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MULTIDIMENSIONAL EXTENSIONS OF PÓLYA–KNOPP–TYPE INEQUALITIES OVER SPHERICAL CONES

CHANG-PAO CHEN, JIN-WEN LAN AND DAH-CHIN LUOR

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Abstract. In this paper, we introduce a new type of limit process to evaluate the modular-type operator norm of an integral operator. This leads us to get multidimensional extensions of Pólya-Knopp-type inequalities with general measures. Our results not only extend Levin-Cochran-Lee-type inequalities from $n = 1$ to general n, but also improve the estimates given there. Moreover, they generalize Carleson's result, which is involved in the proof of Carleman's inequality. Besides these, the Pólya-Knopp-type inequalities for the cases of Laplace transform and generalized Riemann-Liouville operators are derived. For the lower bounds, a parallel theory to the above is also established.

1. Introduction

Let E be a spherical cone in $\mathbb{R}^n$, that is, $E = \cup_{s>0}\, sA$ for some Borel measurable subset A of the unit sphere Σ^{n-1}. For instance, $E = \mathbb{R}^n \setminus \{0\}$ for $A = \Sigma^{n-1}$. In [3], the smallest constant C in (1.1) was investigated:

$$\left\{\int_E \left(\Phi \circ \mathbb{K}f(x)\right)^q d\mu\right\}^{1/q} \leqslant C\left\{\int_E \left(\Phi \circ f(x)\right)^p d\nu\right\}^{1/p} \tag{1.1}$$

for all $f \in D_{\mathbb{K}} \cap L^p_{\Phi}(d\nu)$, where $p,q \neq 0$, μ, ν are two σ-finite Borel measures on E, $\Phi \in CV^+(I)$, $\Phi \circ f(x) = \Phi(f(x))$, and $\mathbb{K}f(x)$ is one of the following two forms:

$$\mathbb{K}f(x) := \int_{\tilde{S}_x} k(x,t)f(t)\,d\sigma(t) \qquad (x \in E) \tag{1.2}$$

and

$$\tilde{\mathbb{K}}f(x) := \int_{E\setminus S_x} k(x,t)f(t)\,d\sigma(t) \qquad (x \in E). \tag{1.2*}$$

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Here $D_{\mathbb{K}}$ is the space of those f such that $\mathbb{K}f(x)$ is well-defined for μ a.e. $x \in E$ and $L^p_\Phi(d\nu)$ is the set of all real-valued Borel measurable f with

$$\|f\|_{\Phi,p,\nu} := \left\{ \int_E \Big(\Phi \circ f(x) \Big)^p d\nu \right\}^{1/p} < \infty. \tag{1.3}$$

Note that the integral in (1.3) is realized as $\sup_{x\in E} |\Phi \circ f(x)|$ for the case $p = \infty$, the class $CV^+(I)$ denotes the set of all nonnegative convex functions defined on an open interval I in $\mathbb{R}$, $\tilde{S}_x = \cup_{0<s\leqslant \|x\|} sA$, $S_x = \tilde{S}_x \setminus \|x\|A$, $k(x,t) \geqslant 0$ is locally integrable over $\mathbb{E} \times \mathbb{E}$, and σ is a σ-finite Borel measure on E.

For $\Phi(s) = |s|$, $L^p_\Phi(d\nu)$ and $\|f\|_{\Phi,p,\nu}$ reduce to the classical Banach space $L^p(d\nu)$ and the usual L_p-norm $\|f\|_{p,\nu}$, respectively. We write $\|f\|_{L^p(E,\nu)}$ for $\|f\|_{p,\nu}$ in the case that the integral region E is emphasized. Let $\|\mathbb{K}\|_{D_{\mathbb{K}}\cap L^p_\Phi(d\nu)\mapsto L^q_\Phi(d\mu)}$ (in brief, $\|\mathbb{K}\|_*$) denote the smallest constant C in (1.1). Clearly,

$$\|\mathbb{K}\|_* = \sup_f \frac{\|\Phi \circ \mathbb{K}f\|_{q,\mu}}{\|\Phi \circ f\|_{p,\nu}},$$

where the supremum is taken over all $f \in D_{\mathbb{K}} \cap L^p_\Phi(d\nu)$ with $\|\Phi \circ f\|_{p,\nu} \neq 0$. The investigation of the value $\|\mathbb{K}\|_*$ goes back to the work of Hardy. In [11, Theorem 327], it was proved that $\|\mathbb{K}\|_* = p/(p-1)$ for the case:

$$1 < p = q < \infty, \quad n = 1, \quad \Phi(s) = |s|, \quad k(x,t) = 1/|B(\|x\|)|, \quad d\sigma = dt,$$

$$A = \Sigma^{n-1}, E = \mathbb{R}^n \setminus \{0\}, d\mu = d\nu = dx,$$

where $B(r)$ is the closed ball centered at the origin with radius r. This result was extended from $n = 1$ to general n by Christ and Grafakos (see [4, Theorem 1]). Recently, it was further extended to the general form of (1.1). In [3], the present authors established the following Muckenhoupt-type estimate for $\|\mathbb{K}\|_*$:

$$\|\mathbb{K}\|_* \leqslant A_{p,q} := \left(\frac{q}{p^*} + \frac{q}{\eta} \right)^{1/q} \left(1 + \frac{p^*}{\eta} \right)^{\eta^*/(p^* q^*)} A_M(p,q), \tag{1.4}$$

where $1 \leqslant p,q \leqslant \infty$, $\eta = \max(p,q)$, $(\cdot)^*$ is the conjugate exponent of $(\cdot)$, and $A_M(p,q)$ is a generalized Muckenhoupt constant defined by the factorization $k(x,t) = g(t)\psi(x,t)$ (see Section 2 for details). In the same paper, we also derived the following inequality:

$$\|\tilde{\mathbb{K}}\|_* \leqslant \tilde{A}_{p,q} := \left(\frac{q}{p^*} + \frac{q}{\eta} \right)^{1/q} \left(1 + \frac{p^*}{\eta} \right)^{\eta^*/(p^* q^*)} \tilde{A}_M(p,q), \tag{1.4*}$$

where $\tilde{A}_M(p,q)$ is the corresponding generalized Muckenhoupt constant. We remark that the inequality (1.4) gives us an n-dimensional weighted extension of Levinson's modular inequality (cf. [16]) and both of (1.4) and (1.4^*) demonstrate the n-dimensional modular forms of [1, 12]. Moreover, these two inequalities generalize the result of Muckenhoupt-Bradley-Maz'ja, and so on, and the estimates given in (1.4) and

(1.4^*) improve the corresponding ones given in [9, 12, 22] and [19, Theorem 5.3]. Besides these, some examples show that we can use different choices of $g(t)$ and $\psi(x,t)$ in the decomposition $k(x,t) = g(t)\psi(x,t)$ to decrease the upper bound estimate in (1.4) to a better value for $\|\mathbb{K}\|_*$ and the estimates in (1.4) can be better than p^*A_{PS} and A_W defined in [20, 23] (cf. [3] for details).

In this paper, we focus on finding out the extensions of the following Pólya-Knopp inequality:

$$\int_0^\infty \exp\left(\frac{1}{x}\int_0^x \log f(t)dt\right)dx \leqslant e\int_0^\infty f(x)dx \qquad (f \geqslant 0).$$

The desired generalizations will be set up in the form (1.1) with $\Phi^\varepsilon \in CV^+(I)$ for some $\varepsilon > 0$ or its reduced forms from some replacements, e.g., $\Phi(s) \longrightarrow e^s$ and $f(t) \longrightarrow \log|f(t)|$. In addition, the constants involved here will be derived from a limit process acting on a family of $A_{p,q}$,which are not a direct result from the choice of $A_{p,q}$ in (1.4). More precisely, C will be obtained from the evaluation of the right side of (1.5) or (1.5^*);

$$\|\mathbb{K}\|_* \leqslant \inf_{\varepsilon \in \mathfrak{F}_\Phi^+}\left\{(A_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon}\right\} \tag{1.5}$$

and

$$\|\tilde{\mathbb{K}}\|_* \leqslant \inf_{\varepsilon \in \mathfrak{F}_\Phi^+}\left\{(\tilde{A}_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon}\right\}, \tag{1.5*}$$

where $0 < p,q < \infty$ and $\mathfrak{F}_\Phi^+ = \{\varepsilon > 0 : \Phi^\varepsilon \in CV^+(I)\}$. Note that (1.5) and (1.5^*) will be established for general $k(x,t)$ (cf. Section 3). These two inequalities provide us with a new type of limit process to get different types of Pólya-Knopp inequalities. In Section 4, we consider the special case: $k(x,t) = 1/\Lambda(x)$ (respectively, $k(x,t) = 1/\tilde{\Lambda}(x)$) with $\Phi^\varepsilon \in CV^+(I)$ (respectively, $(1/\Phi)^\varepsilon \in CV^+(I)$) for small $\varepsilon > 0$, where

$$\Lambda(x) = \int_{\tilde{S}_x} d\sigma \qquad \text{and} \qquad \tilde{\Lambda}(x) = \int_{E\setminus S_x} d\sigma. \tag{1.6}$$

As a consequence, we derive, in Section 5, two general forms of the Levin-Cochran-Lee-type inequalities, which correspond to $d\sigma(t) = \|t\|^{s-1}dt$. Our results give n-dimensional extensions of [6, Theorem 9], [8, Theorem 1], [10], [13, Theorems 3.3 & 4.3], [14, p.51, Example 1.22], [17, Corollary 6], and [23, Proposition 7.5]. Moreover, the best constants involved here improve the known ones appearing in [10], [13, Inequality (4.8)], [14, p.51, Example 1.22], and [23, Inequality (7.37)]. We point out that Carleson's result [2] is a special case. In Section 6, we consider $d\sigma(t) = e^{-\gamma\|t\|}\|t\|^{1-n}dt$. This case is related to the Laplace transform. For such a case, the results in Section 4 lead us to the Pólya-Knopp-type inequality associated with Stepanov's and Heinig's results.

We remark that the inequality (1.5) not only extends the result of P. Jain et al. from $n = 1$ and $k(x,t) = 1/|\tilde{S}_x|$ to $n \geqslant 1$ and general $k(x,t)$, but also improves the known upper bound estimate for the best constant (cf. Section 3). Moreover, the limit process associated with (1.5) also works well for other cases with $k(x,t) \neq 1/\Lambda(x)$

and $k(x,t) \neq 1/\tilde{\Lambda}(x)$, for example, the case of the Riemann-Liouville operator, which corresponds to $k(x,t) = \gamma(\|x\| - \|t\|)^{\gamma-1}/(|A|\|x\|^{\gamma})$ and $d\sigma = \|t\|^{1-n}dt$, where $\gamma > 0$. Applying (1.5) to this case, we get the Pólya-Knopp-type inequality of Riemann-Liouville operator (see Section 7).

