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SHARP BOUNDS FOR TOADER–QI MEAN IN TERMS OF LOGARITHMIC AND IDENTRIC MEANS

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Abstract. In the article, we prove that the double inequality $\lambda\sqrt{L(a,b)I(a,b)} < TQ(a,b) < \mu\sqrt{L(a,b)I(a,b)}$ holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda\leqslant\sqrt{e/\pi}$ and $\mu\geqslant 1$, and give an affirmative answer to the conjecture proposed by Yang in [39], where $L(a,b)=(b-a)/(\log b-\log a)$, $I(a,b)=(b^b/a^a)^{1/(b-a)}/e$ and $TQ(a,b)=\frac{2}{\pi}\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}d\theta$ are respectively the logarithmic, identric and Toader-Qi means of a and b.

1. Introduction

Let $a,b>0$ with $a\neq b$. Then the logarithmic mean $L(a,b)$, identric mean $I(a,b)$ and Toader-Qi mean $TQ(a,b)$ [21, 32] of a and b are defined by

$$L(a,b)=\frac{b-a}{\log b-\log a},\qquad I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \tag{1.1}$$

$$TQ(a,b)=\frac{2}{\pi}\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}d\theta. \tag{1.2}$$

Recently, the logarithmic mean $L(a,b)$ and identric mean $I(a,b)$ have been the subject of intensive research. In particular, many remarkable inequalities for $L(a,b)$ and $I(a,b)$ can be found in the literature [2, 5, 7–12, 14–16, 22, 24, 29–31, 34–39, 41, 42]. In [26, 27, 33], the inequalities between the logarithmic mean, arithmetic mean and classical arithmetic-geometric mean of Gauss are presented. The ratio of identric means leads to the weighted geometric mean

$$\frac{I\left(a^2,b^2\right)}{I(a,b)}=\left(a^ab^b\right)^{1/(a+b)},$$

which was investigated in [23, 25, 28]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [13, 17, 18]. In [13], the authors studied a variant of Jensen's functional equation involving the logarithmic

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mean, which appears in a heat conduction problem. A representation of the logarithmic mean as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in [5]. In [4, 6], it is shown that the logarithmic mean can be expressed in terms of Gauss hypergeometric function ${}_2F_1$. Carlson and Gustafson [6] proved that the reciprocal of the logarithmic mean is strictly totally positive, that is, every $n\times n$ determinant with elements $1/L(a_i,b_j)$, where $0<a_1<a_2<\cdots<a_n$ and $0<b_1<b_2<\cdots<b_n$, is positive for all $n\geqslant 1$.

Very recently, Qi et al. [21] proved that the identity

$$TQ(a,b)=\sqrt{ab}I_0\left(\frac{1}{2}\log\frac{b}{a}\right) \tag{1.3}$$

and the inequalities

$$L(a,b)<TQ(a,b)<\frac{A(a,b)+G(a,b)}{2}<\frac{2A(a,b)+G(a,b)}{3}<I(a,b)$$

hold for all $a,b>0$ with $a\neq b$, where

$$I_0(t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{2^{2n}(n!)^2} \tag{1.4}$$

is the modified Bessel function of the first kind [1], and $A(a,b)=(a+b)/2$ and $G(a,b)=\sqrt{ab}$ are respectively the classical arithmetic and geometric means of a and b.

In [40], Yang proved that the double inequalities

$$\sqrt{\frac{2A(a,b)L(a,b)}{\pi}}<TQ(a,b)<\sqrt{A(a,b)L(a,b)},$$

$$A^{1/4}(a,b)L^{3/4}(a,b)<TQ(a,b)<\frac{1}{4}A(a,b)+\frac{3}{4}L(a,b)$$

and conjectured that the inequality

$$TQ(a,b)<\sqrt{L(a,b)I(a,b)} \tag{1.5}$$

hold for all $a,b>0$ with $a\neq b$.

