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HARDY–LITTLEWOOD–PÓLYA INEQUALITIES AND HAUSDORFF OPERATORS ON BLOCK SPACES

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Abstract. We establish the Hardy-Littlewood-Pólya inequality, the Hardy inequality and the Hilbert inequality on block spaces. Furthermore, we also have the boundedness of the Hausdorff operators on block spaces.

1. Introduction

In this paper, we aim to extend the Hardy-Littlewood-Pólya inequalities and the boundedness of Hausdorff operators to block spaces. Particularly, as consequences of the Hardy-Littlewood-Pólya inequalities, we also obtain the Hardy inequalities and the Hilbert inequality on block spaces.

The Hardy-Littlewood-Pólya inequalities for Lebesgue spaces were established in [20]. These inequalities unify several important results in analysis such as the Hardy inequality, the Hilbert inequalities, the Riemann-Liouville integral and the Weyl integral. The Hardy inequalities have been developed extensively, for detail references for the study of the Hardy inequality and its related topics, the reader is referred to [13, 37].

There exists another generalization of the Hardy inequalities, namely, the Hausdorff operators. The study of the Hausdorff operators dated back to 1917 where the Hausdorff summability method was introduced [19]. The reader is referred to [29] for the history of the development of the Hausdorff operators. The studies of the Hausdorff operators had been extended to several different setting and function spaces, the reader is refereed to a long but incomplete list of references [5, 6, 9, 10, 11, 15, 16, 17, 18, 24, 26, 27, 28, 29, 30, 31, 32, 34, 35, 38, 40].

One of the main results in this paper is the boundedness of the Hausdorff operators on block spaces. Block space is one of the natural generalization of Lebesgue spaces. It arises from the study of Morrey spaces. More precisely, block space is the predual of the Morrey space [1, 4, 25, 41]. In [1], some important results in analysis, such as the capacity theory and the potential theory, had been extended to block spaces. Furthermore, the mapping properties of some vector-valued operators with singular kernels for block spaces are obtained in [23]. Using these mapping properties, the Triebel-Lizorkin-block

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spaces were introduced in [23]. In addition, the block spaces also provide some crucial supporting results for the studies of Sobolev embedding of Triebel-Lizorkin-Morrey-Lorentz spaces and the atomic decompositions of Hardy-Morrey spaces with variable exponents in [21, 22], respectively. The reader is also referred to the surveys [7, 8] for the recent progress on the studies of classical operators in general Morrey type spaces.

Therefore, in this paper, we have two main results for block spaces. We find that both of them rely on the Minkowski inequality and the mapping properties of the dilation operators on block spaces. To establish the Minkowski's inequality, we use the predual space of the block space, namely, the Zorko space. The details of the above mentioned results are given in Section 2.

This paper is organized as follows. Some basic properties of block spaces are given in Section 2. The Hardy-Littlewood-Pólya inequalities on block spaces are established in Section 3. Finally, in Section 4, we present the boundedness of Hausdorff operators on block spaces.

2. Block spaces

In this section, we present some basic properties for block spaces. Precisely, we have the duality of Morrey spaces and block spaces, the Minkowski's inequality on block spaces and the mapping properties of the dilation operators on block spaces.

We begin with the definitions of Morrey spaces and block spaces.

The Morrey space is defined by

$$M_{p,\lambda}(\mathbb{R}^n) = \{f \in L^p_{loc} : \|f\|_{M_{p,\lambda}(\mathbb{R}^n)} < \infty\},$$

where $1 \leqslant p < \infty$ and $0 \leqslant \lambda < n$ and

$$\|f\|_{M_{p,\lambda}(\mathbb{R}^n)} = \sup_{B \in \mathbb{B}} \left(\frac{1}{r^\lambda} \int_B |f(x)|^p dx \right)^{1/p}.$$

The Morrey space was introduced by Morrey [34] in order to study some quasi-linear elliptic partial differential equations. Next, we recall the definition of block spaces [4].