Let $L_*(\mathbb{K})$ be the number defined by the formula:

$$L_*(\mathbb{K}) = \inf_f \frac{\|\Phi \circ \mathbb{K}f\|_{q,\mu}}{\|\Phi \circ f\|_{p,\nu}},$$

where the infimum is taken over all $f \geqslant 0$ with $\|\Phi \circ f\|_{p,\nu} \neq 0$. This number was investigated by Prokhorov [21] for the following case:

$$-\infty < p,q < 0, \quad n = 1, \quad \Phi(s) = |s|, \quad k(x,t) = v(t), \quad d\sigma = \chi_{(a,b)}(t)dt,$$

$$d\mu = u(x)\chi_{(a,b)}(x)dx, \quad d\nu = \chi_{(a,b)}(x)dx.$$

Clearly, $L_*(\mathbb{K}) \leqslant \|\mathbb{K}\|_*$. In Section 3 to Section 7, we shall prove that a parallel theory to $\|\mathbb{K}\|_*$ is also true for $L_*(\mathbb{K})$. For instance, the following inequality holds for $-\infty < p,q < 0$:

$$L_*(\mathbb{K}) \geqslant \sup_{\varepsilon \in \mathfrak{F}_\Phi^-} \left\{ (A_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon} \right\}. \tag{1.7}$$

Here $\mathfrak{F}_\Phi^- = \{\varepsilon < 0 : \Phi^\varepsilon \in CV^+(I)\}$. Similarly, we have

$$L_*(\tilde{\mathbb{K}}) \geqslant \sup_{\varepsilon \in \mathfrak{F}_\Phi^-} \left\{ (\tilde{A}_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon} \right\}. \tag{1.7*}$$

We shall give details in the corresponding sections from Section 3 to Section 7.

Throughout this paper, in no ambiguity, f will be assumed to obey $f \in D_{\mathbb{K}}$ (respectively, $f \in D_{\tilde{\mathbb{K}}}$) when (1.2) (respectively, (1.2*)) is involved. The symbol $|A|$ will denote the surface area of the set A given at the beginning.

2. Preliminary

Let $k(x,t) = g(t)\psi(x,t)$, where $g : E \mapsto (0,\infty)$ and $\psi : E \times E \mapsto [0,\infty)$. Suppose that $d\nu_a/d\sigma > 0$ for σ a. e. on E, where ν_a is the absolutely continuous part of the measure ν with respect to σ. For $1 \leqslant p,q < \infty$, set

$$A_{p,q}(x) = \left\| \frac{g(\cdot)}{d\nu_a/d\sigma} \right\|_{L^{p^*}(\tilde{S}_x,\nu)}^{\eta^*/q^*} \times \left\| \left(\sup_{t \in \tilde{S}_x} \psi(\cdot,t) \right) \right\|_{L^q(E \setminus S_x,\mu)}, \tag{2.1}$$

$$\tilde{A}_{p,q}(x) = \left\| \frac{g(\cdot)}{d\nu_a/d\sigma} \right\|_{L^{p^*}(E \setminus S_x,\nu)}^{\eta^*/q^*} \times \left\| \left(\sup_{t \in E \setminus S_x} \psi(\cdot,t) \right) \right\|_{L^q(\tilde{S}_x,\mu)}, \tag{2.2}$$

and

$$A_M(p,q) = \|A_{p,q}(\cdot)\|_{r,\omega}, \quad \tilde{A}_M(p,q) = \|\tilde{A}_{p,q}(\cdot)\|_{r,\omega},$$

$$d\omega(t) = \left(\frac{g(t)}{d\nu_a/d\sigma}\right)^{p^*} d\nu(t), \tag{2.3}$$

where $1/r = 1/q - 1/\eta$ and $\eta = \max(p,q)$. We remark that

$$\frac{\eta^*}{p^*q^*} = \begin{cases} 1/p^* & \text{if } \eta = q, \\ 1/q^* & \text{if } \eta = p. \end{cases}$$

Moreover, for $p^* = \infty$, $A_M(p,q) = \sup_{x\in E} |A_{p,q}(x)|$ and $\tilde{A}_M(p,q) = \sup_{x\in E} |\tilde{A}_{p,q}(x)|$. In [3, Theorems 3.3 & 3.5], the following result was established.

THEOREM 2.1. *Let* $1 \leqslant p,q < \infty$, $\Phi \in CV^+(I)$, *and* $k(x,t) = g(t)\psi(x,t)$. *Suppose that* (2.4) *and* (2.5) *are satisfied, where*

$$\int_{\tilde{S}_x} k(x,t)d\sigma(t) = 1 \qquad (x \in E) \tag{2.4}$$

and

$$\int_{\tilde{S}_x\setminus S_x} d\omega = 0 \qquad \text{for all} \quad x \in E. \tag{2.5}$$

If $A_M(p,q) < \infty$, *then* (1.4) *is true. Moreover, the condition* (2.5) *is not necessary for the case* $1 \leqslant p \leqslant q < \infty$. *For* $\Phi(s) = |s|$, *the condition* (2.4) *can be removed.*

For the complementary integral operator $\tilde{\mathbb{K}}$, the present authors also derived the following dual result of Theorem 2.1 (see [3, Theorems 3.4 & 3.5]).

THEOREM 2.2. *Let* $p,q,\Phi,k(x,t),g(t)$ *and* $\psi(x,t)$ *be defined as Theorem 2.1. Suppose that* (2.5) *and* (2.6) *are satisfied, where*

$$\int_{E\setminus S_x} k(x,t)d\sigma(t) = 1 \qquad (x \in E). \tag{2.6}$$

If $\tilde{A}_M(p,q) < \infty$, *then* (1.4^*) *is true. Moreover, the condition* (2.5) *is not necessary for the case* $1 \leqslant p \leqslant q < \infty$. *For* $\Phi(s) = |s|$, *the condition* (2.6) *can be removed.*

Clearly, Theorems 2.1 and 2.2 enable us to choose $A_{p,q}$ or $\tilde{A}_{p,q}$ for the value of C so that (1.1) holds. This plays an important role in developing the theory of Pólya-Knopp-type inequalities. We shall see this point later.

3. Main results

Let $0 \leqslant A_{p,q} \leqslant \infty$ be the constant obeying the following property:

$$\left(\int_E \left|\mathbb{K}f(x)\right|^q d\mu\right)^{1/q} \leqslant A_{p,q}\left(\int_E |f(x)|^p d\nu\right)^{1/p} \qquad (f \geqslant 0), \tag{3.1}$$

where $p,q \neq 0$ and $\mathbb{K}f(x)$ is of type (1.2) or (1.2^*). When (1.2^*) occurs, we shall write $\tilde{\mathbb{K}}f(x)$ and $\tilde{A}_{p,q}$ in the places of $\mathbb{K}f(x)$ and $A_{p,q}$, respectively. Throughout this section, we assume that $d\nu_a/d\sigma > 0$ for σ a. e. on E and then $A_{p,q}$ exists. Moreover, the following general form of Pólya-Knopp inequality holds.

THEOREM 3.1. *Let* $\mathbb{K}f(x)$ *be of type* (1.2). *Suppose that* (2.4) *is satisfied. Then* (1.5) *holds for* $0 < p,q < \infty$ *and* (1.7) *is true for* $-\infty < p,q < 0$.

Proof. Let $0 < p,q < \infty$ and $\varepsilon \in \mathfrak{F}_{\Phi}^{+}$. We have $\Phi^{\varepsilon} \in CV^{+}(I)$. The extended Jensen's integral inequality (cf. [3, Lemma 2.5]) and (2.4) together ensure that $\Phi^{\varepsilon} \circ \mathbb{K}f(x) \leqslant \mathbb{K}(\Phi^{\varepsilon} \circ f)(x)$ for all $f \geqslant 0$. We have $q/\varepsilon > 0$. By the definition of $A_{p/\varepsilon,q/\varepsilon}$, we obtain

$$\|\Phi^{\varepsilon} \circ \mathbb{K}f\|_{q/\varepsilon,\mu} \leqslant \|\mathbb{K}(\Phi^{\varepsilon} \circ f)\|_{q/\varepsilon,\mu} \leqslant A_{p/\varepsilon,q/\varepsilon}\|\Phi^{\varepsilon} \circ f\|_{p/\varepsilon,\nu}.$$

This can be rewritten in the following form:

$$\|\Phi \circ \mathbb{K}f\|_{q,\mu}^{\varepsilon} \leqslant A_{p/\varepsilon,q/\varepsilon}\|\Phi \circ f\|_{p,\nu}^{\varepsilon} \qquad (f \geqslant 0). \tag{3.2}$$

Take the $(1/\varepsilon)$ power for both sides of (3.2) and then (1.5) follows. As for the case that $-\infty < p,q < 0$ and $\varepsilon \in \mathfrak{F}_{\Phi}^{-}$, we find that the argument given before (3.2) still works well. This shows that (3.2) is true. Take the $(1/\varepsilon)$ power for both sides of (3.2). Since $1/\varepsilon < 0$, we have to reverse the inequality sign, and then (1.7) follows. The proof is complete. □

For $\tilde{\mathbb{K}}f(x)$, let $0 \leqslant \tilde{A}_{p,q} \leqslant \infty$ be the constant subject to (3.3), where

$$\left(\int_E \left|\tilde{\mathbb{K}}f(x)\right|^q d\mu\right)^{1/q} \leqslant \tilde{A}_{p,q}\left(\int_E |f(x)|^p d\nu\right)^{1/p} \qquad (f \geqslant 0). \tag{3.3}$$

To replace (2.4) by (2.6), the proof of Theorem 3.1 leads us to the following result.

THEOREM 3.2. *Let* $\tilde{\mathbb{K}}f(x)$ *be of type* (1.2^{*}). *Suppose that* (2.6) *is satisfied. Then* (1.5^{*}) *holds for* $0 < p,q < \infty$ *and* (1.7^{*}) *is true for* $-\infty < p,q < 0$.

We make three remarks on Theorems 3.1 and 3.2. First, (1.5) says that $\|\mathbb{K}\|_{*}$ is bounded above by the infimum of a family of $(A_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon}$. This gives us the following limit process:

$$\|\mathbb{K}\|_{*} \leqslant \inf_{\varepsilon \in \mathfrak{F}_{\Phi}^{+}} (A_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon} \leqslant \liminf_{\varepsilon \to 0^{+}} \left\{ (A_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon} \right\}, \tag{3.4}$$

whenever $(0,\varepsilon_0) \subset \mathfrak{F}_{\Phi}^{+}$ for some $\varepsilon_0 > 0$. To evaluate the right side of (3.4) by a particular choice of $A_{p,q}$, e.g. the one given in (1.4), we shall get an upper bound for $\|\mathbb{K}\|_{*}$. Under the change $f(t) \longrightarrow \log|f(t)|$, (3.4) will be transformed into a general form of the Pólya-Knopp inequality for $\mathbb{K}f(x)$. We have

$$\liminf_{\varepsilon \to 0^{+}} \left\{ (A_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon} \right\} = q^{1/q} e^{1/\eta} \liminf_{\varepsilon \to 0^{+}} \left\{ \varepsilon^{-1/q} \left(A_M(p/\varepsilon, q/\varepsilon) \right)^{1/\varepsilon} \right\},$$

where $\eta = \max(p,q)$. To compare this value with [13, Eqs. (3.4) & (4.3)] (see also [23, Theorem 2.2]), we see that (3.4) not only extends the result of P. Jain et al. from

$n = 1$ and $k(x,t) = 1/|\tilde{S}_x|$ to $n \geqslant 1$ and general $k(x,t)$, but also improves the known estimate for the best constant. For the case $\tilde{\mathbb{K}}f(x)$, (1.5^*) ensures that

$$\|\tilde{\mathbb{K}}\|_* \leqslant \inf_{\varepsilon \in \mathfrak{F}_\Phi^+} (\tilde{A}_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon} \leqslant \liminf_{\varepsilon \to 0^+} \left\{ (\tilde{A}_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon} \right\}, \tag{3.4*}$$

whenever $(0,\varepsilon_0) \subset \mathfrak{F}_\Phi^+$ for some $\varepsilon_0 > 0$. This will lead us to a general form of the Pólya-Knopp inequality for $\tilde{\mathbb{K}}f(x)$. A detailed discussion for special cases will be given in Section 4 to Section 7.