Let $b>a>0$ and $t=(\log b-\log a)/2>0$. Then from (1.1) we clearly see that the logarithmic mean $L(a,b)$ and identric mean $I(a,b)$ can be expressed as

$$L(a,b)=\sqrt{ab}\frac{\sinh t}{t},\ \ I(a,b)=\sqrt{ab}e^{t/\tanh t-1}, \tag{1.6}$$

and (1.2) and (1.3) lead to

$$\begin{aligned}\frac{TQ(a,b)}{\sqrt{ab}}&=\frac{2}{\pi}\int_0^{\pi/2}e^{t\cos(2\theta)}d\theta=I_0(t)\\&=\frac{2}{\pi}\int_0^{\pi/2}\cosh(t\cos\theta)d\theta=\frac{2}{\pi}\int_0^{\pi/2}\cosh(t\sin\theta)d\theta.\end{aligned} \tag{1.7}$$

The main purpose of this paper is to present the best possible parameters λ and μ such that the double inequality

$$\lambda\sqrt{L(a,b)I(a,b)} < TQ(a,b) < \mu\sqrt{L(a,b)I(a,b)}$$

holds for all $a,b>0$ with $a\neq b$ and give a positive answer to the conjecture given in (1.5).

2. Lemmas

In order to prove our main result we need several lemmas, which we present in this section.

LEMMA 2.1. (See [19]) *Let* $\binom{n}{k}$ *be the number of different ways to choose* k *elements from a given set with* n *distinct elements, that is*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Then

$$\sum_{k=0}^{\infty}\binom{n}{k}^2 = \binom{2n}{n}.$$

LEMMA 2.2. (See [19]) *Let* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *be two real sequences with* $b_n>0$ *and* $\lim_{n\to\infty} a_n/b_n = s$. *Then the power series* $\sum_{n=0}^{\infty} a_n t^n$ *is convergent for all* $t\in\mathbb{R}$ *and*

$$\lim_{t\to\infty}\frac{\sum_{n=0}^{\infty} a_n t^n}{\sum_{n=0}^{\infty} b_n t^n} = s$$

if the power series $\sum_{n=0}^{\infty} b_n t^n$ *is convergent for all* $t\in\mathbb{R}$.

LEMMA 2.3. (See [20]) *The double inequality*

$$\frac{1}{(x+a)^{1-a}} < \frac{\Gamma(x+a)}{\Gamma(x+1)} < \frac{1}{x^{1-a}}$$

holds for all $x>0$ *and* $a\in(0,1)$, *where* $\Gamma(x)=\int_0^{\infty} e^{-t}t^{x-1}dt$ *is the classical Euler gamma function.*

LEMMA 2.4. (See [3]) *Let* $A(t)=\sum_{k=0}^{\infty} a_k t^k$ *and* $B(t)=\sum_{k=0}^{\infty} b_k t^k$ *be two real power series converging on* $(-r,r)$ $(r>0)$ *with* $b_k>0$ *for all* k. *If the non-constant sequence* $\{a_k/b_k\}$ *is increasing (decreasing) for all* k, *then the function* $A(t)/B(t)$ *is strictly increasing (decreasing) on* $(0,r)$.

LEMMA 2.5. *Let $I_0(t)$ be defined by (1.4). Then the identity*

$$I_0^2(t) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^4} t^{2n}$$

holds for all $t \in \mathbb{R}$.

Proof. From (1.4) and Lemma 2.1 together with the Cauchy product we have

$$\begin{aligned} I_0^2(t) &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{1}{2^{2k}(k!)^2} \frac{1}{2^{2(n-k)}[(n-k)!]^2} \right) t^{2n} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2^{2n}(n!)^2} \sum_{k=0}^{n} \frac{(n!)^2}{(k!)^2[(n-k)!]^2} \right) t^{2n} = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^4} t^{2n}. \quad \square \end{aligned}$$

LEMMA 2.6. *Let $I_0(t)$ be defined by (1.4) and*

$$B(t) = \sqrt{\frac{15[\sinh(2t) + \sinh t]}{44t} - \frac{1}{44}\left(1 + \frac{1}{2}t^2\right)}. \tag{2.1}$$

Then the double inequality

$$\sqrt{\frac{44}{15\pi}} B(t) < I_0(t) < B(t)$$

holds for all $t > 0$.

