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WEIGHTED HARDY TYPE INEQUALITIES ON THE HEISENBERG GROUP $\mathbb{H}^n$

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Abstract. In the present article, we provide a sufficient condition on a pair of nonnegative weight functions V and W on the Heisenberg group $\mathbb{H}^n$, so that we establish a general L^p Hardy type inequality involving these weights with a remainder term. The method we use here is practical enough to get more weighted Hardy type inequalities. We also obtain new results on two-weight Hardy and Hardy-Poincaré type inequalities with remainder terms on $\mathbb{H}^n$. Our findings improve and include many previously known results in special cases.

1. Introduction

It is well known that on the Euclidean space $\mathbb{R}^n$, the Hardy inequality asserts that

$$\int_{\mathbb{R}^n} |\nabla\phi(x)|^p dx \geqslant \left|\frac{n-p}{p}\right|^p \int_{\mathbb{R}^n} \frac{|\phi(x)|^p}{|x|^p} dx, \tag{1}$$

and holds for every $\phi \in C_0^\infty(\mathbb{R}^n)$ if $1 \leqslant p < n$, and for every $\phi \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ if $p > n$. Garcia Azorero and Peral Alonso [4] proved that the constant on the right-hand side of (1) is sharp. However, for $p > 1$, it is never achieved in the corresponding Sobolev spaces $W_0^{1,p}(\mathbb{R}^n)$ and $W_0^{1,p}(\mathbb{R}^n \backslash \{0\})$ respectively, where $W_0^{1,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ in the norm $\|\phi\|_{W^{1,p}} := \left(\int_{\mathbb{R}^n} (|\nabla\phi(x)|^p + |\phi(x)|^p)\, dx\right)^{1/p}$.

Inequality (1) was first discovered by Hardy [26] in one dimensional case on the positive half-line $(0,\infty)$. Later on, it has been extended to higher dimensions and large number of papers dealing with new proofs, various extensions, refinements and generalizations have appeared in the Euclidean setting, see for example, [32], [7], [4], [2], [14], [5], [6], [19], [1], [17], [20], [9], [21], [22], [33], [25], [13] and the references therein. From the points of all these developments, it is natural to ask whether Hardy type inequalities can hold on the nilpotent Lie groups, especially, on the Heisenberg group $\mathbb{H}^n$?

In this direction, the first result was obtained by Garofalo and Lanconelli. That is, in [18], they established the following L^2 Hardy inequality on $\mathbb{H}^n$:

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}\phi(z,l)|^2 dzdl \geqslant \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{H}^n} \frac{|z|^2}{|z|^4 + l^2} \phi^2(z,l)\, dzdl, \tag{2}$$

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where $\phi \in C_0^\infty(\mathbb{H}^n\backslash\{(0,0)\})$ and $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$. Here and hereafter we write $z = (x,y) \in \mathbb{R}^n \times \mathbb{R}^n$, $l \in \mathbb{R}$, $|z|^2 = |x|^2 + |y|^2$ and $(0,0)$ is the neutral element of $\mathbb{H}^n$ (see Section 2 for definitions and preliminaries).

Since the work of Garofalo and Lanconelli, there has been continuously growing interest in the study of Hardy type inequalities on $\mathbb{H}^n$, see e.g., [31], [24], [10], [35], [11], [12], [23], [28], [3], [27], [29], [34] and the references therein. For instance, by using the Picone-type identity associated with the p-sub-Laplacian, Niu et al. got an L^p analogue of (2) in [31]:

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}\phi|^p dzdl \geqslant \left(\frac{Q-p}{p}\right)^p \int_{\mathbb{H}^n} \frac{|z|^p}{\left(|z|^4+l^2\right)^{\frac{p}{2}}} |\phi|^p dzdl, \tag{3}$$

where $\phi \in C_0^\infty(\mathbb{H}^n\backslash\{(0,0)\})$ and $Q > p > 1$. A different proof of (2) with the sharp constant $(\frac{Q-2}{2})^2$ was given by Goldstein and Zhang [24].

On the other hand, as far as we know, weighted Hardy type inequalities on $\mathbb{H}^n$ were first studied in [11]. In that paper, D'Ambrosio proved that for every $\phi \in C_0^\infty(\mathbb{H}^n)$, the inequality

$$\int_{\mathbb{H}^n} \frac{|z|^{\beta-p}}{\left(|z|^4+l^2\right)^{\frac{\alpha-2p}{4}}} |\nabla_{\mathbb{H}^n}\phi|^p dzdl \geqslant \left(\frac{Q+\beta-\alpha}{p}\right)^p \int_{\mathbb{H}^n} \frac{|z|^\beta}{\left(|z|^4+l^2\right)^{\frac{\alpha}{4}}} |\phi|^p dzdl \tag{4}$$

holds, where $p > 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy the conditions $Q > \alpha - \beta$ and $Q > 2 + p - \beta$. Moreover, the constant appeared on the right-hand side of (4) is sharp.

It is worth mentioning that a constructive method to derive Hardy type inequalities was presented by D'Ambrosio in the paper [12] for a quite general class of second order operators, including the sub-elliptic operator defined on $\mathbb{H}^n$. Namely, in the context of the Heisenberg group, let Φ be any positive weight, for every $\phi \in C_0^\infty(\mathbb{H}^n)$ the Hardy type inequality

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}\phi|^p dzdl \geqslant c \int_{\mathbb{H}^n} \frac{|\nabla_{\mathbb{H}^n}\Phi|^p}{\Phi^p} |\phi|^p dzdl$$

is valid, provided $-\nabla_{\mathbb{H}^n} \cdot \left(|\nabla_{\mathbb{H}^n}\Phi|^{p-2} \nabla_{\mathbb{H}^n}\Phi\right) \geqslant 0$. Here, the proof relies on the divergence theorem and on the suitable choice of a vector field.