DEFINITION 1. Let $1 \leqslant p < \infty$ and $0 \leqslant \lambda < n$. A measurable function b is called a (p,λ)-block if it is supported in a ball $B(x_0,r)$, $x_0 \in \mathbb{R}^n$, $r > 0$, and

$$\|b\|_{L^p(\mathbb{R}^n)} \leqslant r^{-\frac{\lambda}{p}}. \tag{1}$$

We write $b \in b_{p,\lambda}$ if b is a (p,λ)-block.

Define $\mathscr{B}_{p,\lambda}(\mathbb{R}^n)$ by

$$\mathscr{B}_{p,\lambda}(\mathbb{R}^n) = \left\{ \sum_{k=1}^{\infty} \lambda_k b_k : \sum_{k=1}^{\infty} |\lambda_k| < \infty \text{ and } b_k \text{ is a } (p,\lambda)\text{-block} \right\}. \tag{2}$$

The space $\mathscr{B}_{p,\lambda}$ is endowed with the norm

$$\|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} = \inf\left\{\sum_{k=1}^{\infty}|\lambda_k| : \quad f = \sum_{k=1}^{\infty}\lambda_k b_k\right\}, \tag{3}$$

where the infimum is taken over all such decompositions of f.

Notice that there is another family of function spaces also named as "block spaces", see [33, 39].

The dual space of the block space $\mathscr{B}_{p',\lambda}(\mathbb{R}^n)$ is the Morrey space $M_{p,\lambda}(\mathbb{R}^n)$, see [4, 25, 41]. In fact, this duality property can be further extended. To present this extension, we recall a function space introduced by Zorko in [41].

Let $C_0(\mathbb{R}^n)$ denote the class of continuous function with compact support in $\mathbb{R}^n$. We now recall the definition of Zorko spaces [41].

DEFINITION 2. Let $1 \leqslant p < \infty$ and $0 \leqslant \lambda < n$. The Zorko spaces $Z_{p,\lambda}(\mathbb{R}^n)$ is the closure of $C_0(\mathbb{R}^n)$ in $M_{p,\lambda}(\mathbb{R}^n)$.

The following result is given in [1, Section 3].

THEOREM 1. *Let $1 < p < \infty$ and $0 \leqslant \lambda < n$. We have*

$$(Z_{p,\lambda}(\mathbb{R}^n))^* = \mathscr{B}_{p',\lambda}(\mathbb{R}^n) \quad \textit{and} \quad (\mathscr{B}_{p,\lambda}(\mathbb{R}^n))^* = M_{p',\lambda}(\mathbb{R}^n).$$

As stated in [1, Section 3.2], the triple $(Z_{p,\lambda}(\mathbb{R}^n), \mathscr{B}_{p',\lambda}(\mathbb{R}^n), M_{p,\lambda}(\mathbb{R}^n))$ has a relationship akin to the triple (VMO, H^1, BMO).

Since $M_{p',0}(\mathbb{R}^n) = L^{p'}(\mathbb{R}^n)$, Theorem 1 yields that

$$\mathscr{B}_{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n).$$

Thus, block space is a generalization of Lebesgue space.

Moreover, we have the Hölder inequality for $\mathscr{B}_{p,\lambda}(\mathbb{R}^n)$ and $M_{p',\lambda}(\mathbb{R}^n)$.

LEMMA 1. *Let $1 < p < \infty$ and $0 \leqslant \lambda < n$. We have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant \|f\|_{M_{p,\lambda}(\mathbb{R}^n)}\|g\|_{\mathscr{B}_{p',\lambda}(\mathbb{R}^n)}.$$

The proof of the above result follows from the definitions of $M_{p,\lambda}(\mathbb{R}^n)$ and $\mathscr{B}_{p',\lambda}(\mathbb{R}^n)$ and the Hölder inequality for Lebesgue spaces, see [12].