Proof. Let the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be defined by

$$a_n = \frac{(2n)!}{2^{2n}(n!)^4},$$

$$b_0 = 1, \qquad b_1 = \frac{1}{2}, \qquad b_n = \frac{15\left(2^{2n+1}+1\right)}{44(2n+1)!} \quad (n \geqslant 2).$$

Then simple computations lead to

$$\frac{a_0}{b_0} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = 1 > \frac{a_3}{b_3} = \frac{385}{387} > \frac{a_4}{b_4} = \frac{2695}{2736}, \tag{2.2}$$

$$\begin{aligned} \frac{a_n}{b_n} &= \frac{44}{15} \frac{2^{2n}}{2^{2n+1}+1} \frac{(2n)!(2n+1)!}{2^{4n}(n!)^4} \\ &= \frac{44}{15} \frac{2^{2n}(2n+1)}{2^{2n+1}+1} \left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(n+1)} \right)^2 \\ &= \frac{44}{15\pi} \frac{2^{2n}(2n+1)}{2^{2n+1}+1} \left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)} \right)^2, \end{aligned} \tag{2.3}$$

$$\frac{\frac{a_{n+1}}{b_{n+1}}}{\frac{a_n}{b_n}}-1=-\frac{2^{2n+1}-\left(3n^2+6n+2\right)}{(n+1)^2\left(2^{2n+3}+1\right)} \tag{2.4}$$

and

$$\begin{aligned}&2^{2n+1}-\left(3n^2+6n+2\right)\\&>2\left(1+2n+\frac{2n(2n-1)}{2}\right)-\left(3n^2+6n+2\right)\\&=n(n-4)\geqslant 0\end{aligned} \tag{2.5}$$

for all $n\geqslant 4$.

From Lemma 2.3 and (2.2)–(2.5) we clearly see that the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ is decreasing and

$$\frac{44}{15\pi}\frac{2^{2n}(2n+1)}{\left(2^{2n+1}+1\right)\left(n+\frac{1}{2}\right)}<\frac{a_n}{b_n}<\frac{44}{15\pi}\frac{2^{2n}(2n+1)}{\left(2^{2n+1}+1\right)n} \tag{2.6}$$

for all $n\geqslant 4$.

Inequality (2.6) leads to

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{44}{15\pi}. \tag{2.7}$$

It follows from Lemma 2.5 and (2.1) together with the monotonicity of the sequence $\{a_n/b_n\}_{n=0}^{\infty}$ that

$$\frac{I_0^2(t)}{B^2(t)}=\frac{\sum_{n=0}^{\infty}a_nt^{2n}}{\frac{15}{44}\sum_{n=0}^{\infty}\frac{\left(2^{2n+1}+1\right)}{(2n+1)!}t^{2n}-\frac{1}{44}\left(1+\frac{1}{2}t^2\right)}=\frac{\sum_{n=0}^{\infty}a_nt^{2n}}{\sum_{n=0}^{\infty}b_nt^{2n}} \tag{2.8}$$

and the function $I_0^2(t)/B^2(t)$ is strictly decreasing on the interval $(0,\infty)$.

Lemma 2.2, (2.2), (2.7), (2.8) and the monotonicity of $I_0^2(t)/B^2(t)$ lead to the conclusion that

$$\frac{44}{15\pi}=\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{t\to\infty}\frac{I_0^2(t)}{B^2(t)}<\frac{I_0^2(t)}{B^2(t)}<\lim_{t\to 0^+}\frac{I_0^2(t)}{B^2(t)}=\frac{a_0}{b_0}=1 \tag{2.9}$$

for all $t>0$.

Therefore, Lemma 2.6 follows easily from (2.9). □

LEMMA 2.7. *The double inequality*

$$\begin{aligned}&\frac{15}{44}(2\cosh t+1)-\frac{1}{44}\left(1+\frac{1}{2}t^2\right)\frac{t}{\sinh t}<e^{t/\tanh t-1}\\&<\frac{44}{15e}\left[\frac{15}{44}(2\cosh t+1)-\frac{1}{44}\left(1+\frac{1}{2}t^2\right)\frac{t}{\sinh t}\right]\end{aligned} \tag{2.10}$$

holds for all $t>0$.