The second remark is that the limit process associated with (3.4) differs from the well-known scheme by means of the formula $(G_{\mathbb{K}}f)(x) = \lim_{\varepsilon \to 0^+} [\mathbb{K}(f^\varepsilon)]^{1/\varepsilon}(x)$ (cf. e.g. [23, Section 7]). The essential part of this process is to evaluate the value: $\liminf_{\varepsilon \to 0^+} \left\{ (A_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon} \right\}$, whenever $(0,\varepsilon_0) \subset \mathfrak{F}_\Phi^+$ for some $\varepsilon_0 > 0$. Clearly, such a limit is completely determined by those $A_{p/\varepsilon,q/\varepsilon}$ with ε sufficiently small. With the help of Theorem 2.1, we shall choose ε_0 so that the following conditions hold for all $0 < \varepsilon \leqslant \varepsilon_0$:

$$\Phi^\varepsilon \in CV^+(I), \quad p/\varepsilon \geqslant 1, \quad q/\varepsilon \geqslant 1, \quad \text{and} \quad A_M(p/\varepsilon, q/\varepsilon) < \infty.$$

Under this choice, the number $A_{p/\varepsilon,q/\varepsilon}$ in (3.4) can be evaluated by (1.4) with p/ε and q/ε in the places of p and q. For $\tilde{A}_{p/\varepsilon,q/\varepsilon}$ in (3.4^*), we shall take into a similar consideration. The details are given in Section 4 to Section 7.

Third, the infimum in (1.5) or (1.5^*) may not take place as $\varepsilon \longrightarrow 0^+$. Section 7 will give such an example (see the paragraph after Corollary 7.1).

At the end of this section, we consider the characterization of $\Phi(x)$. Let $\Phi(x) > 0$ on I. We know that $(\Phi^\varepsilon)'' = \varepsilon\Phi^{\varepsilon-2}((\varepsilon-1)(\Phi')^2 + \Phi\Phi'')$, so $\Phi^\varepsilon \in CV^+(I)$, if (3.5) holds for $\varepsilon > 0$ or (3.5^*) is true for $\varepsilon < 0$:

$$\Phi(x)\Phi''(x) \geqslant (\Phi'(x))^2 \quad \text{for all } x \in I, \tag{3.5}$$

$$\Phi(x)\Phi''(x) \leqslant (\Phi'(x))^2 \quad \text{for all } x \in I. \tag{3.5*}$$

It is easy to see that $\Phi(x) = e^x$ satisfies (3.5) and (3.5^*). Thus, Theorems 3.1 and 3.2 work well at least for the case $\Phi(x) = e^x$.

4. The case $k(x,t) = 1/\Lambda(x)$ or $1/\tilde{\Lambda}(x)$

Theorems 3.1 and 3.2 have many applications. In this section, we deal with the case $k(x,t) = 1/\Lambda(x)$ or $1/\tilde{\Lambda}(x)$, where $\Lambda(x)$ or $\tilde{\Lambda}(x)$ are defined by (1.6), and then consider their applications in the next sections. Set

$$\Lambda(\infty) = \int_E d\sigma \qquad \text{and} \qquad \tilde{\Lambda}(0) = \int_E d\sigma.$$

Theorem 3.1 gives the following consequence.

COROLLARY 4.1. *Let $\theta \in \mathbb{R}$, $0 < \Lambda(x) < \infty$, and $w : E \mapsto (0,\infty)$. Suppose that $\sigma(\tilde{S}_x \setminus S_x) = 0$ for all x and (4.1) is satisfied, where*

$$\frac{w(x_2)}{w(x_1)} \leqslant \left(\frac{\Lambda(x_2)}{\Lambda(x_1)}\right)^{\theta} \qquad (\|x_1\| \leqslant \|x_2\|). \tag{4.1}$$

(*i*) *If $0 < p,q < \infty$, $\dfrac{\alpha+1}{q} \geqslant \dfrac{\beta+1}{p}$, $\Phi^{\varepsilon} \in CV^{+}(I)$ for all sufficiently small $\varepsilon > 0$, and $\Lambda(\infty)^{(\alpha+1)/q-(\beta+1)/p} < \infty$, then for all $f : E \mapsto \bar{I}$,*

$$\left(\int_E \left\{\Phi\left(\frac{1}{\Lambda(x)}\int_{\tilde{S}_x} f(t)d\sigma(t)\right)\right\}^q \Lambda(x)^{\alpha} w(x)^{q/p}\, d\sigma(x)\right)^{1/q}$$
$$\leqslant C_{p,q}\Lambda(\infty)^{(\alpha+1)/q-(\beta+1)/p}\left\{\int_E (\Phi\circ f(x))^p \Lambda(x)^{\beta} w(x)\, d\sigma(x)\right\}^{1/p}, \tag{4.2}$$

where

$$C_{p,q} = \begin{cases} e^{(\beta+\theta)/p+1/q} & (p \leqslant q); \\ e^{(\beta+\theta+1)/p}\left(\frac{p-q}{(\alpha+1)p-(\beta+1)q}\right)^{1/q-1/p} & (q < p). \end{cases} \tag{4.3}$$

(*ii*) *If $-\infty < p,q < 0$, $\dfrac{\alpha+1}{q} \leqslant \dfrac{\beta+1}{p}$, $(1/\Phi)^{\varepsilon} \in CV^{+}(I)$ for all sufficiently small $\varepsilon > 0$, and $\Lambda(\infty)^{(\alpha+1)/q-(\beta+1)/p} > 0$, then (4.2) holds with a reversed sign of inequality and the corresponding constant $C_{p,q}$ is obtained from (4.3) by making the changes: $p \leqslant q \longrightarrow q \leqslant p$ and $q < p \longrightarrow p < q$.*

(*iii*) *The condition $\sigma(\tilde{S}_x \setminus S_x) = 0$ can be removed from (i) (respectively, (ii)) for the case $p \leqslant q$ (respectively, $q \leqslant p$).*

Proof. This is the special case $g(t) = 1$ and $\psi(x,t) = 1/\Lambda(x)$ of Theorem 3.1. Set $d\mu(x) = \Lambda(x)^{\alpha} w(x)^{q/p} d\sigma(x)$ and $d\nu(x) = \Lambda(x)^{\beta} w(x) d\sigma(x)$. Consider (*i*). Choose $\varepsilon_0 > 0$ so small that for $0 < \varepsilon < \varepsilon_0$,

$$\Phi^{\varepsilon} \in CV^{+}(I), \quad p_{\varepsilon} \geqslant 1, \quad q_{\varepsilon} \geqslant 1, \quad \text{and} \quad \theta < p_{\varepsilon}\left(1 - \frac{\alpha+1}{q_{\varepsilon}}\right), \tag{4.4}$$

where $p_{\varepsilon} = p/\varepsilon$ and $q_{\varepsilon} = q/\varepsilon$. We claim that $A_M(p_{\varepsilon}, q_{\varepsilon}) < \infty$ for such ε. Let $A_{\varepsilon}(x)$ denote the number $A_{p,q}(x)$ defined by (2.1) with p_{ε} and q_{ε} instead of p and q, respectively. For $p \leqslant q$, we have

$$A_{\varepsilon}(x) = \left(\int_{\tilde{S}_x} \Lambda(t)^{\beta(1-p_{\varepsilon}^{*})} w(t)^{1-p_{\varepsilon}^{*}} d\sigma(t)\right)^{1/p_{\varepsilon}^{*}}$$
$$\times \left(\int_{E\setminus S_x} \Lambda(s)^{\alpha-q_{\varepsilon}} w(s)^{q_{\varepsilon}/p_{\varepsilon}} d\sigma(s)\right)^{1/q_{\varepsilon}}. \tag{4.5}$$

The conditions $\dfrac{\alpha+1}{q_\varepsilon} \geqslant \dfrac{\beta+1}{p_\varepsilon}$ and $\theta < p_\varepsilon\left(1-\frac{\alpha+1}{q_\varepsilon}\right)$ together imply $(\theta+\beta)(1-p_\varepsilon^*)+1>0$. By (4.1),

$$\int_{\tilde{S}_x} \Lambda(t)^{\beta(1-p_\varepsilon^*)} w(t)^{1-p_\varepsilon^*} d\sigma(t) \leqslant \frac{\Lambda(x)^{\beta(1-p_\varepsilon^*)+1} w(x)^{1-p_\varepsilon^*}}{(\theta+\beta)(1-p_\varepsilon^*)+1} \tag{4.6}$$

(cf. the proof of [3, Corollary 4.1]). On the other hand, $\alpha-q_\varepsilon+1+q_\varepsilon\theta/p_\varepsilon<0$, and so by (4.1) again, we obtain

$$\int_{E\setminus S_x} \Lambda(s)^{\alpha-q_\varepsilon} w(s)^{q_\varepsilon/p_\varepsilon} d\sigma(s) \leqslant \frac{\Lambda(x)^{\alpha-q_\varepsilon+1} w(x)^{q_\varepsilon/p_\varepsilon}}{q_\varepsilon-\alpha-1-(q_\varepsilon\theta/p_\varepsilon)}. \tag{4.7}$$

Putting $(4.5)-(4.7)$ together yields

$$A_M(p_\varepsilon,q_\varepsilon) = \sup_{x\in E}|A_\varepsilon(x)| \leqslant \left(\frac{1}{(\theta+\beta)(1-p_\varepsilon^*)+1}\right)^{1/p_\varepsilon^*}\left(\frac{1}{q_\varepsilon-\alpha-1-(q_\varepsilon\theta/p_\varepsilon)}\right)^{1/q_\varepsilon} \times \Lambda(\infty)^{(\alpha+1)/q_\varepsilon-(\beta+1)/p_\varepsilon} < \infty.$$

With the help of Theorems 2.1 and 3.1, we infer that

$$\begin{aligned}\|\mathbb{K}\|_* \leqslant (A_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon} &= \left(\frac{q_\varepsilon}{p_\varepsilon^*}+1\right)^{1/(\varepsilon q_\varepsilon)}\left(1+\frac{p_\varepsilon^*}{q_\varepsilon}\right)^{1/(\varepsilon p_\varepsilon^*)}\Big(A_M(p_\varepsilon,q_\varepsilon)\Big)^{1/\varepsilon} \\ &\leqslant \frac{\left(1-\frac{\varepsilon}{p}+\frac{\varepsilon}{q}\right)^{1/\varepsilon-1/p+1/q}\Lambda(\infty)^{(\alpha+1)/q-(\beta+1)/p}}{\left(1-\frac{\varepsilon(\beta+1)}{p}-\frac{\varepsilon\theta}{p}\right)^{1/\varepsilon-1/p}\left(1-\frac{\varepsilon(\alpha+1)}{q}-\frac{\varepsilon\theta}{p}\right)^{1/q}} \\ &\longrightarrow e^{(\beta+\theta)/p+1/q}\Lambda(\infty)^{(\alpha+1)/q-(\beta+1)/p} \quad \text{as} \quad \varepsilon\to 0^+.\end{aligned} \tag{4.8}$$