With the above duality results, we now establish the Minkowski's inequality in $\mathscr{B}_{p,\lambda}(\mathbb{R}^n)$.

THEOREM 2. (Minkowski's inequality) *Let $1 < p < \infty$, $0 \leqslant \lambda < n$ and μ be a signed σ-finite Borel measure on $\mathbb{R}^m$. Let $f(x,s)$ be a measurable function on $\mathbb{R}^n \times \mathbb{R}^m$ such that $\|f(\cdot,s)\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \in L^1(|\mu|)$. We have*

$$\left\|\int_{\mathbb{R}^m} f(\cdot,s)d\mu\right\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \leqslant \int_{\mathbb{R}^m} \|f(\cdot,s)\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}d|\mu|. \tag{4}$$

Proof. Write

$$F(x) = \int_{\mathbb{R}^m} f(x,s) d\mu.$$

Let $g \in Z_{p',\lambda}(\mathbb{R}^n)$ with $\|g\|_{Z_{p',\lambda}(\mathbb{R}^n)} \leqslant 1$. By using the Hölder inequality for $\mathscr{B}_{p,\lambda}(\mathbb{R}^n)$ and $M_{p',\lambda}(\mathbb{R}^n)$, we find that

$$\begin{aligned}\left|\int_{\mathbb{R}^n} F(x)g(x)dx\right| &\leqslant \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} |f(x,s)||g(x)|dxd|\mu| \\ &\leqslant \int_{\mathbb{R}^m} \|f(\cdot,s)\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}\|g\|_{M_{p',\lambda}(\mathbb{R}^n)}d|\mu| \\ &= \int_{\mathbb{R}^m} \|f(\cdot,s)\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}\|g\|_{Z_{p',\lambda}(\mathbb{R}^n)}d|\mu| \\ &\leqslant \int_{\mathbb{R}^m} \|f(\cdot,s)\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}d|\mu|.\end{aligned}$$

By taking supremum over $g \in Z_{p',\lambda}(\mathbb{R}^n)$ with $\|g\|_{Z_{p',\lambda}(\mathbb{R}^n)} \leqslant 1$, Theorem 1 yields that $F \in \mathscr{B}_{p,\lambda}(\mathbb{R}^n)$ and (4) is valid. $\square$

Next, we extend the notion of Boyd's indices to $\mathscr{B}_{p,\lambda}(\mathbb{R}^n)$. For each $s \in \mathbb{R}\backslash\{0\}$ and for any measurable function f on $\mathbb{R}^n$, let D_s be the dilation operator defined by

$$(D_s f)(x) = f(x/s), \quad x \in \mathbb{R}^n.$$

LEMMA 2. *Let $1 < p < \infty$, $0 \leqslant \lambda < n$. Then,*

$$\|D_t f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} = |t|^{\frac{n+\lambda}{p}} \|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}. \tag{5}$$

Proof. For any $f \in \mathscr{B}_{p,\lambda}(\mathbb{R}^n)$ and $\varepsilon > 0$, there exist families of (p,λ)-blocks $\{b_k\}_{k\in\mathbb{N}}$ with supports $\{B(x_k,r_k)\}_{k\in\mathbb{N}}$ and scalars $\{\lambda_k\}_{k\in\mathbb{N}}$ such that

$$f = \sum_{k\in\mathbb{N}} \lambda_k b_k$$

and $\sum_{k\in\mathbb{N}} |\lambda_k| < (1+\varepsilon)\|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}$.