In this article, one of our main goals is to give an alternative method of construction of general weighted Hardy type inequalities on $\mathbb{H}^n$ via utilizing a differential inequality. More precisely, motivated by an idea of Frank and Seiringer in [17], we show that if $W \in L^1_{loc}(\mathbb{H}^n)$ and $V \in C^\infty(\mathbb{H}^n)$ are nonnegative functions and $\Phi \in C^\infty(\mathbb{H}^n)$ is a positive function such that

$$-\nabla_{\mathbb{H}^n} \cdot \left(V(z,l)\,|\nabla_{\mathbb{H}^n}\Phi|^{p-2} \nabla_{\mathbb{H}^n}\Phi\right) \geqslant W(z,l)\,\Phi^{p-1}$$

almost everywhere in $\mathbb{H}^n$, then for every $\phi \in C_0^\infty(\mathbb{H}^n)$ there holds

$$\int_{\mathbb{H}^n} V(z,l)\,|\nabla_{\mathbb{H}^n}\phi|^p dzdl \geqslant \int_{\mathbb{H}^n} W(z,l)\,|\phi|^p dzdl + c_p \int_{\mathbb{H}^n} V(z,l) \left|\nabla_{\mathbb{H}^n}\frac{\phi}{\Phi}\right|^p \Phi^p dzdl,$$

where $p \geqslant 2$ and $c_p > 0$. A similar inequality with a different nonnegative remainder term also exists for the case $1 < p < 2$. The method we use here is constitutive in the sense that, by using proper functions instead of V and Φ, it allows to get several weighted Hardy type inequalities including previously known and also new results (see Applications of Theorem 1). Moreover, we would like to mention in particular that our approach automatically yields a remainder term. For the proof of the above improved Hardy type inequality, we shall mainly use the well known convexity inequalities (14) and (16), and the integration by parts formula.

The second goal of this paper is to obtain an improved version of (3) containing the weight functions g and δ, and the homogeneous norm ρ on $\mathbb{H}^n$ which is defined as in (6). Then we turn our attention to specific weights instead of g and δ that provide the hypotheses in the Theorem 2 and we derive weighted Hardy type inequalities with different nonnegative reminder terms. We shall also consider a new form of the improved L^p Hardy-Poincaré inequality (29) on $\mathbb{H}^n$ with a general weight.

Outline of the article The plan of our paper is summarized in the following sentences. Section 2 is concerned with some basic notations, definitions and preliminaries. In section 3, we start by proving a general form of L^p Hardy type inequality (3) involving two nonnegative weight functions V and W. We note that our result leads us to obtain several weighted Hardy type inequalities. Then under a differential assumption on the weight functions g and δ, and the homogeneous norm ρ, we acquire a two-weight L^p Hardy type inequality. Finally, in section 4, we focus on improved weighted L^p Hardy-Poincaré type inequality.

2. Preliminaries and notations

We first introduce some basic notations, definitions and preliminaries on $\mathbb{H}^n$ that will be used throughout the article. For further details on this topic we refer the interested readers to [8], [18] and the references therein.

DEFINITION 1. Heisenberg group $\mathbb{H}^n$ is a Lie group whose underlying manifold is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, $n \in \mathbb{N}$ with the following group structure:

$$w \circ w' = \left(x + x', y + y', l + l' + 2\sum_{i=1}^n \left(x_i y_i' - y_i x_i'\right)\right)$$

where $w := (x, y, l) = (x_1, \ldots, x_n, y_1, \ldots, y_n, l)$ is a point of $\mathbb{H}^n$.

The left-invariant vector fields for this group structure are

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial l}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial l}, \quad i = 1, \ldots, n. \tag{5}$$

These vector fields generate the Lie algebra of $\mathbb{H}^n$ and the commutators of the vector fields $\{X_1, Y_1, \ldots, X_n, Y_n\}$ satisfy the relation

$$[X_i, Y_i] = -4\frac{\partial}{\partial l}, \quad i = 1, \ldots, n$$

with all other brackets being equal to zero. The Kohn's sub-Laplacian on $\mathbb{H}^n$ is expressed by

$$\begin{aligned}\Delta_{\mathbb{H}^n} &= \sum_{i=1}^{n}\left(X_i^2+Y_i^2\right)\\ &= \sum_{i=1}^{n}\left(\frac{\partial^2}{\partial x_i^2}+\frac{\partial^2}{\partial y_i^2}+4y_i\frac{\partial^2}{\partial x_i\partial l}-4x_i\frac{\partial^2}{\partial y_i\partial l}+4\left(x_i^2+y_i^2\right)\frac{\partial^2}{\partial l^2}\right)\end{aligned}$$

and the sub-elliptic gradient is $2n$ dimensional vector field given by

$$\nabla_{\mathbb{H}^n} = (X_1,\dots,X_n,Y_1,\dots,Y_n).$$

The homogeneous norm on $\mathbb{H}^n$ is defined as follows

$$\|w\|_{\mathbb{H}^n} = \|(x,y,l)\|_{\mathbb{H}^n} = \left(\left(\sum_{i=1}^{n}\left(x_i^2+y_i^2\right)\right)^2+l^2\right)^{1/4}, \tag{6}$$