Hence, (4.2) holds for $p\leqslant q$. Next, consider $q<p$. We have $A_M(p_\varepsilon,q_\varepsilon)=\|A_\varepsilon(x)\|_{r_\varepsilon,\omega_\varepsilon}$, where $1/r_\varepsilon = 1/q_\varepsilon-1/p_\varepsilon$, $d\omega_\varepsilon(x)=\Lambda(x)^{\beta(1-p_\varepsilon^*)}w(x)^{1-p_\varepsilon^*}d\sigma(x)$, and

$$A_\varepsilon(x) = \left(\int_{\tilde{S}_x}\Lambda(t)^{\beta(1-p_\varepsilon^*)}w(t)^{1-p_\varepsilon^*}d\sigma(t)\right)^{1/q_\varepsilon^*}\left(\int_{E\setminus S_x}\Lambda(s)^{\alpha-q_\varepsilon}w(s)^{q_\varepsilon/p_\varepsilon}d\sigma(s)\right)^{1/q_\varepsilon}.$$

By (4.6) and (4.7), we obtain

$$A_\varepsilon(x) \leqslant \left(\frac{1}{(\theta+\beta)(1-p_\varepsilon^*)+1}\right)^{1/q_\varepsilon^*}\left(\frac{1}{q_\varepsilon-\alpha-1-(q_\varepsilon\theta/p_\varepsilon)}\right)^{1/q_\varepsilon} \times \Lambda(x)^{\beta(1-p_\varepsilon^*)/q_\varepsilon^*+\alpha/q_\varepsilon} w(x)^{(1-p_\varepsilon^*)/q_\varepsilon^*+1/p_\varepsilon},$$

which implies

$$A_M(p_\varepsilon,q_\varepsilon) \leqslant \left(\frac{1}{(\theta+\beta)(1-p_\varepsilon^*)+1}\right)^{1/q_\varepsilon^*}\left(\frac{1}{q_\varepsilon-\alpha-1-(q_\varepsilon\theta/p_\varepsilon)}\right)^{1/q_\varepsilon} \times\left(\int_E \Lambda(x)^{\beta(1-p_\varepsilon^*)(r_\varepsilon/q_\varepsilon^*+1)+\alpha r_\varepsilon/q_\varepsilon} w(x)^{(1-p_\varepsilon^*)(r_\varepsilon/q_\varepsilon^*+1)+r_\varepsilon/p_\varepsilon} d\sigma(x)\right)^{1/r_\varepsilon}. \tag{4.9}$$

We have $\beta(1-p_\varepsilon^*)(r_\varepsilon/q_\varepsilon^*+1)+\alpha r_\varepsilon/q_\varepsilon=(\alpha p_\varepsilon-\beta q_\varepsilon)/(p_\varepsilon-q_\varepsilon)$ and $(1-p_\varepsilon^*)(r_\varepsilon/q_\varepsilon^*+1)+r_\varepsilon/p_\varepsilon=0$. Hence, the integral in (4.9) becomes $\int_E \Lambda(x)^{(\alpha p_\varepsilon-\beta q_\varepsilon)/(p_\varepsilon-q_\varepsilon)}d\sigma(x)$. After evaluating the last integral, we conclude that

$$A_M(p_\varepsilon,q_\varepsilon)\leqslant\left(\frac{1}{(\theta+\beta)(1-p_\varepsilon^*)+1}\right)^{1/q_\varepsilon^*}\left(\frac{1}{q_\varepsilon-\alpha-1-(q_\varepsilon\theta/p_\varepsilon)}\right)^{1/q_\varepsilon}$$
$$\times\left(\frac{p_\varepsilon-q_\varepsilon}{(\alpha+1)p_\varepsilon-(\beta+1)q_\varepsilon}\right)^{1/q_\varepsilon-1/p_\varepsilon}\Lambda(\infty)^{(\alpha+1)/q_\varepsilon-(\beta+1)/p_\varepsilon}<\infty.$$

With the help of Theorems 2.1 and 3.1, a similar argument to (4.8) will lead us to (4.2) for the case $q<p$. For case (ii), it follows from Theorem 3.1 that

$$L_*(\mathbb{K})\geqslant\limsup_{\varepsilon\longrightarrow 0^+}\left\{A_{p/(-\varepsilon),q/(-\varepsilon)}\right\}^{1/(-\varepsilon)}=\limsup_{\varepsilon\longrightarrow 0^+}\left\{A_{(-p)/\varepsilon,(-q)/\varepsilon}\right\}^{-1/\varepsilon}. \tag{4.10}$$

We have $0<-p,-q<\infty$ and $\dfrac{\alpha+1}{-q}\geqslant\dfrac{\beta+1}{-p}$. Replace p,q by $-p,-q$, respectively. Then the above argument also enables us to get an upper bound estimate for $(A_{-p/\varepsilon,-q/\varepsilon})^{1/\varepsilon}$ directly by using the one for $(A_{p/\varepsilon,q/\varepsilon})^{1/\varepsilon}$. Plugging this into (4.10), we get (ii). As for (iii), it follows from Theorem 2.1. This completes the proof. □

Corollary 4.1 generalizes many well-known inequalities, such as Levin-Cochran-Lee-type inequalities and Carleson's result. We shall discuss them in Section 5.

Like Corollary 4.1, we have the following consequence of Theorem 3.2.

COROLLARY 4.2. *Let $\theta\in\mathbb{R}$, $0<\tilde\Lambda(x)<\infty$, and $w:E\mapsto(0,\infty)$. Suppose that $\sigma(\tilde S_x\setminus S_x)=0$ for all x and (4.11) is satisfied, where*

$$\frac{w(x_2)}{w(x_1)}\geqslant\left(\frac{\tilde\Lambda(x_2)}{\tilde\Lambda(x_1)}\right)^{\theta}\qquad(\|x_1\|\leqslant\|x_2\|). \tag{4.11}$$

(i) If $0<p,q<\infty$, $\dfrac{\alpha+1}{q}\geqslant\dfrac{\beta+1}{p}$, $\Phi^\varepsilon\in CV^+(I)$ for all sufficiently small $\varepsilon>0$, and $\tilde\Lambda(0)^{(\alpha+1)/q-(\beta+1)/p}<\infty$, then for all $f:E\mapsto\bar I$,

$$\left(\int_E\left\{\Phi\left(\frac{1}{\tilde\Lambda(x)}\int_{E\setminus S_x}f(t)\,d\sigma(t)\right)\right\}^q\tilde\Lambda(x)^\alpha w(x)^{q/p}\,d\sigma(x)\right)^{1/q}$$
$$\leqslant C_{p,q}\tilde\Lambda(0)^{(\alpha+1)/q-(\beta+1)/p}\left\{\int_E(\Phi\circ f(x))^p\tilde\Lambda(x)^\beta w(x)\,d\sigma(x)\right\}^{1/p}, \tag{4.12}$$

where $C_{p,q}$ is defined by (4.3).

(ii) If $-\infty<p,q<0$, $\dfrac{\alpha+1}{q}\leqslant\dfrac{\beta+1}{p}$, $(1/\Phi)^\varepsilon\in CV^+(I)$ for all sufficiently small $\varepsilon>0$, and $\tilde\Lambda(0)^{(\alpha+1)/q-(\beta+1)/p}>0$, then (4.12) holds with a reversed sign of inequality and the corresponding constant $C_{p,q}$ is obtained from (4.3) by making the changes: $p\leqslant q\longrightarrow q\leqslant p$ and $q<p\longrightarrow p<q$.

(*iii*) *The condition* $\sigma(\tilde{S}_x \setminus S_x) = 0$ *can be removed from* (*i*) (*respectively,* (*ii*)) *for the case* $p \leqslant q$ (*respectively,* $q \leqslant p$).

Proof. This corollary can be derived by modifying the proof of Corollary 4.1 with the change $A_\varepsilon(x) \longrightarrow \tilde{A}_\varepsilon(x)$. For instance, (4.5) for $p \leqslant q$ is changed to

$$\tilde{A}_\varepsilon(x) = \left(\int_{E\setminus S_x} \tilde{\Lambda}(t)^{\beta(1-p_\varepsilon^*)} w(t)^{1-p_\varepsilon^*} d\sigma(t)\right)^{1/p_\varepsilon^*} \left(\int_{\tilde{S}_x} \tilde{\Lambda}(s)^{\alpha-q_\varepsilon} w(s)^{q_\varepsilon/p_\varepsilon} d\sigma(s)\right)^{1/q_\varepsilon}.$$

We leave the details to the readers. □

The applications of Corollary 4.2 to particular cases will be given in next sections.

5. Levin-Cochran-Lee-type inequalities

This type of inequalities correspond to the case $d\sigma(t) = \|t\|^{s-1}dt$ of Corollaries 4.1 and 4.2 (cf. [8]). They are extensions of the classical Pólya-Knopp inequality. In the following, we shall further extend them to the cases:

$$d\sigma(t) = \|t\|^{s-1}\chi_{\tilde{S}_b}(t)dt \qquad (b \in E \cup \{\infty\})$$

or

$$d\sigma(t) = \|t\|^{s-1}\chi_{E\setminus S_b}(t)dt \qquad (b \in E \cup \{0\}),$$

where $\tilde{S}_\infty = E$ and $S_0 = \emptyset$. As a consequence of Corollary 4.1, we get the following extension of the Levin-Cochran-Lee inequality.

COROLLARY 5.1. *Let* $s > 1-n$, $\theta \in \mathbb{R}$, $b \in E \cup \{\infty\}$, $w : \tilde{S}_b \mapsto (0,\infty)$, *and*

$$\frac{w(x_2)}{w(x_1)} \leqslant \left(\frac{\|x_2\|}{\|x_1\|}\right)^\theta \qquad (\|x_1\| \leqslant \|x_2\|). \tag{5.1}$$

(*i*) *If* $0 < p,q < \infty$, $\dfrac{\alpha+1}{q} \geqslant \dfrac{\beta+1}{p}$, *and* $\Phi^\varepsilon \in CV^+(I)$ *for all sufficiently small* $\varepsilon > 0$, *then for all* $f : \tilde{S}_b \mapsto \overline{I}$,

$$\left(\int_{\tilde{S}_b} \left\{\Phi\left(\frac{s+n-1}{\|x\|^{s+n-1}|A|}\int_{\tilde{S}_x} \|t\|^{s-1} f(t)\,dt\right)\right\}^q \|x\|^{\alpha+1-n} w(x)^{q/p}\,dx\right)^{1/q}$$
$$\leqslant C_{p,q} e^{\frac{\beta+\theta+1}{(s+n-1)p}} \|b\|^{(\alpha+1)/q-(\beta+1)/p} \left\{\int_{\tilde{S}_b} (\Phi\circ f(x))^p \|x\|^{\beta+1-n} w(x)\,dx\right\}^{1/p}, \tag{5.2}$$

where

$$C_{p,q} = \begin{cases} \left(\frac{s+n-1}{e|A|}\right)^{1/p-1/q} & (p \leqslant q); \\ \left(\frac{|A|(p-q)}{(\alpha+1)p-(\beta+1)q}\right)^{1/q-1/p} & (q < p). \end{cases} \tag{5.3}$$

(ii) *If* $-\infty < p,q < 0$, $\dfrac{\alpha+1}{q} \leqslant \dfrac{\beta+1}{p}$, *and* $(1/\Phi)^{\varepsilon} \in CV^{+}(I)$ *for all sufficiently small* $\varepsilon > 0$, *then* (5.2) *holds with a reversed sign of inequality and the corresponding constant* $C_{p,q}$ *is obtained from* (5.3) *by making the changes:* $p \leqslant q \longrightarrow q \leqslant p$ *and* $q < p \longrightarrow p < q$.