We see that $D_t b_k$ is a (p,λ)-block with support $B(tx_k,|t|r_k)$ and

$$\|D_t b_k\|_{L^p(\mathbb{R}^n)} \leqslant |t|^{\frac{n}{p}} r_k^{-\frac{\lambda}{p}} = |t|^{\frac{n+\lambda}{p}} (|t|r_k)^{-\frac{\lambda}{p}}. \tag{6}$$

Write

$$D_t f = \sum_{k\in\mathbb{N}} \lambda_k D_t b_k = \sum_{k\in\mathbb{N}} \gamma_k c_k$$

where

$$\gamma_k = \lambda_k |t|^{\frac{n+\lambda}{p}} \quad \text{and} \quad c_k = |t|^{-\frac{n+\lambda}{p}} D_t b_k.$$

In view of (6), $\{c_k\}_{k\in\mathbb{N}}$ is a family of (p,λ)-blocks. Therefore, $D_t f \in \mathscr{B}_{p,\lambda}(\mathbb{R}^n)$ with

$$\|D_t f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \leqslant \sum_{k\in\mathbb{N}} |\gamma_k| = \sum_{k\in\mathbb{N}} |\lambda_k||t|^{\frac{n+\lambda}{p}} \leqslant (1+\varepsilon)|t|^{\frac{n+\lambda}{p}} \|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}.$$

As $\varepsilon > 0$ is arbitrary, we find that

$$\|D_t f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \leqslant |t|^{\frac{n+\lambda}{p}} \|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}. \tag{7}$$

Since $D_{1/t}D_t f = f$, (7) yields

$$\|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} = \|D_{1/t}D_t f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \leqslant |t|^{-\frac{n+\lambda}{p}} \|D_t f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}.$$

Therefore,

$$\|D_t f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} = |t|^{\frac{n+\lambda}{p}} \|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}. \quad \square$$

The above result gives us the operator norm of the dilation operators D_t on block spaces. This result is crucial on the study of Hardy-Littlewood-Pólya inequalities in the coming sections.

Moreover, the mapping property of the dilation operators is also related to Sobolev type embedding, the reader is referred to [21] for the use of the mapping property of the dilation operators on the Sobolev embedding of Triebel-Lizorkin-Morrey-Lorentz spaces.

3. Hardy-Littlewood-Pólya inequalities

In this section, we establish the Hardy-Littlewood-Pólya inequalities on block spaces. We also apply these inequalities to extend the Hardy inequalities and the Hilbert inequality on block spaces.

Since the classical Hardy-Littlewood-Pólya inequalities for Lebesgue spaces are established for Lebesgue spaces on $(0,\infty)$. Therefore, we consider block spaces defined on $(0,\infty)$. Notice that the results on the previous section also apply to the block spaces on $(0,\infty)$, $\mathscr{B}_{p,\lambda}(0,\infty)$.

THEOREM 3. (Hardy-Littlewood-Pólya inequalities) *Let $1 < p < \infty$, $0 \leqslant \lambda < 1$ and $K(\cdot,\cdot)$ be a measurable function on $(0,\infty)\times(0,\infty)$. If K satisfies*

1. $K(\theta s, \theta t) = \theta^{-1}K(s,t)$, $\theta > 0$,
2. $\int_0^\infty |K(v,1)| v^{-\frac{1+\lambda}{p}} dv < \infty$,

then, the linear operator

$$Tf(t) = \int_0^\infty K(s,t) f(s) ds$$

is bounded on $\mathscr{B}_{p,\lambda}(0,\infty)$.

Proof. Let $v = s/t$. We have

$$|Tf(t)| \leqslant \int_0^\infty |K(vt,t)||(D_{1/v}f)(t)| t dv = \int_0^\infty |K(v,1)||(D_{1/v}f)(t)| dv.$$

Applying the norm $\|\cdot\|_{\mathscr{B}_{p,\lambda}(0,\infty)}$ on both sides of the above inequality, Theorem 2 yields

$$\|Tf\|_{\mathscr{B}_{p,\lambda}(0,\infty)} \leqslant \int_0^\infty |K(v,1)| \|(D_{1/v}f)\|_{\mathscr{B}_{p,\lambda}(0,\infty)} dv.$$