which is smooth away from the neutral element of $\mathbb{H}^n$. In what follows, we shall use the following notations

$$z=(x,y),\ r=|z|=\left(\sum_{i=1}^{n}\left(x_i^2+y_i^2\right)\right)^{1/2},\ \rho=\rho(w)=\|w\|_{\mathbb{H}^n}=\left(r^4+l^2\right)^{1/4}$$

and denote the neutral element of $\mathbb{H}^n$ by $0=(0,0)$. The Heisenberg dilation $\delta_\lambda : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ is given by $\delta_\lambda(z,l)=(\lambda z,\lambda^2 l)$ for each real number $\lambda>0$. The Jacobian determinant of δ_λ with respect to the Lebesgue measure is equal to λ^Q, where

$$Q=2n+2$$

is the homogeneous dimension of $\mathbb{H}^n$. Hence the change of variable formula gives that

$$d\delta_\lambda(z,l)=\lambda^Q dzdl=\lambda^Q dw,$$

where $dw=dzdl$ denotes the Lebesgue measure on $\mathbb{R}^{2n+1}$. The norm function ρ on $\mathbb{H}^n$ is highly related with the fundamental solution of $-\Delta_{\mathbb{H}^n}$. Folland [15] proved that the fundamental solution of $-\Delta_{\mathbb{H}^n}$ with pole at zero is

$$\Psi(w)=\frac{C_Q}{\rho^{Q-2}},$$

where $C_Q>0$ is a number depending only on Q. The open ball centered at the origin with radius R will be represented by

$$B_R=\{w\in\mathbb{H}^n : \rho<R\}.$$

Now let us mention, without proofs, some useful formulas which we shall use throughout the computations in this paper. A direct calculation yields

$$X_i\rho=\frac{r^2}{\rho^3}x_i+\frac{l}{\rho^3}y_i,\quad Y_i\rho=\frac{r^2}{\rho^3}y_i-\frac{l}{\rho^3}x_i,\quad i=1,\dots,n$$

for all $w \in \mathbb{H}^n \setminus \{0\}$. Let $\phi = \phi(\rho)$ be a smooth *radial function* on $\mathbb{H}^n$, that is only depends on the function ρ in (6), then

$$|\nabla_{\mathbb{H}^n} \phi(\rho)| = \frac{r}{\rho} |\phi'(\rho)|$$

and

$$\Delta_{\mathbb{H}^n} \phi(\rho) = \frac{r^2}{\rho^2} \left(\phi''(\rho) + \frac{Q-1}{\rho} \phi'(\rho) \right)$$

for all $w \in \mathbb{H}^n \setminus \{0\}$. In particular, when $\phi(\rho) = \rho^\alpha$ we have

$$|\nabla_{\mathbb{H}^n} \rho^\alpha| = |\alpha| r \rho^{\alpha-2} \tag{7}$$

and

$$\Delta_{\mathbb{H}^n} \rho^\alpha = \alpha (Q + \alpha - 2) r^2 \rho^{\alpha-4}, \tag{8}$$

where $w \in \mathbb{H}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$. Together with the above definitions and formulas (7) and (8), one can obtain the following identities

$$\nabla_{\mathbb{H}^n} \rho \cdot \nabla_{\mathbb{H}^n} r = \frac{r^3}{\rho^3} \quad \text{in } \mathbb{H}^n \setminus \{0\} \tag{9}$$

and

$$\nabla_{\mathbb{H}^n} \cdot \left(\frac{\rho^3}{r^2} \nabla_{\mathbb{H}^n} \rho \right) = Q \quad \text{in } \mathbb{H}^n \setminus \mathscr{Z}, \tag{10}$$

where $\mathscr{Z} := \{ w = (z,l) \in \mathbb{H}^n : z = 0, l \in \mathbb{R} \}$.

3. Weighted Hardy type inequalities and their improved versions

For certain class of functions, an alternative way of constructing Hardy type inequalities is based on the careful usage of differential equations or differential inequalities. In this regard, we now give the following result.

THEOREM 1. *Let $W \in L^1_{loc}(\mathbb{H}^n)$ and $V \in C^1(\mathbb{H}^n)$ be nonnegative functions. Assume $\Phi \in C^\infty(\mathbb{H}^n)$ is a positive function satisfying the differential inequality*

$$-\nabla_{\mathbb{H}^n} \cdot \left(V(w) |\nabla_{\mathbb{H}^n} \Phi|^{p-2} \nabla_{\mathbb{H}^n} \Phi \right) \geqslant W(w) \Phi^{p-1}, \tag{11}$$

almost everywhere in $\mathbb{H}^n$. There exists a positive constant c_p depending only on p such that; if $p \geqslant 2$, then

$$\int_{\mathbb{H}^n} V(w) |\nabla_{\mathbb{H}^n} \phi|^p dw \geqslant \int_{\mathbb{H}^n} W(w) |\phi|^p dw + c_p \int_{\mathbb{H}^n} V(w) \left| \nabla_{\mathbb{H}^n} \frac{\phi}{\Phi} \right|^p \Phi^p dw \tag{12}$$

and if $1 < p < 2$, *then*

$$\int_{\mathbb{H}^n} V(w) |\nabla_{\mathbb{H}^n} \phi|^p dw \geqslant \int_{\mathbb{H}^n} W(w) |\phi|^p dw + c_p \int_{\mathbb{H}^n} V(w) \frac{\left|\nabla_{\mathbb{H}^n} \frac{\phi}{\Phi}\right|^2 \Phi^2}{\left(\left|\frac{\phi}{\Phi} \nabla_{\mathbb{H}^n} \Phi\right| + \left|\nabla_{\mathbb{H}^n} \frac{\phi}{\Phi}\right| \Phi\right)^{2-p}} dw \tag{13}$$

for all $\phi \in C_0^\infty(\mathbb{H}^n)$.