Proof. Consider (i). The case $b = \infty$ of (5.2) can be obtained from the case $\|b\| < \infty$ by considering the limits of both sides of (5.2) with b_m in place of b, where $b_m \in E$ and $\|b_m\| \uparrow \|b\|$ as $m \uparrow \infty$. Hence, we can assume $b \in E$. In this case, $\|b\| < \infty$. We shall apply Corollary 4.1 (i) to $d\sigma(t) = \|t\|^{s-1}\chi_{\tilde{S}_b}(t)dt$ and the triple $(\theta^*, \alpha^*, \beta^*)$, where $\theta^* = \theta/(s+n-1)$, $\alpha^* = (\alpha+1)/(s+n-1)-1$, and $\beta^* = (\beta+1)/(s+n-1)-1$. For this case, $\sigma(\tilde{S}_x \setminus S_x) = 0$ for all x. We have $\dfrac{\alpha^*+1}{q} \geqslant \dfrac{\beta^*+1}{p} \Longleftrightarrow \dfrac{\alpha+1}{q} \geqslant \dfrac{\beta+1}{p}$. Moreover,

$$\Lambda(x) = \begin{cases} \|x\|^{s+n-1}|A|/(s+n-1) & \text{for } x \in \tilde{S}_b, \\ \|b\|^{s+n-1}|A|/(s+n-1) & \text{for } x \in E \setminus \tilde{S}_b. \end{cases}$$

Define $w^* : E \mapsto (0,\infty)$ by the formulas: $w^*(x) = w(x)$ for $x \in \tilde{S}_b$ and $w^*(x) = w(x\|b\|/\|x\|)$ for $x \in E \setminus \tilde{S}_b$. By (5.1), we find that (4.1) holds, whenever (w,Λ,θ) is replaced by (w^*,Λ,θ^*). Inserting the exact values of $\Lambda(x), \alpha^*, \beta^*$, and θ^* into the places of $\Lambda(x), \alpha, \beta$, and θ in (4.2) and (4.3), we get (5.2), and so (i) follows. For (ii), it can be proved in the same way by using Corollary 4.1 (ii). $\square$

As indicated in Section 3, the first (respectively, second) part of Corollary 5.1 can apply to those Φ obeying (3.5) (respectively, (3.5^*)). In particular, the case $\Phi(x) = e^x$, $w(x) = 1, \theta = 0$ and $f(t) \longrightarrow \log|f(t)|$ of (5.2) gives the form:

$$\left(\int_{\tilde{S}_b}\left\{\exp\left(\frac{s+n-1}{\|x\|^{s+n-1}|A|}\int_{\tilde{S}_x}\|t\|^{s-1}\log|f(t)|\,dt\right)\right\}^{q}\|x\|^{\alpha+1-n}dx\right)^{1/q}$$
$$\leqslant C_{p,q}e^{\frac{\beta+1}{(s+n-1)p}}\|b\|^{(\alpha+1)/q-(\beta+1)/p}\left\{\int_{\tilde{S}_b}|f(x)|^{p}\|x\|^{\beta+1-n}dx\right\}^{1/p}, \tag{5.4}$$

where $s > 1-n$, $0 < p,q < \infty$, $(\alpha+1)/q \geqslant (\beta+1)/p$, and $C_{p,q}$ is given by (5.3). Obviously, the case $n = 1$ of (5.4) generalizes [8, Theorem 1], [13, Theorems 3.3 & 4.3], and [14, p. 51, Example 1.22]. We can say more. We have $e^{1-x} \leqslant 1/x$ for $x \geqslant 1$, so $e^{1/q-1/p} \leqslant (p/q)^{1/q}$ for $p \leqslant q$. This indicates that (5.4) improves the corresponding constant in [14, p. 51, Example 1.22]. We know that $e^{(\beta+1)/p} < e^{\beta/p+1/q}$ for $q < p$. Hence, the estimate given in (5.4) is also better than the one appearing in [13, Inequality (4.8)].

For $n \geqslant 1$, (5.4) also generalizes [6, Inequality (23)], [10, Proposition 3.6] and [23, Proposition 7.5]. Indeed, it improves the upper bounds given there. We illustrate these below. We have $|\tilde{S}_x| = \|x\|^n|A|/n$. This enables us to transform the case $s = n(\varepsilon-1)+1$, $\alpha = n(\lambda+1)-1$, and $\beta = n(\delta+1)-1$ of (5.4) into the following form:

$$\left(\int_{\tilde{S}_b}\left\{\exp\left(\frac{\varepsilon}{|\tilde{S}_x|^{\varepsilon}}\int_{\tilde{S}_x}|\tilde{S}_t|^{\varepsilon-1}\log|f(t)|\,dt\right)\right\}^q|\tilde{S}_x|^{\lambda}\,dx\right)^{1/q}$$
$$\leqslant C_{p,q}e^{(\delta+1)/(\varepsilon p)}|\tilde{S}_b|^{(\lambda+1)/q-(\delta+1)/p}\left\{\int_{\tilde{S}_b}|f(x)|^p|\tilde{S}_x|^{\delta}\,dx\right\}^{1/p}, \tag{5.5}$$

where $\varepsilon>0$, $0<p,q<\infty$, $(\lambda+1)/q\geqslant(\delta+1)/p$, and

$$C_{p,q}=\begin{cases}(\varepsilon/e)^{1/p-1/q} & (p\leqslant q);\\ \left(\frac{p-q}{(\lambda+1)p-(\delta+1)q}\right)^{1/q-1/p} & (q<p).\end{cases} \tag{5.6}$$

Hence, Eq. (5.5) generalizes [10, Proposition 3.6] and [23, Proposition 7.5]. Moreover, $e^{1/q-1/p}\leqslant(p/q)^{1/q}$ for $p\leqslant q$, and consequently, (5.5) improves the upper bound for the best constant given there.

As indicated in [3, Section 5], for $p=q=1$, (5.4) can be improved in the following way:

$$\int_{\tilde{S}_b}\exp\left(\frac{s+n-1}{\|x\|^{s+n-1}|A|}\int_{\tilde{S}_x}\|t\|^{s-1}\log|f(t)|\,dt\right)\|x\|^{\gamma-1}\,dx$$
$$\leqslant\frac{s+n-1}{s}e^{\gamma/(s+n-1)}\left\{\int_{\tilde{S}_b}\|x\|^{\gamma-1}|f(x)|\left\{1-\left(\frac{\|x\|}{\|b\|}\right)^s\right\}dx, \tag{5.7}$$

where $\gamma\in\mathbb{R}$. This gives an n-dimensional extension of [5, Theorem $3\,(i)$] and [7, Corollary $2\,(i)$].

We go back to (5.2). Consider the particular case:

$$\Phi(x)=e^x,\quad b=\infty,\quad n=s=1,\quad f(t)\longrightarrow -f(t),$$
$$0<p=q<\infty,\quad \alpha=\beta.$$

For such a case, (5.2) takes the form:

$$\int_0^\infty\exp\left(\frac{-p}{x}\int_0^x f(t)\,dt\right)x^{\alpha}w(x)\,dx\leqslant e^{\alpha+\theta+1}\int_0^\infty e^{-pf(x)}x^{\alpha}w(x)\,dx,$$

where $\alpha\in\mathbb{R}$ and θ satisfies (5.1). We know that

$$F(x)=F(x)-F(0)=\int_0^x F'(t)dt\qquad (x>0)$$

for all convex function F on $(0,\infty)$ with $\lim_{x\to 0^+}F(x)=F(0)=0$ (cf. [24, Theorem 7.43]). By using this fact, the above inequality can be rewritten in the form:

$$\int_0^\infty e^{-pF(x)/x}x^{\alpha}w(x)\,dx\leqslant e^{\alpha+\theta+1}\int_0^\infty e^{-pF'(x)}x^{\alpha}w(x)\,dx.$$

This is an extension of Carleson's result [2].

Consider the case $d\sigma(t)=\|t\|^{s-1}\chi_{E\setminus S_b}(t)dt$, where $b\in E\cup\{0\}$. We have

$$\tilde{\Lambda}(x)=\begin{cases}\|b\|^{s+n-1}|A|/(-s-n+1) & \text{for } x\in S_b,\\ \|x\|^{s+n-1}|A|/(-s-n+1) & \text{for } x\in E\setminus S_b,\end{cases}$$

where $s<1-n$. By modifying the proof of Corollary 5.1, we can easily obtain the following consequence of Corollary 4.2.

COROLLARY 5.2. *Let $s<1-n$, $\theta\in\mathbb{R}$, $b\in E\cup\{0\}$, $w:E\setminus S_b\mapsto(0,\infty)$, and*

$$\frac{w(x_2)}{w(x_1)}\geqslant\left(\frac{\|x_2\|}{\|x_1\|}\right)^{\theta}\qquad(\|x_1\|\leqslant\|x_2\|).\tag{5.8}$$

(*i*) *If $0<p,q<\infty$, $\dfrac{\alpha+1}{q}\leqslant\dfrac{\beta+1}{p}$, and $\Phi^{\varepsilon}\in CV^{+}(I)$ for all sufficiently small $\varepsilon>0$, then for all $f:E\setminus S_b\mapsto\bar{I}$,*

$$\begin{aligned}&\left(\int_{E\setminus S_b}\left\{\Phi\left(\frac{-s-n+1}{\|x\|^{s+n-1}|A|}\int_{E\setminus S_x}\|t\|^{s-1}f(t)\,dt\right)\right\}^{q}\|x\|^{\alpha+1-n}w(x)^{q/p}\,dx\right)^{1/q}\\ &\leqslant C_{p,q}e^{\frac{\beta+\theta+1}{(s+n-1)p}}\|b\|^{(\alpha+1)/q-(\beta+1)/p}\left\{\int_{E\setminus S_b}(\Phi\circ f(x))^{p}\|x\|^{\beta+1-n}w(x)\,dx\right\}^{1/p},\end{aligned}\tag{5.9}$$

where

$$C_{p,q}=\begin{cases}\left(\frac{-s-n+1}{e|A|}\right)^{1/p-1/q} & (p\leqslant q);\\ \left(\frac{|A|(p-q)}{(\beta+1)q-(\alpha+1)p}\right)^{1/q-1/p} & (q<p).\end{cases}\tag{5.10}$$

(*ii*) *If $-\infty<p,q<0$, $\dfrac{\alpha+1}{q}\geqslant\dfrac{\beta+1}{p}$, and $(1/\Phi)^{\varepsilon}\in CV^{+}(I)$ for all sufficiently small $\varepsilon>0$, then (5.9) holds with a reversed sign of inequality and the corresponding constant $C_{p,q}$ is obtained from (5.10) by making the changes: $p\leqslant q\longrightarrow q\leqslant p$ and $q<p\longrightarrow p<q$.*