Consequently, (5) assures that

$$\|Tf\|_{\mathscr{B}_{p,\lambda}(0,\infty)} \leqslant C\|f\|_{\mathscr{B}_{p,\lambda}(0,\infty)} \int_0^\infty |K(v,1)| v^{-\frac{1+\lambda}{p}} dv \leqslant C\|f\|_{\mathscr{B}_{p,\lambda}(0,\infty)}$$

for some $C > 0$. Thus, T is bounded on $\mathscr{B}_{p,\lambda}(0,\infty)$. □

When $\lambda = 0$, Item (2) of the above theorem becomes

$$\int_0^\infty |K(v,1)| v^{-\frac{1}{p}} dv < \infty$$

which is the condition for the Hardy-Littlewood-Pólya inequalities on Lebesgue spaces [14, Theorem 6.20].

As a consequence of the above theorem, we have the Hardy inequalities on block spaces.

THEOREM 4. (Hardy's Inequality) *Let* $1 < p < \infty$, $0 \leqslant \lambda < 1$.

1. *If* $p > 1 + \lambda$, *then*
$$Tf(t) = \frac{1}{t} \int_0^t f(s) ds$$
is bounded on $\mathscr{B}_{p,\lambda}(0,\infty)$.

2. *The operator*
$$Sf(t) = \int_t^\infty \frac{f(s)}{s} ds$$
is bounded on $\mathscr{B}_{p,\lambda}(0,\infty)$.

Proof. Let $K(s,t) = t^{-1}\chi_E(s,t)$ where $E = \{(s,t) : s < t\}$. It satisfies Item (1) of Theorem 3. Furthermore, we have

$$\int_0^\infty |K(v,1)| v^{-\frac{1+\lambda}{p}} dv = \int_0^1 v^{-\frac{1+\lambda}{p}} dv = \left. \frac{v^{-\frac{1+\lambda}{p}+1}}{-\frac{1+\lambda}{p}+1} \right|_0^1 < \infty.$$

According to Theorem 3, we find that

$$\|Tf\|_{\mathscr{B}_{p,\lambda}(0,\infty)} \leqslant C\|f\|_{\mathscr{B}_{p,\lambda}(0,\infty)}.$$

To establish the boundedness of the operator S, let $K(s,t) = s^{-1}\chi_E(s,t)$ where $E = \{(s,t) : s > t\}$. It also satisfies Item (1) of Theorem 3. Similarly, we have

$$\int_0^\infty |K(v,1)| v^{-\frac{1+\lambda}{p}} dv = \int_1^\infty v^{-\frac{1+\lambda}{p}-1} dv < \infty.$$

Thus, Theorem 3 concludes that

$$\|Sf\|_{\mathscr{B}_{p,\lambda}(0,\infty)} \leqslant C\|f\|_{\mathscr{B}_{p,\lambda}(0,\infty)}$$

for some $C > 0$. □

We also have the Hilbert inequality on block spaces.

THEOREM 5. (Hilbert's inequality) *Let* $1 < p < \infty$, $0 \leqslant \lambda < 1$. *If* $p > 1+\lambda$, *then*

$$Tf(t) = \int_0^\infty \frac{f(s)}{t+s} ds$$

is bounded on $\mathscr{B}_{p,\lambda}(0,\infty)$.

Proof. Let $K(s,t) = (s+t)^{-1}$. It obviously fulfills Item (1) of Theorem 3. Moreover, we have

$$\int_0^\infty (v+1)^{-1} v^{-\frac{1+\lambda}{p}} dv \leqslant \int_1^\infty v^{-\frac{1+\lambda}{p}-1} dv + \int_0^1 v^{-\frac{1+\lambda}{p}} dv < \infty.$$

Hence, Theorem 3 yields the boundedness of T on $\mathscr{B}_{p,\lambda}(0,\infty)$. □

4. Multidimensional Hausdorff operators

In this section, we extend the boundedness of Hausdorff operators to block spaces. The boundedness of Hausdorff operators had already been extended to a number of function spaces such as the Hardy spaces, the Herz spaces and the Morrey spaces. The reader is referred to [2, 5, 6, 9, 15, 24, 26, 27, 28, 30, 31, 32, 35, 40] for details.