Proof. Suppose that $\phi = \Phi\varphi$, where $\varphi \in C_0^\infty(\mathbb{H}^n)$ and $0 < \Phi \in C^\infty(\mathbb{H}^n)$. We now recall the following convexity inequality that will be used systematically in this paper (see [30]): Let $a, b \in \mathbb{R}^n$ and $p \geqslant 2$, then it follows that

$$|a+b|^p \geqslant |a|^p + c_p |b|^p + p|a|^{p-2} a \cdot b, \tag{14}$$

where c_p is a positive constant depending only on p and the symbol "$\cdot$" denotes the usual inner product in $\mathbb{R}^n$. Taking $a = \varphi \nabla_{\mathbb{H}^n} \Phi$ and $b = \Phi \nabla_{\mathbb{H}^n} \varphi$ in the inequality (14), direct calculation shows that

$$\begin{aligned} |\nabla_{\mathbb{H}^n} \phi|^p &= |\varphi \nabla_{\mathbb{H}^n} \Phi + \Phi \nabla_{\mathbb{H}^n} \varphi|^p \\ &\geqslant |\varphi|^p |\nabla_{\mathbb{H}^n} \Phi|^p + c_p \Phi^p |\nabla_{\mathbb{H}^n} \varphi|^p + \Phi |\nabla_{\mathbb{H}^n} \Phi|^{p-2} \nabla_{\mathbb{H}^n} \Phi \cdot \nabla_{\mathbb{H}^n} (|\varphi|^p). \end{aligned} \tag{15}$$

Multiplying both sides of (15) by $V(w)$ and applying integration by parts formula to the last term on the right hand side of the inequality over $\mathbb{H}^n$ give us

$$\begin{aligned} \int_{\mathbb{H}^n} V(w) |\nabla_{\mathbb{H}^n} \phi|^p dw &\geqslant \int_{\mathbb{H}^n} V(w) |\nabla_{\mathbb{H}^n} \Phi|^p |\varphi|^p dw + c_p \int_{\mathbb{H}^n} V(w) |\nabla_{\mathbb{H}^n} \varphi|^p \Phi^p dw \\ &\quad - \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n} \cdot \left(V(w) \Phi |\nabla_{\mathbb{H}^n} \Phi|^{p-2} \nabla_{\mathbb{H}^n} \Phi \right) |\varphi|^p dw \\ &= - \int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n} \cdot \left(V(w) |\nabla_{\mathbb{H}^n} \Phi|^{p-2} \nabla_{\mathbb{H}^n} \Phi \right) \Phi |\varphi|^p dw \\ &\quad + c_p \int_{\mathbb{H}^n} V(w) |\nabla_{\mathbb{H}^n} \varphi|^p \Phi^p dw. \end{aligned}$$

Next, by using the given weighted p-Laplacian inequality (11), we get

$$\int_{\mathbb{H}^n} V(w) |\nabla_{\mathbb{H}^n} \phi|^p dw \geqslant \int_{\mathbb{H}^n} W(w) |\varphi|^p \Phi^p dw + c_p \int_{\mathbb{H}^n} V(w) |\nabla_{\mathbb{H}^n} \varphi|^p \Phi^p dw.$$

After back substitution $\varphi = \frac{\phi}{\Phi}$, we acquire the desired result (12). Similar to the derivation of (12) above, the inequality (13) can be derived without any difficulty by using the following convexity inequality with the same choices of a and b:

$$|a+b|^p \geqslant |a|^p + p|a|^{p-2} a \cdot b + c_p \frac{|b|^2}{(|a| + |b|)^{2-p}} \tag{16}$$

where $a, b \in \mathbb{R}^n$, $1 < p < 2$ and $c_p > 0$ (see, for example, [30]). Hence, this completes the proof of Theorem 1. □

REMARK 1. We note that if $p = 2$, then (12) is an equality with $c_2 = 1$.

Applications of Theorem 1

Let $\varepsilon > 0$ be given, we set $r_\varepsilon := \left(\varepsilon^2 + r^2\right)^{1/2}$ and $\rho_\varepsilon := \left(r_\varepsilon^4 + l^2\right)^{1/4}$. We now apply Theorem 1 to recover some previously known weighted Hardy type inequalities and to get also other new results on $\mathbb{H}^n$. For instance, considering the following two functions

$$V_\varepsilon = r_\varepsilon^{\beta-p}\rho_\varepsilon^{2p-\alpha} \text{ and } \Phi_\varepsilon = \rho_\varepsilon^{-\frac{Q+\beta-\alpha}{p}},$$

which satisfy the assumptions of the above theorem, we have the inequality (4) by letting $\varepsilon \longrightarrow 0$:

COROLLARY 1. *Let $p > 1$ and $\alpha, \beta \in \mathbb{R}$ satisfy the conditions $Q > \alpha - \beta$ and $Q > 2 + p - \beta$. Then for every $\phi \in C_0^\infty(\mathbb{H}^n)$, there holds*

$$\int_{\mathbb{H}^n} \frac{r^{\beta-p}}{\rho^{\alpha-2p}} |\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant \left(\frac{Q+\beta-\alpha}{p}\right)^p \int_{\mathbb{H}^n} \frac{r^\beta}{\rho^\alpha} |\phi|^p dw. \tag{17}$$

REMARK 2. It was shown by D'Ambrosio [11] that the constant $\left(\frac{Q+\beta-\alpha}{p}\right)^p$ appeared in (17) is sharp.