We indicate before that any Φ obeying (3.5) satisfies $\Phi^{\varepsilon}\in CV^{+}(I)$ for all $\varepsilon>0$. Therefore, the first part of Corollary 5.2 can apply to such Φ. Similarly, the second part of the same corollary also works well for those Φ with the property (3.5^{*}). In particular, Corollary 5.2 can apply to the case that $\Phi(x)=e^{x}$, $w(x)=1,\theta=0$ and $f(t)\longrightarrow\log|f(t)|$. By (5.9),

$$\begin{aligned}&\left(\int_{E\setminus S_b}\left\{\exp\left(\frac{-s-n+1}{\|x\|^{s+n-1}|A|}\int_{E\setminus S_x}\|t\|^{s-1}\log|f(t)|\,dt\right)\right\}^{q}\|x\|^{\alpha+1-n}dx\right)^{1/q}\\ &\leqslant C_{p,q}e^{\frac{\beta+1}{(s+n-1)p}}\|b\|^{(\alpha+1)/q-(\beta+1)/p}\left\{\int_{E\setminus S_b}|f(x)|^{p}\|x\|^{\beta+1-n}dx\right\}^{1/p},\end{aligned}\tag{5.11}$$

where $s<1-n$, $0<p,q<\infty$, $(\alpha+1)/q\leqslant(\beta+1)/p$, and $C_{p,q}$ is given by (5.10). Obviously, the inequality in [17, Corollary 6] is the case $n=1$, $p=q=1$, $\alpha=\beta=\gamma$, and $b=0$ of (5.11). Consider the change of variables: $s=n(\varepsilon-1)+1$, $\alpha=n(\lambda+1)-1$, and $\beta=n(\delta+1)-1$. We can rewrite (5.11) in the following form:

$$\left(\int_{E\setminus S_b}\left\{\exp\left(\frac{-\varepsilon}{|\tilde{S}_x|^{\varepsilon}}\int_{E\setminus S_x}|\tilde{S}_t|^{\varepsilon-1}\log|f(t)|\,dt\right)\right\}^q|\tilde{S}_x|^{\lambda}\,dx\right)^{1/q}$$
$$\leqslant C_{p,q}e^{(\delta+1)/(\varepsilon p)}|\tilde{S}_b|^{(\lambda+1)/q-(\delta+1)/p}\left\{\int_{E\setminus S_b}|f(x)|^p|\tilde{S}_x|^{\delta}\,dx\right\}^{1/p}, \tag{5.12}$$

where $\varepsilon<0$, $0<p,q<\infty$, $(\lambda+1)/q\leqslant(\delta+1)/p$, and

$$C_{p,q}=\begin{cases}(-\varepsilon/e)^{1/p-1/q} & (p\leqslant q);\\ \left(\frac{p-q}{(\delta+1)q-(\lambda+1)p}\right)^{1/q-1/p} & (q<p).\end{cases} \tag{5.13}$$

Clearly, Eq. (5.12) generalizes [10, Proposition 4.4] and improves the upper bound of the best constant. Moreover, the case $p=q=1$, $A=\Sigma^{n-1}$, $b=0$, and $\lambda=\delta=\gamma-1$ of (5.12) reduces to [6, Inequality (24)]. As indicated in [3, Section 5], for $p=q=1$, (5.11) can be improved in the following way:

$$\int_{E\setminus S_b}\exp\left(\frac{-s-n+1}{\|x\|^{s+n-1}|A|}\int_{E\setminus S_x}\|t\|^{s-1}\log|f(t)|\,dt\right)\|x\|^{\gamma-1}\,dx$$
$$\leqslant\frac{s+n-1}{s}e^{\gamma/(s+n-1)}\left\{\int_{E\setminus S_b}\|x\|^{\gamma-1}|f(x)|\left\{1-\left(\frac{\|x\|}{\|b\|}\right)^s\right\}dx.\right. \tag{5.14}$$

This gives an n-dimensional extension of [5, Theorem 3 (ii)] and [7, Corollary 2 (ii)].

6. Pólya-Knopp forms of Stepanov's and Heinig's results

In [15, p. 26], V. V. Stepanov proved the case $n=1$ and $p=q$ of the following inequality:

$$\left(\int_E\left|\int_{E\setminus S_x}e^{\gamma(\|x\|-\|t\|)}\|t\|^{1-n}f(t)dt\right|^q\|x\|^{1-n}dx\right)^{1/q}$$
$$\leqslant\left(\frac{(1/p^*+1/q)|A|}{\gamma}\right)^{1/p^*+1/q}\left(\int_E|f(x)|^p\|x\|^{1-n}dx\right)^{1/p}, \tag{6.1}$$

where $\gamma>0$. This result is connected to the Laplace transform. It has been extended by the present authors to $1\leqslant p\leqslant q<\infty$ and the n-dimensional case (cf. [3]). For $1\leqslant q<p<\infty$, we also gave in [3] an n-dimensional extension of Heinig's result (cf. [12, Corollary 2.3 (b)]). In the following, we shall further establish their Pólya-Knopp form. This result follows from the case $d\sigma=e^{-\gamma\|t\|}\|t\|^{1-n}dt$ of Corollary 4.2.

COROLLARY 6.1. *Let $\gamma>0$, $\theta\in\mathbb{R}$, and $w:E\mapsto(0,\infty)$. Assume that (6.2) is satisfied:*

$$\frac{w(x_2)}{w(x_1)}\geqslant e^{-\gamma(\|x_2\|-\|x_1\|)\theta}\qquad(\|x_1\|\leqslant\|x_2\|). \tag{6.2}$$

(i) *If* $0<p,q<\infty$, $\alpha/q\leqslant\beta/p$, *and* $\Phi^{\varepsilon}\in CV^{+}(I)$ *for all sufficiently small* $\varepsilon>0$, *then for all* $f:E\mapsto\overline{I}$,

$$\left(\int_E\left\{\Phi\left(\frac{\gamma}{|A|}\int_{E\setminus S_x}e^{\gamma(\|x\|-\|t\|)}f(t)\|t\|^{1-n}dt\right)\right\}^q e^{\alpha\|x\|}w(x)^{q/p}\,\|x\|^{1-n}dx\right)^{1/q}$$
$$\leqslant C_{p,q}|A|^{1/q-1/p}\left\{\int_E(\Phi\circ f(x))^p e^{\beta\|x\|}w(x)\,\|x\|^{1-n}dx\right\}^{1/p}, \tag{6.3}$$

where

$$C_{p,q}=\begin{cases}e^{(\gamma\theta-\gamma-\beta)/(\gamma p)+1/q}\gamma^{1/p-1/q} & (p\leqslant q);\\ e^{(\gamma\theta-\beta)/(\gamma p)}\left(\frac{p-q}{\beta q-\alpha p}\right)^{1/q-1/p} & (q<p).\end{cases} \tag{6.4}$$

(ii) *If* $-\infty<p,q<0$, $\alpha/q\geqslant\beta/p$, *and* $(1/\Phi)^{\varepsilon}\in CV^{+}(I)$ *for all sufficiently small* $\varepsilon>0$, *then* (6.3) *holds with a reversed sign of inequality and the corresponding constant* $C_{p,q}$ *is obtained from* (6.4) *by making the changes:* $p\leqslant q\longrightarrow q\leqslant p$ *and* $q<p\longrightarrow p<q$.

Proof. Let $d\sigma(t)=e^{-\gamma\|t\|}\|t\|^{1-n}dt$. Then $\tilde{\Lambda}(x)=|A|e^{-\gamma\|x\|}/\gamma$, and consequently, $(6.2)\Longrightarrow(4.10)$. Let $\lambda=-1-\alpha/\gamma$ and $\delta=-1-\beta/\gamma$. We know that $\dfrac{\lambda+1}{q}\geqslant\dfrac{\delta+1}{p}\Longleftrightarrow\alpha/q\leqslant\beta/p$. Hence, after replacing the indices α and β in Corollary 4.2 by λ and δ, respectively, (i) follows from Corollary 4.2 (i). Similarly, we can get (ii) from Corollary 4.2 (ii). $\square$

Take $\Phi(x)=e^x$ and $f(t)\longrightarrow\log|f(t)|$. Then (6.3) reduces to the form

$$\left(\int_E\left\{\exp\left(\frac{\gamma}{|A|}\int_{E\setminus S_x}e^{\gamma(\|x\|-\|t\|)}\log|f(t)|\|t\|^{1-n}dt\right)\right\}^q e^{\alpha\|x\|}w(x)^{q/p}\,\|x\|^{1-n}dx\right)^{1/q}$$
$$\leqslant C_{p,q}|A|^{1/q-1/p}\left\{\int_E|f(x)|^p e^{\beta\|x\|}w(x)\,\|x\|^{1-n}dx\right\}^{1/p}. \tag{6.5}$$

This inequality corresponds to the following case of [23, Inequality (7.6)]:

$$S_x\longrightarrow E\setminus S_x,\qquad k(x,t)=e^{-\gamma\|t\|}\|t\|^{1-n},$$
$$u(x)=e^{\alpha\|x\|}w(x)^{q/p}\,\|x\|^{1-n},\quad v(x)=e^{\beta\|x\|}w(x)\,\|x\|^{1-n}.$$

Clearly, it is hard to derive such a result by using the scheme given there.

7. Pólya-Knopp form for Riemann-Liouville operators

The operator under consideration is the so-called generalized Riemann-Liouville operator $\mathbb{K}$, which is of the form (1.2) obeying the conditions:

$$k(x,t)=\gamma(\|x\|-\|t\|)^{\gamma-1}/(|A|\|x\|^{\gamma})\quad\text{and}\quad d\sigma=\|t\|^{1-n}dt.$$

This operator reduces to the classical one for $n=1$. In [1, 3, 12, 18], the best constant for this operator has been investigated. However, its Pólya-Knopp form is still unknown.

The purpose of this section is to derive such a result. Clearly, the desired result can not be obtained directly from Corollary 4.1 (i), because the kernel $k(x,t)$ is not of the form $1/\Lambda(x)$. In the following, we point out that the theory developed in Section 3 still works well for this case.