We use the definition of multidimensional Hausdorff operators from [27, 28].

Let $A = A(u) = (a_{ij})_{i,j=1}^n = (a_{ij}(u))_{i,j=1}^n$ be an $n \times n$ matrix with the entries $a_{ij}(u)$ being measurable functions of u. The matrix $A(u)$ is non-degenerate almost everywhere. Recall that $xA(u)$, $x \in \mathbb{R}^n$, is the row n-vector obtained by multiplying the row n-vector x by the matrix $A(u)$.

Let $\Phi(u)$ be a measurable function. The multidimensional Hausdorff operator associated with $A(u)$ and $\Phi(u)$ is given by

$$(\mathscr{H}f)(x) = (\mathscr{H}_\Phi f)(x) = (\mathscr{H}_{\Phi,A} f)(x) = \int_{\mathbb{R}^n} \Phi(u) f(xA(u)) du.$$

Notice that, as given in [6, 27, 35], the Hausdorff operator is defined in term of Borel measure rather than the Lebesgue measure.

The adjoint operator $\mathscr{H}^*$ is defined as

$$(\mathscr{H}^* f)(x) = (\mathscr{H}^*_{\Phi,A} f)(x) = \int_{\mathbb{R}^n} \Phi(u) |\det A^{-1}(u)| f(xA^{-1}(u)) du.$$

For any measurable function f on $\mathbb{R}^n$, let $D_{A(u)}$ be the dilation operator defined by

$$(D_{A(u)} f)(x) = f(xA(u)), \quad x \in \mathbb{R}^n.$$

In order to study the multidimensional Hausdorff operators $\mathscr{H}_{\Phi,A}$ and $\mathscr{H}^*_{\Phi,A}$, we need the following result which is an extension of Lemma 2.

Even though the proof of the subsequent lemma is similar to the proof of Lemma 2, for completeness, we present the details.

LEMMA 3. *Let* $1 < p < \infty$, $0 \leqslant \lambda < n$. *Then,*

$$\|D_{A(u)}f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \leqslant |detA(u)|^{-1/p}\|A^{-1}(u)\|^{\frac{\lambda}{p}}|\|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \tag{8}$$

where $\|A^{-1}(u)\|$ *is the operator norm of* $A^{-1}(u)$.

Proof. Let $f \in \mathscr{B}_{p,\lambda}(\mathbb{R}^n)$. According to the definition of $\mathscr{B}_{p,\lambda}(\mathbb{R}^n)$, for any $\varepsilon > 0$, there exist families of (p,λ)-blocks $\{b_k\}_{k\in\mathbb{N}}$ with supports $\{B(x_k,r_k)\}_{k\in\mathbb{N}}$ and scalars $\{\lambda_k\}_{k\in\mathbb{N}}$ such that

$$f = \sum_{k\in\mathbb{N}} \lambda_k b_k \quad \text{and} \quad \sum_{k\in\mathbb{N}} |\lambda_k| < (1+\varepsilon)\|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}.$$

We find that $D_{A(u)}b_k$ is a (p,λ)-block,

$$\text{supp}D_{A(u)}b_k \subseteq B(x_kA^{-1}(u), \|A^{-1}(u)\|r_k)$$

and

$$\begin{aligned}\|D_{A(u)}b_k\|_{L^p(\mathbb{R}^n)} &\leqslant |\text{det}A(u)|^{-\frac{1}{p}} r_k^{-\frac{\lambda}{p}} \\ &= |\text{det}A(u)|^{-\frac{1}{p}}\|A^{-1}(u)\|^{\frac{\lambda}{p}}(\|A^{-1}(u)\|^{\frac{\lambda}{p}} r_k)^{-\frac{\lambda}{p}}.\end{aligned} \tag{9}$$