By choosing the following two pairs

$$V = 1 \text{ and } \Phi_\varepsilon = r_\varepsilon^{-\frac{Q-p-2}{p}},$$

and letting $\varepsilon \longrightarrow 0$, we obtain the subsequent L^p Hardy type inequality involving the weight $1/r^p$ due to D'Ambrosio [11]:

COROLLARY 2. *Let $Q - 2 > p > 1$. Then the inequality*

$$\int_{\mathbb{H}^n} |\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant \left(\frac{Q-p-2}{p}\right)^p \int_{\mathbb{H}^n} \frac{|\phi|^p}{r^p} dw$$

is valid for every $\phi \in C_0^\infty(\mathbb{H}^n)$.

We now take the subsequent pair

$$V_\varepsilon = \rho_\varepsilon^\alpha \text{ and } \Phi_\varepsilon = \left(1 + \rho_\varepsilon^{\frac{p}{p-1}}\right)^{-\frac{Q+\alpha-p}{p}},$$

and then pass to the limit as $\varepsilon \longrightarrow 0$, we derive the following weighted L^p Hardy type inequality:

COROLLARY 3. *Let $\alpha \in \mathbb{R}$, $1 < p < Q$ and $Q + \alpha - p > 0$. Then, for every function $\phi \in C_0^\infty(\mathbb{H}^n)$, the inequality holds true*

$$\int_{\mathbb{H}^n} \rho^\alpha |\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant \left(\frac{Q+\alpha-p}{p-1}\right)^{p-1} (Q+\alpha) \int_{\mathbb{H}^n} \frac{\rho^{\alpha-p} r^p}{\left(1+\rho^{\frac{p}{p-1}}\right)^p} |\phi|^p dw.$$

Another application of Theorem 1 with the special functions

$$V_\varepsilon = \left(1+\rho_\varepsilon^{\frac{p}{p-1}}\right)^{\alpha(p-1)} \text{ and } \Phi_\varepsilon = \left(1+\rho_\varepsilon^{\frac{p}{p-1}}\right)^{1-\alpha}$$

leads us to the following result by letting $\varepsilon \longrightarrow 0$:

COROLLARY 4. *Let $Q > p > 1$ and $\alpha > 1$. Then, for all $\phi \in C_0^\infty(\mathbb{H}^n)$, we have*

$$\int_{\mathbb{H}^n} \left(1+\rho^{\frac{p}{p-1}}\right)^{\alpha(p-1)} |\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant C(Q,\alpha,p) \int_{\mathbb{H}^n} \frac{\left(1+\rho^{\frac{p}{p-1}}\right)^{(\alpha-1)(p-1)} r^p}{\rho^p} |\phi|^p dw,$$

where $C(Q,\alpha,p) := Q\left(\frac{\alpha p-p}{p-1}\right)^{p-1}$.

It is worth noting that, in the Euclidean setting, Abdellaoui et al. [1] showed that if $1<p<n$ and $-\infty<\alpha<\frac{n-p}{p}$, then for all $\phi \in C_0^\infty(\mathbb{R}^n)$ there holds

$$\int_{\mathbb{R}^n} |x|^{-p\alpha} |\nabla\phi|^p dx \geqslant \left(\frac{n-p(\alpha+1)}{p}\right)^p \int_{\mathbb{R}^n} |x|^{-p(\alpha+1)} |\phi|^p dx. \tag{18}$$

Now by considering these two units

$$V_\varepsilon = \rho_\varepsilon^{-p\alpha} \text{ and } \Phi_\varepsilon = \rho_\varepsilon^{-\frac{Q-p(\alpha+1)}{p}},$$

and passing to the limit as $\varepsilon \longrightarrow 0$, we reach the Heisenberg analogue of the inequality (18) which is in fact a special case of (4) due to D'Ambrosio [11].

COROLLARY 5. *Let $\alpha \in \mathbb{R}$, $1<p<Q$ and $Q>p(\alpha+1)$. Then the inequality*

$$\int_{\mathbb{H}^n} \rho^{-p\alpha}|\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant \left(\frac{Q-p(\alpha+1)}{p}\right)^p \int_{\mathbb{H}^n} \rho^{-p(\alpha+2)} r^p |\phi|^p dw$$

is valid for every function $\phi \in C_0^\infty(\mathbb{H}^n)$.

With the help of a differential assumption on the weight functions g and δ, and the homogeneous norm ρ on $\mathbb{H}^n$, we now prove the following Hardy type inequality which contains two remainder terms.