Proof. Let

$$f_1(t) = \log\left(e^{t/\tanh t - 1}\right) - \log\left[\frac{15}{44}(2\cosh t + 1) - \frac{1}{44}\left(1 + \frac{1}{2}t^2\right)\frac{t}{\sinh t}\right] \quad (2.11)$$
$$= \frac{t\cosh t}{\sinh t} - \log\left[\frac{15}{44}(2\cosh t + 1) - \frac{1}{44}\left(1 + \frac{1}{2}t^2\right)\frac{t}{\sinh t}\right] - 1,$$
$$f_2(t) = 15\cosh(3t) + 62\cosh(2t) + 3t^2\cosh(2t) - 2t^3\sinh(2t)$$
$$-64t\sinh(2t) - 15t\cosh t - 60t\sinh t + 2t^4 + t^2 - 62$$

and

$$u_n = 30 \times 3^{2n} - \left(4n^3 - 12n^2 + 133n - 124\right)2^{2n} - 30(8n+1).$$

Then elaborated computations lead to

$$f_1(0^+) = 0, \quad \lim_{t\to\infty} f_1(t) = \log\frac{44}{15e}, \quad (2.12)$$

$$f_1'(t) = \frac{f_2(t)}{2\left[30\sinh(2t) + 30\sinh t - t^3 - 2t\right]\sinh^2 t}, \quad (2.13)$$

$$f_2(t) = \sum_{n=3}^{\infty}\frac{u_n}{2(2n)!}t^{2n}, \quad (2.14)$$

$$u_3 = 3520, \quad (2.15)$$

$$u_{n+1} - 9u_n = \left[20(n-2)^3 + 12(n-2)^2 + 521(n-2) + 34\right]2^{2n} + 1920n > 0 \quad (2.16)$$

for $n \geqslant 3$.

It is not difficult to verify that

$$30\sinh(2t) + 30\sinh t - t^3 - 2t > 0 \quad (2.17)$$

for $t > 0$.

It follows from (2.15) and (2.16) that $u_n > 0$ for all $n \geqslant 3$. Then (2.13) and (2.14) together with (2.17) lead to the conclusion that $f_1(t)$ is strictly increasing on the interval $(0,\infty)$.

From (2.12) and the monotonicity of $f_1(t)$ we clearly see that

$$0 < f_1(t) < \log\frac{44}{15e} \quad (2.18)$$

for all $t > 0$.

Therefore, inequality (2.10) follows easily from (2.11) and (2.18). □

3. Main results

THEOREM 3.1. *The double inequality*

$$\lambda\sqrt{L(a,b)I(a,b)} < TQ(a,b) < \mu\sqrt{L(a,b)I(a,b)} \tag{3.1}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\lambda\leqslant\sqrt{e/\pi}$ *and* $\mu\geqslant 1$.

Proof. Since $L(a,b)$, $TQ(a,b)$ and $I(a,b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $b>a>0$. Let $t=(\log b-\log a)/2>0$. Then from (1.6) and (1.7) we clearly see that inequality (3.1) is equivalent to

$$\lambda^2\frac{\sinh t}{t}e^{t/\tanh t-1} < I_0^2(t) < \mu^2\frac{\sinh t}{t}e^{t/\tanh t-1}. \tag{3.2}$$

Let $B(t)$ be defined by (2.1). Then it follows from Lemma 2.7 that

$$B^2(t) < \frac{\sinh t}{t}e^{t/\tanh t-1} < \frac{44}{15e}B^2(t),$$

which leads to

$$\frac{15e}{44}\frac{\sinh t}{t}e^{t/\tanh t-1} < B^2(t) < \frac{\sinh t}{t}e^{t/\tanh t-1}. \tag{3.3}$$