COROLLARY 7.1. *Let $\gamma\geqslant 1$, $\theta\in\mathbb{R}$, $b\in E\cup\{\infty\}$, and $w:\tilde{S}_b\mapsto(0,\infty)$. Suppose that (5.1) is satisfied. If $0<p,q<\infty$, $\frac{\alpha+1}{q}\geqslant\frac{\beta+1}{p}$, and $\Phi^{\varepsilon}\in CV^{+}(I)$ for all sufficiently small $\varepsilon>0$, then for all $f:\tilde{S}_b\mapsto\bar{I}$,*

$$\left(\int_{\tilde{S}_b}\left\{\Phi\left(\frac{\gamma}{|A|\|x\|^{\gamma}}\int_{\tilde{S}_x}(\|x\|-\|t\|)^{\gamma-1}f(t)\|t\|^{1-n}dt\right)\right\}^{q}\|x\|^{\alpha+1-n}w(x)^{q/p}dx\right)^{1/q}$$
$$\leqslant C_{p,q}\|b\|^{(\alpha+1)/q-(\beta+1)/p}\left(\int_{\tilde{S}_b}(\Phi\circ f(x))^{p}\|x\|^{\beta+1-n}w(x)dx\right)^{1/p},\tag{7.1}$$

where

$$C_{p,q}=\begin{cases}\left(\frac{1}{e|A|}\right)^{1/p-1/q}\left(\inf_{0<\varepsilon<s}\dfrac{\gamma^{1/\varepsilon}\left(1-\frac{\varepsilon}{p}+\frac{\varepsilon}{q}\right)^{1/q-1/p}}{\left(1-\frac{\varepsilon(\beta+1)}{p}-\frac{\varepsilon\theta}{p}\right)^{1/\varepsilon-1/p}\left(1-\frac{\varepsilon(\alpha+1)}{q}-\frac{\varepsilon\theta}{p}\right)^{1/q}}\right) & (p\leqslant q);\\[2ex] \left(\frac{|A|(p-q)}{(\alpha+1)p-(\beta+1)q}\right)^{1/q-1/p}\left(\inf_{0<\varepsilon<s}\dfrac{\gamma^{1/\varepsilon}}{\left(1-\frac{\varepsilon(\beta+1)}{p}-\frac{\varepsilon\theta}{p}\right)^{1/\varepsilon-1/q}\left(1-\frac{\varepsilon(\alpha+1)}{q}-\frac{\varepsilon\theta}{p}\right)^{1/q}}\right) & (q<p)\end{cases}$$

and s is the largest positive constant subject to the conditions:

$$0<s\leqslant\min\{p,q\}\qquad\text{and}\qquad\frac{\theta}{p}+\frac{\alpha+1}{q}\leqslant\frac{1}{s}.\tag{7.2}$$

Proof. We have $\tilde{S}_\infty=\cup_m\tilde{S}_{b_m}$, where $b_m\in E$ and $\|b_m\|\uparrow\infty$ as $m\uparrow\infty$. After taking limits for both sides of (7.1), we can assume $b\neq\infty$. In this case, $b\in E$. This corollary corresponds to the following case of Theorem 3.1 (i):

$$k(x,t)=\gamma(\|x\|-\|t\|)^{\gamma-1}/(|A|\|x\|^{\gamma}),\quad g(t)=1,\quad \psi(x,t)=k(x,t),$$

$$d\mu=\chi_{\tilde{S}_b}(x)\|x\|^{\alpha+1-n}(w^*(x))^{q/p}dx,$$

$$d\nu=\chi_{\tilde{S}_b}(x)\|x\|^{\beta+1-n}w^*(x)dx,\quad d\sigma=\chi_{\tilde{S}_b}\|t\|^{1-n}dt,$$

where $w^*(x)=w(x)$ for $x\in\tilde{S}_b$, $w^*(x)=w(x\|b\|/\|x\|)$ for $x\in E\setminus\tilde{S}_b$. By using the spherical coordinates, (2.4) is satisfied. Clearly, $d\nu_a/d\sigma=\|x\|^{\beta}w^*(x)>0$ on E, and (2.5) holds. From $\Phi^{\varepsilon_2} = (\Phi^{\varepsilon_1})^{\varepsilon_2/\varepsilon_1}$, we know that the hypothesis on Φ implies $\Phi^{\varepsilon} \in CV^+(I)$ for all $\varepsilon > 0$. Let $p_\varepsilon = p/\varepsilon$ and $q_\varepsilon = q/\varepsilon$. By the definition of s (cf. (7.2)), the following hold for all $0 < \varepsilon < s$:

$$\Phi^{\varepsilon} \in CV^+(I), \quad p_\varepsilon \geqslant 1, \quad q_\varepsilon \geqslant 1, \quad \text{and} \quad \theta < p_\varepsilon\left(1 - \frac{\alpha+1}{q_\varepsilon}\right).$$

Let $A_\varepsilon(x)$ denote the number defined by (2.1) with p_ε and q_ε instead of p and q, respectively. By definition, we have $A_\varepsilon(x) = 0$ for all $x \in E \setminus \tilde{S}_b$. As for $x \in \tilde{S}_b$, we divide the argument into two cases: $p \leqslant q$ and $q < p$. First, consider the case $p \leqslant q$. It is clear that $\gamma \geqslant 1$ implies $\sup_{t \in \tilde{S}_x} \psi(z,t) = \gamma/(|A|\|z\|)$ for all $z \in E \setminus S_x$, which leads us to

$$A_\varepsilon(x) = \frac{\gamma}{|A|}\left(\int_{\tilde{S}_x} \|t\|^{\beta(1-p_\varepsilon^*)+1-n} w(t)^{1-p_\varepsilon^*} dt\right)^{1/p_\varepsilon^*} \times \left(\int_{\tilde{S}_b \setminus S_x} \|z\|^{\alpha - q_\varepsilon + 1 - n} w(z)^{q_\varepsilon/p_\varepsilon} dz\right)^{1/q_\varepsilon} \qquad (x \in \tilde{S}_b). \tag{7.3}$$

Like (4.5) – (4.7), we can obtain

$$A_M(p_\varepsilon, q_\varepsilon) = \sup_{x \in \tilde{S}_b} |A_\varepsilon(x)| \leqslant \gamma |A|^{1/q_\varepsilon - 1/p_\varepsilon}\left(\frac{1}{(\beta+\theta)(1-p_\varepsilon^*)+1}\right)^{1/p_\varepsilon^*} \times \left(\frac{1}{q_\varepsilon - \alpha - 1 - (q_\varepsilon\theta/p_\varepsilon)}\right)^{1/q_\varepsilon} \|b\|^{(\alpha+1)/q_\varepsilon - (\beta+1)/p_\varepsilon} < \infty.$$

With the help of Theorems 2.1 and 3.1, we infer that

$$\|\mathbb{K}\|_* \leqslant \inf_{0<\varepsilon<s} A_{p/\varepsilon, q/\varepsilon}^{1/\varepsilon} \leqslant |A|^{1/q-1/p}\|b\|^{(\alpha+1)/q-(\beta+1)/p} \times \inf_{0<\varepsilon<s} \frac{\gamma^{1/\varepsilon}\left(1 - \frac{\varepsilon}{p} + \frac{\varepsilon}{q}\right)^{1/\varepsilon - 1/p + 1/q}}{\left(1 - \frac{\varepsilon(\beta+1)}{p} - \frac{\varepsilon\theta}{p}\right)^{1/\varepsilon - 1/p}\left(1 - \frac{\varepsilon(\alpha+1)}{q} - \frac{\varepsilon\theta}{p}\right)^{1/q}}. \tag{7.4}$$

We have assumed that $p \leqslant q$. By an elementary calculation, the function $\varepsilon \mapsto \left(1 - \frac{\varepsilon}{p} + \frac{\varepsilon}{q}\right)^{1/\varepsilon}$ is decreasing on $0 < \varepsilon < s$. For such ε,

$$\left(1 - \frac{\varepsilon}{p} + \frac{\varepsilon}{q}\right)^{1/\varepsilon} \leqslant \lim_{\tau \to 0^+}\left(1 - \frac{\tau}{p} + \frac{\tau}{q}\right)^{1/\tau} = e^{1/q-1/p}.$$

Plugging this into (7.4), we get $\|\mathbb{K}\|_* \leqslant C_{p,q}\|b\|^{(\alpha+1)/q-(\beta+1)/p}$. By the definition of $\|\mathbb{K}\|_*$, (7.1) follows. Next, consider the case $0 < q < p < \infty$. We have $A_M(p_\varepsilon, q_\varepsilon) =$

$\|A_\varepsilon(x)\|_{r_\varepsilon,\omega_\varepsilon}$, where $1/r_\varepsilon = 1/q_\varepsilon - 1/p_\varepsilon$ and $d\omega_\varepsilon(x) = \|x\|^{\beta(1-p_\varepsilon^*)}w(x)^{1-p_\varepsilon^*}d\sigma(x)$ on $\tilde{S}_b$. Moreover, $A_\varepsilon(x) = 0$ for all $x \in E\setminus\tilde{S}_b$. For $x \in \tilde{S}_b$, (7.3) will be replaced by

$$A_\varepsilon(x) = \frac{\gamma}{|A|}\left(\int_{\tilde{S}_x}\|t\|^{\beta(1-p_\varepsilon^*)+1-n}w(t)^{1-p_\varepsilon^*}dt\right)^{1/q_\varepsilon^*} \times\left(\int_{\tilde{S}_b\setminus S_x}\|z\|^{\alpha-q_\varepsilon+1-n}w(z)^{q_\varepsilon/p_\varepsilon}dz\right)^{1/q_\varepsilon}.$$

By (5.1),

$$A_\varepsilon(x) \leqslant \gamma\left(\frac{1}{(\theta+\beta)(1-p_\varepsilon^*)+1}\right)^{1/q_\varepsilon^*}\left(\frac{1}{q_\varepsilon-\alpha-1-(q_\varepsilon\theta/p_\varepsilon)}\right)^{1/q_\varepsilon} \times\|x\|^{\beta(1-p_\varepsilon^*)/q_\varepsilon^*+\alpha/q_\varepsilon}w(x)^{(1-p_\varepsilon^*)/q_\varepsilon^*+1/p_\varepsilon} \qquad (x\in\tilde{S}_b).$$

Following the argument given after (4.9), we can easily prove

$$A_M(p_\varepsilon,q_\varepsilon) \leqslant\gamma\left(\frac{1}{(\theta+\beta)(1-p_\varepsilon^*)+1}\right)^{1/q_\varepsilon^*}\left(\frac{1}{q_\varepsilon-\alpha-1-(q_\varepsilon\theta/p_\varepsilon)}\right)^{1/q_\varepsilon} \times\left(\frac{|A|(p_\varepsilon-q_\varepsilon)}{(\alpha+1)p_\varepsilon-(\beta+1)q_\varepsilon}\right)^{1/q_\varepsilon-1/p_\varepsilon}\|b\|^{(\alpha+1)/q_\varepsilon-(\beta+1)/p_\varepsilon} < \infty.$$

With the help of Theorems 2.1 and 3.1, we infer that

$$\|\mathbb{K}\|_* \leqslant \inf_{0<\varepsilon<s} A_{p/\varepsilon,q/\varepsilon}^{1/\varepsilon} \leqslant C_{p,q}\|b\|^{(\alpha+1)/q-(\beta+1)/p}.$$

This leads us to (7.1). The proof is complete. □

We remark that the case $p \leqslant q$ of Corollary 7.1 gives an example for which the infimum in (1.5) does not occur as $\varepsilon \to 0^+$. This follows from the observation:

$$\frac{\gamma^{1/\varepsilon}\left(1-\frac{\varepsilon}{p}+\frac{\varepsilon}{q}\right)^{1/q-1/p}}{\left(1-\frac{\varepsilon(\beta+1)}{p}-\frac{\varepsilon\theta}{p}\right)^{1/\varepsilon-1/p}\left(1-\frac{\varepsilon(\alpha+1)}{q}-\frac{\varepsilon\theta}{p}\right)^{1/q}} \approx \gamma^{1/\varepsilon}e^{(\beta+1+\theta)/p} \longrightarrow\infty \quad \text{as} \quad \varepsilon\to 0^+.$$

Up to now, the exact place for such ε is still unknown.