THEOREM 2. *Let $\alpha \in \mathbb{R}$, $2 \leqslant p < Q+\alpha$ and c_p is a positive constant depending only on p. Let g be a nonnegative C^1-function and δ be a positive C^∞-function such that $-\nabla_{\mathbb{H}^n} \cdot (g(w)\rho^{p-Q}\delta^{2-p}|\nabla_{\mathbb{H}^n}\delta|^{p-2}\nabla_{\mathbb{H}^n}\delta) \geqslant 0$ almost everywhere in $\mathbb{H}^n$. Then the following inequality is valid for all $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$:*

$$\begin{aligned}\int_{\mathbb{H}^n} g(w)\rho^\alpha|\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant \left(\tfrac{Q+\alpha-p}{p}\right)^p &\int_{\mathbb{H}^n} g(w)\rho^{\alpha-2p} r^p |\phi|^p dw \\ &+\left(\tfrac{Q+\alpha-p}{p}\right)^{p-1} \int_{\mathbb{H}^n} \rho^{\alpha-2p+3} r^{p-2} (\nabla_{\mathbb{H}^n}\rho \cdot \nabla_{\mathbb{H}^n} g)|\phi|^p dw \\ &+\frac{c_p}{p^p}\int_{\mathbb{H}^n} g(w)\rho^\alpha\delta^{-p}|\nabla_{\mathbb{H}^n}\delta|^p|\phi|^p dw. \end{aligned} \tag{19}$$

Proof. Let $\phi \in C_0^\infty(\mathbb{H}^n \setminus \{0\})$ and define $\psi := \rho^{-\beta}\phi$ with $\beta < 0$ that will be chosen later. Then we have

$$\nabla_{\mathbb{H}^n}\left(\rho^\beta \psi\right) = \beta\rho^{\beta-1}\psi\nabla_{\mathbb{H}^n}\rho + \rho^\beta\nabla_{\mathbb{H}^n}\psi. \tag{20}$$

Taking $a = \beta\rho^{\beta-1}\psi\nabla_{\mathbb{H}^n}\rho$ and $b = \rho^\beta\nabla_{\mathbb{H}^n}\psi$ in the convexity inequality (14) and using the identity (7), we deduce that

$$\begin{aligned} |\nabla_{\mathbb{H}^n}\phi|^p \geqslant\ & |\beta|^p \rho^{p(\beta-2)} r^p |\psi|^p + c_p\rho^{p\beta}|\nabla_{\mathbb{H}^n}\psi|^p \\ & +\beta|\beta|^{p-2}\rho^{p(\beta-2)+3} r^{p-2}\nabla_{\mathbb{H}^n}\rho \cdot \nabla_{\mathbb{H}^n}\left(|\psi|^p\right). \end{aligned} \tag{21}$$

Multiplying both sides of (21) by $g(w)\rho^\alpha$ and applying integration by parts formula over $\mathbb{H}^n$ give

$$\begin{aligned} \int_{\mathbb{H}^n} g(w)\rho^\alpha|\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant\ & |\beta|^p \int_{\mathbb{H}^n} g(w)\rho^{\alpha+p(\beta-2)} r^p |\psi|^p dw \\ & -\beta|\beta|^{p-2}\int_{\mathbb{H}^n} \nabla_{\mathbb{H}^n}\cdot\left(g(w)\rho^{\alpha+p(\beta-2)+3} r^{p-2}\nabla_{\mathbb{H}^n}\rho\right)|\psi|^p dw \\ & +c_p\int_{\mathbb{H}^n} g(w)\rho^{\alpha+p\beta}|\nabla_{\mathbb{H}^n}\psi|^p dw. \end{aligned} \tag{22}$$

Together with the identities (7), (8) and (9), direct calculation yields

$$\begin{aligned} \nabla_{\mathbb{H}^n}\cdot\left(g(w)\rho^{\alpha+p(\beta-2)+3} r^{p-2}\nabla_{\mathbb{H}^n}\rho\right) =\ & (Q+\alpha+\beta p - p)\, g(w)\rho^{\alpha+p(\beta-2)} r^p \\ & +\rho^{\alpha+p(\beta-2)+3} r^{p-2}\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n} g. \end{aligned} \tag{23}$$

By substituting (23) into (22), one can obtain

$$\begin{aligned} \int_{\mathbb{H}^n} g(w)\rho^\alpha|\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant\ & f(Q,\alpha,p;\beta)\int_{\mathbb{H}^n} g(w)\rho^{\alpha+p(\beta-2)} r^p |\psi|^p dw \\ & -\beta|\beta|^{p-2}\int_{\mathbb{H}^n} \rho^{\alpha+p(\beta-2)+3} r^{p-2}\left(\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n} g\right)|\psi|^p dw \\ & +c_p\int_{\mathbb{H}^n} g(w)\rho^{\alpha+p\beta}|\nabla_{\mathbb{H}^n}\psi|^p dw, \end{aligned} \tag{24}$$

where $f(Q,\alpha,p;\beta) = |\beta|^p - \beta|\beta|^{p-2}(Q+\alpha+\beta p - p)$. Let us choose $\beta = \frac{p-\alpha-Q}{p}$, clearly $\beta < 0$, then the inequality (24) takes the following form

$$\begin{aligned} \int_{\mathbb{H}^n} g(w)\rho^\alpha|\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant\ & \left(\tfrac{Q+\alpha-p}{p}\right)^p \int_{\mathbb{H}^n} g(w)\rho^{-p-Q} r^p |\psi|^p dw \\ & +\left(\tfrac{Q+\alpha-p}{p}\right)^{p-1}\int_{\mathbb{H}^n} \rho^{3-p-Q} r^{p-2}\left(\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n} g\right)|\psi|^p dw \\ & +c_p\int_{\mathbb{H}^n} g(w)\rho^{p-Q}|\nabla_{\mathbb{H}^n}\psi|^p dw. \end{aligned} \tag{25}$$