From Lemma 2.6 and (3.3) we have

$$\frac{e}{\pi}\frac{\sinh t}{t}e^{t/\tanh t-1} < I_0^2(t) < \frac{\sinh t}{t}e^{t/\tanh t-1}. \tag{3.4}$$

Note that

$$\lim_{t\to 0^+}\frac{\sinh t}{t} = \lim_{t\to 0^+} I_0(t) = \lim_{t\to 0^+} e^{t/\tanh t-1} = 1, \tag{3.5}$$

$$\lim_{t\to\infty}\frac{\cosh t}{e^{t/\tanh t-1}} = \frac{e}{2}\lim_{t\to\infty}\frac{1+e^{-2t}}{e^{\frac{2te^{-t}}{e^t+e^{-t}}}} = \frac{e}{2}. \tag{3.6}$$

It follows from Lemmas 2.2, 2.3 and 2.5 that

$$\frac{1}{n+1/2} < \frac{\Gamma^2(n+1/2)}{\Gamma^2(n+1)} < \frac{1}{n},$$

$$\begin{aligned}\lim_{t\to\infty}\frac{I_0^2(t)}{\sinh(2t)/(2t)} &= \lim_{t\to\infty}\frac{\sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}(n!)^4}t^{2n}}{\sum_{n=0}^{\infty}\frac{2^{2n}}{(2n+1)!}t^{2n}} = \lim_{n\to\infty}\frac{(2n)!(2n+1)!}{2^{4n}(n!)^4} \\ &= \lim_{n\to\infty}\left[(2n+1)\left(\frac{(2n-1)!!}{2^n n!}\right)^2\right] \\ &= \lim_{n\to\infty}\left[\frac{2n+1}{\Gamma^2(1/2)}\left(\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right)^2\right] \\ &= \lim_{n\to\infty}\left[\frac{2(n+1/2)}{\pi}\left(\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right)^2\right] = \frac{2}{\pi}.\end{aligned} \tag{3.7}$$

Equations (3.5)–(3.7) lead to

$$\lim_{t\to0^+}\frac{I_0^2(t)}{\frac{\sinh t}{t}e^{t/\tanh t-1}}=1,\qquad \lim_{t\to\infty}\frac{I_0^2(t)}{\frac{\sinh t}{t}e^{t/\tanh t-1}}=\frac{e}{\pi}. \tag{3.8}$$

Therefore, inequality (3.2) holds for all $t>0$ if and only if $\lambda\leqslant\sqrt{e/\pi}$ and $\mu\geqslant1$ follows from (3.4) and (3.8). □

REMARK 3.1. Theorem 3.1 gives a positive answer to the conjecture given in (1.5).

Let a_n, b_n and $B(t)$ be defined as in Lemma 2.6. Then $a_0=b_0=1$, $a_1=b_1=1/2$, $a_2=b_2=3/32$, $a_3/b_3=385/387$ and the sequence $\{a_n/b_n\}$ is decreasing for $n\geqslant0$. It follows from Lemmas 2.2 and 2.4 together with (2.7) we clearly see that

$$\frac{44}{15\pi}=\lim_{n\to\infty}\frac{a_n}{b_n}<\frac{I_0^2(t)-\left(a_0+a_1t^2+a_2t^4\right)}{B^2(t)-(b_0+b_1t^2+b_2t^4)}=\frac{\sum_{n=3}^\infty a_nt^{2n}}{\sum_{n=3}^\infty b_nt^{2n}}<\frac{a_3}{b_3}=\frac{385}{387} \tag{3.9}$$

for all $t>0$.

Inequality (3.9) leads to Corollary 3.1 immediately.

COROLLARY 3.1. *The double inequality*

$$\sqrt{\frac{\sinh(2t)+\sinh t}{\pi t}+\frac{\pi-3}{\pi}\left(1+\frac{1}{2}t^2+\frac{15\pi-44}{160(\pi-3)}t^4\right)}<I_0(t)$$
$$<\sqrt{\frac{175[\sinh(2t)+\sinh t]}{516t}-\frac{3}{172}\left(1+\frac{1}{2}t^2-\frac{1}{36}t^4\right)}$$

holds for all $t>0$.

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