Consequently,

$$D_{A(u)}f = \sum_{k\in\mathbb{N}} \lambda_k D_{A(u)}b_k = \sum_{k\in\mathbb{N}} \gamma_k c_k$$

where

$$\gamma_k = \lambda_k|\text{det}A(u)|^{-\frac{1}{p}}\|A^{-1}(u)\|^{\frac{\lambda}{p}} \quad \text{and} \quad c_k = |\text{det}A(u)|^{\frac{1}{p}}\|A^{-1}(u)\|^{-\frac{\lambda}{p}} D_{A(u)}b_k.$$

According to (9), $\{c_k\}_{k\in\mathbb{N}}$ is a family of (p,λ)-blocks. Therefore, $D_{A(u)}f \in \mathscr{B}_{p,\lambda}(\mathbb{R}^n)$ with

$$\begin{aligned}\|D_{A(u)}f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \leqslant \sum_{k\in\mathbb{N}} |\gamma_k| &= \sum_{k\in\mathbb{N}} |\lambda_k|\text{det}A(u)|^{-\frac{1}{p}}\|A^{-1}(u)\|^{\frac{\lambda}{p}} \\ &\leqslant (1+\varepsilon)|\text{det}A(u)|^{-\frac{1}{p}}\|A^{-1}(u)\|^{\frac{\lambda}{p}}\|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, we find that

$$\|D_{A(u)}f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \leqslant |\text{det}A(u)|^{-\frac{1}{p}}\|A^{-1}(u)\|^{\frac{\lambda}{p}}\|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}. \tag{10}$$

□

The following theorem extends the results in [27, 28] to block spaces.

THEOREM 6. *Let* $1 < p < \infty$, $0 \leqslant \lambda < 1$.

1. If

$$\|\Phi\|_{\lambda,p,A} = \int_{\mathbb{R}^n} |\Phi(u)||detA(u)|^{-\frac{1}{p}} \|A^{-1}(u)\|^{\frac{\lambda}{p}} du < \infty,$$

then

$$\|H_\mu f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \leqslant \|\Phi\|_{\lambda,p,A} \|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}.$$

2. If

$$\|\Phi\|^*_{\lambda,p,A} = \int_{\mathbb{R}^n} |\Phi(u)||detA^{-1}(u)|^{1-\frac{1}{p}} \|A(u)\|^{\frac{\lambda}{p}} du < \infty,$$

then

$$\|H^*_\mu f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \leqslant \|\Phi\|^*_{\lambda,p,A} \|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}.$$

Proof. In view of the Minkowski inequality for $\mathscr{B}_{p,\lambda}(\mathbb{R}^n)$ and Lemma 3, we find that

$$\begin{aligned}\|\mathscr{H}_{\Phi,A} f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} &\leqslant \int_{\mathbb{R}^n} |\Phi(u)| \|D_{A(u)} f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} du \\ &\leqslant \left(\int_{\mathbb{R}^n} |\Phi(u)||\mathrm{det} A(u)|^{-\frac{1}{p}} \|A^{-1}(u)\|^{\frac{\lambda}{p}} du \right) \|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \\ &= \|\Phi\|_{\lambda,p,A} \|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\|H^*_\mu f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} &\leqslant \int_{\mathbb{R}^n} |\Phi(u)||\mathrm{det} A^{-1}(u)| \|D_{A^{-1}(u)} f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} du \\ &\leqslant \left(\int_{\mathbb{R}^n} |\Phi(u)||\mathrm{det} A^{-1}(u)|^{1-\frac{1}{p}} \|A(u)\|^{\frac{\lambda}{p}} du \right) \|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)} \\ &= \|\Phi\|^*_{\lambda,p,A} \|f\|_{\mathscr{B}_{p,\lambda}(\mathbb{R}^n)}. \quad \square\end{aligned}$$

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