We now concentrate on the integral expression $\int_{\mathbb{H}^n} g(w)\rho^{p-Q}|\nabla_{\mathbb{H}^n}\psi|^p dw$ on the right hand side of the inequality (25). Let us define a new function $\varphi := \delta^{-1/p}\psi$, where

$0<\delta\in C^{\infty}(\mathbb{H}^n\setminus\{0\})$ and $\psi\in C_0^{\infty}(\mathbb{H}^n\setminus\{0\})$. It follows from the convexity inequality (14) that

$$\begin{aligned}|\nabla_{\mathbb{H}^n}\psi|^p &= \left|p^{-1}\delta^{\frac{1-p}{p}}\varphi\nabla_{\mathbb{H}^n}\delta+\delta^{\frac{1}{p}}\nabla_{\mathbb{H}^n}\varphi\right|^p \\ &\geqslant p^{-p}\delta^{1-p}|\nabla_{\mathbb{H}^n}\delta|^p|\varphi|^p+p^{1-p}\delta^{2-p}|\nabla_{\mathbb{H}^n}\delta|^{p-2}\nabla_{\mathbb{H}^n}\delta\cdot\nabla_{\mathbb{H}^n}(|\varphi|^p).\end{aligned} \tag{26}$$

Multiplying both sides of (26) by $g(w)\rho^{p-Q}$ and integrating over $\mathbb{H}^n$, we get

$$\begin{aligned}\int_{\mathbb{H}^n} g(w)\rho^{p-Q}|\nabla_{\mathbb{H}^n}\psi|^p dw \geqslant p^{-p}&\int_{\mathbb{H}^n} g(w)\rho^{p-Q}\delta^{1-p}|\nabla_{\mathbb{H}^n}\delta|^p|\varphi|^p dw \\ &+p^{1-p}\int_{\mathbb{H}^n} g(w)\rho^{p-Q}\delta^{2-p}|\nabla_{\mathbb{H}^n}\delta|^{p-2}\nabla_{\mathbb{H}^n}\delta\cdot\nabla_{\mathbb{H}^n}(|\varphi|^p)dw.\end{aligned}$$

Here, first applying integration by parts formula and then using the differential assumption

$$-\nabla_{\mathbb{H}^n}\cdot\left(g(w)\rho^{p-Q}\delta^{2-p}|\nabla_{\mathbb{H}^n}\delta|^{p-2}\nabla_{\mathbb{H}^n}\delta\right)\geqslant 0,$$

we conclude that

$$\int_{\mathbb{H}^n} g(w)\rho^{p-Q}|\nabla_{\mathbb{H}^n}\psi|^p dw \geqslant p^{-p}\int_{\mathbb{H}^n} g(w)\rho^{p-Q}\delta^{1-p}|\nabla_{\mathbb{H}^n}\delta|^p|\varphi|^p dw. \tag{27}$$

Taking back substitution $\varphi=\delta^{-1/p}\rho^{\frac{Q+\alpha-p}{p}}\phi$ into the inequality (27), we have

$$\int_{\mathbb{H}^n} g(w)\rho^{p-Q}|\nabla_{\mathbb{H}^n}\psi|^p dw \geqslant p^{-p}\int_{\mathbb{H}^n} g(w)\rho^{\alpha}\delta^{-p}|\nabla_{\mathbb{H}^n}\delta|^p|\phi|^p dw. \tag{28}$$

Finally, first substituting (28) into (25) and then using the equality $\psi=\rho^{\frac{Q+\alpha-p}{p}}\phi$, we derive the desired result (19). □

REMARK 3. We note that the result stated in Theorem 2 holds also for $1<p<2$ with a different reminder term and in this case we use the convexity inequality (16).

One virtue of our approach is that it automatically gives remainder terms. To be specific, choosing different model functions instead of δ and g that satisfy assumptions of the Theorem 2, we obtain the following weighted L^p Hardy type inequalities with different nonnegative reminder terms.

COROLLARY 6. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$ in $\mathbb{H}^n$ and let $g\equiv 1$, $\delta:=\log(\frac{R}{\rho})$, $R>\sup_{\Omega}(\rho)$. Then for all $\phi\in C_0^{\infty}(\Omega\setminus\{0\})$, we have*

$$\int_{\Omega}\rho^{\alpha}|\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant \left(\frac{Q+\alpha-p}{p}\right)^p\int_{\Omega}\rho^{\alpha-2p}r^p|\phi|^p dw+\frac{c_p}{p^p}\int_{\Omega}\frac{\rho^{\alpha-2p}r^p}{\log^p\left(\frac{R}{\rho}\right)}|\phi|^p dw,$$

where $Q+\alpha>p\geqslant 2$, $\alpha\in\mathbb{R}$ and $c_p>0$.

COROLLARY 7. *Let B_R be a ball with center zero and radius R in $\mathbb{H}^n$ and let $g \equiv 1$, $\delta := R-\rho$. Then for all $\phi \in C_0^\infty(B_R\setminus\{0\})$, we have*

$$\int_{B_R} \rho^\alpha |\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant \left(\frac{Q+\alpha-p}{p}\right)^p \int_{B_R} \rho^{\alpha-2p} r^p |\phi|^p dw + \frac{c_p}{p^p}\int_{B_R} \frac{\rho^{\alpha-p} r^p}{(R-\rho)^p} |\phi|^p dw,$$

where $Q+\alpha > p \geqslant 2$, $\alpha \in \mathbb{R}$ and $c_p > 0$.

