}\right)|\psi(z)\varphi(z)|}{\left(1-|\varphi(z)|^2\right)^2\log\frac{2}{1-|\varphi(z)|}} = 0.$$

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