4. Weighted Hardy-Poincaré type inequality

We now prove the following two-weight Hardy-Poincaré type inequality with a remainder term.

THEOREM 3. *Let g be a nonnegative C^1-function, $Q > p > 1$, $Q+\alpha > 0$ and $\alpha \in \mathbb{R}$. Then for any $\phi \in C_0^\infty(\mathbb{H}^n\setminus\mathscr{Z})$, there holds*

$$\int_{\mathbb{H}^n} g(w)\frac{\rho^{\alpha+3p}}{r^{2p}} |\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant \left(\tfrac{Q+\alpha}{p}\right)^p \int_{\mathbb{H}^n} g(w)\rho^\alpha |\phi|^p dw + \left(\tfrac{Q+\alpha}{p}\right)^{p-1}\int_{\mathbb{H}^n} \frac{\rho^{\alpha+3}}{r^2}(\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n} g)|\phi|^p dw. \tag{29}$$

Proof. The volume growth formula (10) gives

$$\nabla_{\mathbb{H}^n}\cdot\left(g(w)\frac{\rho^3}{r^2}\nabla_{\mathbb{H}^n}\rho\right) = \frac{\rho^3}{r^2}\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n} g + g(w)Q \quad \text{in } \mathbb{H}^n\setminus\mathscr{Z}. \tag{30}$$

Then multiplying both sides of (30) by $\rho^\alpha|\phi|^p$ and integrating over $\mathbb{H}^n$ yield

$$Q\int_{\mathbb{H}^n} g(w)\rho^\alpha|\phi|^p dw = \int_{\mathbb{H}^n}\rho^\alpha|\phi|^p\nabla_{\mathbb{H}^n}\cdot\left(g(w)\frac{\rho^3}{r^2}\nabla_{\mathbb{H}^n}\rho\right)dw - \int_{\mathbb{H}^n}\frac{\rho^{\alpha+3}}{r^2}(\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n} g)|\phi|^p dw.$$

As an immediate consequence of integration by parts formula we have

$$\begin{aligned} Q\int_{\mathbb{H}^n} g(w)\rho^\alpha|\phi|^p dw &= -\int_{\mathbb{H}^n} g(w)\frac{\rho^3}{r^2}\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n}(\rho^\alpha|\phi|^p)dw \\ &\quad -\int_{\mathbb{H}^n}\frac{\rho^{\alpha+3}}{r^2}(\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n} g)|\phi|^p dw \\ &= -\alpha\int_{\mathbb{H}^n} g(w)\rho^\alpha|\phi|^p dw - \int_{\mathbb{H}^n}\frac{\rho^{\alpha+3}}{r^2}(\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n} g)|\phi|^p dw \\ &\quad -p\int_{\mathbb{H}^n} g(w)\frac{\rho^{\alpha+3}}{r^2}(\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n}\phi)|\phi|^{p-1}dw. \end{aligned}$$

After rearranging the terms, the above inequality can be written in the following form

$$M \leqslant -p\int_{\mathbb{H}^n} g(w)\frac{\rho^{\alpha+3}}{r^2}(\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n}\phi)|\phi|^{p-1}dw, \tag{31}$$

where

$$M=(Q+\alpha)\int_{\mathbb{H}^n} g(w)\,\rho^{\alpha}|\phi|^p dw+\int_{\mathbb{H}^n}\frac{\rho^{\alpha+3}}{r^2}\left(\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n}g\right)|\phi|^p dw.$$

Applying successively the Hölder and Young inequalities on the right hand side of (31), we get

$$\begin{aligned}M &\leqslant p\left(\int_{\mathbb{H}^n} g(w)\,\rho^{\alpha}|\phi|^p dw\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{H}^n} g(w)\,\frac{\rho^{\alpha+3p}}{r^{2p}}|\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n}\phi|^p dw\right)^{\frac{1}{p}}\\ &\leqslant (p-1)\,\varepsilon^{\frac{-p}{p-1}}\int_{\mathbb{H}^n} g(w)\,\rho^{\alpha}|\phi|^p dw+\varepsilon^p\int_{\mathbb{H}^n} g(w)\,\frac{\rho^{\alpha+3p}}{r^{2p}}|\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n}\phi|^p dw\end{aligned}$$

for any $\varepsilon>0$. Hence

$$\begin{aligned}\int_{\mathbb{H}^n} g(w)\,\frac{\rho^{\alpha+3p}}{r^{2p}}|\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n}\phi|^p dw \geqslant f(Q,\alpha,p;\varepsilon)&\int_{\mathbb{H}^n} g(w)\,\rho^{\alpha}|\phi|^p dw\\ &+\varepsilon^{-p}\int_{\mathbb{H}^n}\frac{\rho^{\alpha+3}}{r^2}\left(\nabla_{\mathbb{H}^n}\rho\cdot\nabla_{\mathbb{H}^n}g\right)|\phi|^p dw,\end{aligned}$$

where $f(Q,\alpha,p;\varepsilon)=\varepsilon^{-p}\left[Q+\alpha+(1-p)\,\varepsilon^{\frac{-p}{p-1}}\right]$. Note that the function f attains the maximum for $\varepsilon_0=\left(\frac{Q+\alpha}{p}\right)^{\frac{1-p}{p}}$ and this maximum value is equal to $f(Q,\alpha,p;\varepsilon_0)=\left(\frac{Q+\alpha}{p}\right)^p$. Therefore we obtain the desired inequality (29). □

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