Next, consider the corresponding result of Corollary 7.1 to the case (1.7). We assume that $-\infty < p,q < 0$, $\dfrac{\alpha+1}{q} \leqslant \dfrac{\beta+1}{p}$, and $(1/\Phi)^\varepsilon \in CV^+(I)$ for all sufficiently small $\varepsilon > 0$. Then $(1/\Phi)^\varepsilon \in CV^+(I)$ for all $\varepsilon > 0$ and Theorem 3.1 ensures

$$L_*(\mathbb{K}) \geqslant \sup_{\varepsilon>0}\left\{A_{p/(-\varepsilon),q/(-\varepsilon)}\right\}^{1/(-\varepsilon)} = \sup_{\varepsilon>0}\left\{A_{(-p)/\varepsilon,(-q)/\varepsilon}\right\}^{-1/\varepsilon}. \tag{7.5}$$

Clearly, the term $\{\cdots\}^{-1/\varepsilon}$ in (7.5) is same as $\left\{A_{p/\varepsilon,q/\varepsilon}\right\}^{-1/\varepsilon}$ with the replacements: $p \longrightarrow -p$ and $q \longrightarrow -q$. This allows us to estimate (7.5) by using the corresponding values given in the proof of Corollary 7.1. After a suitable calculation, the reverse inequality of (7.1) holds, where

$$C_{p,q}=\begin{cases} \left(\frac{1}{e|A|}\right)^{1/p-1/q}\left(\sup_{0<\varepsilon<s}\frac{\gamma^{-1/\varepsilon}\left(1+\frac{\varepsilon}{p}-\frac{\varepsilon}{q}\right)^{1/q-1/p}}{\left(1+\frac{\varepsilon(\beta+1)}{p}+\frac{\varepsilon\theta}{p}\right)^{-1/\varepsilon-1/p}\left(1+\frac{\varepsilon(\alpha+1)}{q}+\frac{\varepsilon\theta}{p}\right)^{1/q}}\right) & (q\leqslant p);\\ \left(\frac{|A|(p-q)}{(\alpha+1)p-(\beta+1)q}\right)^{1/q-1/p}\left(\sup_{0<\varepsilon<s}\frac{\gamma^{-1/\varepsilon}}{\left(1+\frac{\varepsilon(\beta+1)}{p}+\frac{\varepsilon\theta}{p}\right)^{-1/\varepsilon-1/q}\left(1+\frac{\varepsilon(\alpha+1)}{q}+\frac{\varepsilon\theta}{p}\right)^{1/q}}\right) & (p<q)\end{cases}$$

and s is the largest positive constant subject to the conditions:

$$0<s\leqslant\min\{-p,-q\}\qquad\text{and}\qquad\frac{\theta}{-p}+\frac{\alpha+1}{-q}\leqslant\frac{1}{s}. \tag{7.6}$$

For $0<\gamma<1$, we shall show that (7.1) still holds, but the constant $C_{p,q}$ becomes more complicated. It involves the number $\mathscr{B}_{p,q}(*,*)=\mathscr{B}((\gamma-1)q+1,q-\alpha-1-q\theta/p)$, where $\mathscr{B}(\lambda,\rho)$ denotes the Beta function defined by

$$\mathscr{B}(\lambda,\rho)=\int_0^1 t^{\lambda-1}(1-t)^{\rho-1}dt=\frac{\Gamma(\lambda)\Gamma(\rho)}{\Gamma(\lambda+\rho)}\qquad(\lambda>0,\rho>0).$$

COROLLARY 7.2. *Let* $0<\gamma<1$, $\theta\in\mathbb{R}$, $b\in E\cup\{\infty\}$, *and* $w:\tilde{S}_b\mapsto(0,\infty)$. *Suppose that* (5.1) *is satisfied. If* $0<p,q<\infty$, $\dfrac{\alpha+1}{q}\geqslant\dfrac{\beta+1}{p}$,

$$\frac{1}{1-\gamma}>\max\left\{\frac{q}{p},\frac{q\theta}{p}+\alpha+1\right\}, \tag{7.7}$$

and $\Phi^{\varepsilon}\in CV^{+}(I)$ *for all sufficiently small* $\varepsilon>0$, *then* (7.1) *holds for all* $f:\tilde{S}_b\mapsto\overline{I}$, *where*

$$C_{p,q}=\begin{cases} \left(\frac{1}{e|A|}\right)^{1/p-1/q}q^{1/q}\left(\inf_{q(1-\gamma)<\varepsilon<s}\frac{\gamma^{1/\varepsilon}\left(1-\frac{\varepsilon}{p}+\frac{\varepsilon}{q}\right)^{1/q-1/p}\varepsilon^{-1/q}\mathscr{B}^{1/q}_{p/\varepsilon,q/\varepsilon}(*,*)}{\left(1-\frac{\varepsilon(\beta+1)}{p}-\frac{\varepsilon\theta}{p}\right)^{1/\varepsilon-1/p}}\right) & (p\leqslant q);\\ \left(\frac{|A|(p-q)}{(\alpha+1)p-(\beta+1)q}\right)^{1/q-1/p}q^{1/q}\left(\inf_{q(1-\gamma)<\varepsilon<s}\frac{\gamma^{1/\varepsilon}\varepsilon^{-1/q}\mathscr{B}^{1/q}_{p/\varepsilon,q/\varepsilon}(*,*)}{\left(1-\frac{\varepsilon(\beta+1)}{p}-\frac{\varepsilon\theta}{p}\right)^{1/\varepsilon-1/q}}\right) & (q<p)\end{cases}$$

and s *is the largest positive constant subject to* (7.2).

Proof. Let $p_\varepsilon = p/\varepsilon$ and $q_\varepsilon = q/\varepsilon$. We first remark that (7.7) implies the following two inequalities:

$$(1-\gamma)q < p \quad \text{and} \quad \frac{1}{(1-\gamma)q} > \frac{\theta}{p} + \frac{\alpha+1}{q}.$$

To compare them with the definition of s, we infer that $(1-\gamma)q < s$. Next, for $(1-\gamma)q < \varepsilon < s$, we have

$$(\gamma-1)q_\varepsilon + 1 > 0 \quad \text{and} \quad q_\varepsilon - \alpha - 1 - \frac{q_\varepsilon \theta}{p_\varepsilon} > q\left(\frac{1}{s} - \frac{\alpha+1}{q} - \frac{\theta}{p}\right) > 0.$$

This implies that $\mathscr{B}_{p_\varepsilon,q_\varepsilon}(*,*)$ is finite for this range of ε.

Now, we come back to the proof of this corollary. We modify the proof of Corollary 7.1 in the following way. From $0<\gamma<1$, we get

$$\sup_{t\in\tilde{S}_x} \psi(z,t) = \frac{\gamma}{|A|}\left(1-\frac{\|x\|}{\|z\|}\right)^{\gamma-1}\frac{1}{\|z\|} \qquad (x\in\tilde{S}_b; z\in E\setminus S_x),$$

and so by (5.1), the following estimate holds for the case $0<p\leqslant q<\infty$:

$$\begin{aligned} A_\varepsilon(x) &= \frac{\gamma}{|A|}\left(\int_{\tilde{S}_x}\|t\|^{\beta(1-p_\varepsilon^*)+1-n} w(t)^{1-p_\varepsilon^*}dt\right)^{1/p_\varepsilon^*} \\ &\quad \times \left(\int_{\tilde{S}_b\setminus S_x}\left(1-\frac{\|x\|}{\|z\|}\right)^{(\gamma-1)q_\varepsilon}\|z\|^{\alpha-q_\varepsilon+1-n}w(z)^{q_\varepsilon/p_\varepsilon}dz\right)^{1/q_\varepsilon} \\ &\leqslant \gamma|A|^{1/q_\varepsilon-1/p_\varepsilon}\|x\|^{(\alpha+1)/q_\varepsilon-(\beta+1)/p_\varepsilon}\left(\frac{1}{(\beta+\theta)(1-p_\varepsilon^*)+1}\right)^{1/p_\varepsilon^*}\mathscr{B}^{1/q_\varepsilon}_{p_\varepsilon,q_\varepsilon}(*,*). \end{aligned}$$

This leads us to

$$\begin{aligned} A_M(p_\varepsilon,q_\varepsilon) = \sup_{x\in\tilde{S}_b}|A_\varepsilon(x)| &\leqslant \gamma|A|^{1/q_\varepsilon-1/p_\varepsilon}\left(\frac{1}{(\beta+\theta)(1-p_\varepsilon^*)+1}\right)^{1/p_\varepsilon^*} \\ &\quad \times \mathscr{B}^{1/q_\varepsilon}_{p_\varepsilon,q_\varepsilon}(*,*)\|b\|^{(\alpha+1)/q_\varepsilon-(\beta+1)/p_\varepsilon} < \infty. \end{aligned}$$

With the help of Theorems 2.1 and 3.1, we conclude that

$$\|K\|_* \leqslant \inf_{0<\varepsilon<s} A^{1/\varepsilon}_{p/\varepsilon,q/\varepsilon} \leqslant C_{p,q}\|b\|^{(\alpha+1)/q-(\beta+1)/p} \qquad (p\leqslant q).$$

For $0<q<p<\infty$, $A_\varepsilon(x)$ is modified in the following way:

$$\begin{aligned} A_\varepsilon(x) &= \frac{\gamma}{|A|}\left(\int_{\tilde{S}_x}\|t\|^{\beta(1-p_\varepsilon^*)+1-n} w(t)^{1-p_\varepsilon^*}dt\right)^{1/q_\varepsilon^*} \\ &\quad \times \left(\int_{\tilde{S}_b\setminus S_x}\left(1-\frac{\|x\|}{\|z\|}\right)^{(\gamma-1)q_\varepsilon}\|z\|^{\alpha-q_\varepsilon+1-n}w(z)^{q_\varepsilon/p_\varepsilon}dz\right)^{1/q_\varepsilon} \\ &\leqslant \gamma\|x\|^{\beta(1-p_\varepsilon^*)/q_\varepsilon^*+\alpha/q_\varepsilon}w(x)^{(1-p_\varepsilon^*)/q_\varepsilon^*+1/p_\varepsilon}\left(\frac{1}{(\beta+\theta)(1-p_\varepsilon^*)+1}\right)^{1/q_\varepsilon^*}\mathscr{B}^{1/q_\varepsilon}_{p_\varepsilon,q_\varepsilon}(*,*), \end{aligned}$$

where $x \in \tilde{S}_b$. By a similar argument to (4.9), we infer that

$$\begin{aligned} A_M(p_\varepsilon, q_\varepsilon) &\leqslant \gamma \mathscr{B}^{1/q_\varepsilon}_{p_\varepsilon, q_\varepsilon}(*,*) \left(\frac{1}{(\beta+\theta)(1-p^*_\varepsilon)+1} \right)^{1/q^*_\varepsilon} \\ &\quad \times \left(\int_{\tilde{S}_b} \|x\|^{(\alpha p_\varepsilon - \beta q_\varepsilon)/(p_\varepsilon - q_\varepsilon)+1-n} dx \right)^{1/r_\varepsilon} \\ &= \gamma \mathscr{B}^{1/q_\varepsilon}_{p_\varepsilon, q_\varepsilon}(*,*) \left(\frac{1}{(\beta+\theta)(1-p^*_\varepsilon)+1} \right)^{1/q^*_\varepsilon} |A|^{1/r_\varepsilon} \|b\|^{(\alpha+1)/q_\varepsilon - (\beta+1)/p_\varepsilon} \\ &\quad \times \left(\frac{p_\varepsilon - q_\varepsilon}{(\alpha+1)p_\varepsilon - (\beta+1)q_\varepsilon} \right)^{1/r_\varepsilon}, \end{aligned}$$

and so

$$\|K\|_* \leqslant \inf_{0<\varepsilon<s} A^{1/\varepsilon}_{p/\varepsilon, q/\varepsilon} \leqslant C_{p,q} \|b\|^{(\alpha+1)/q - (\beta+1)/p} \qquad (q < p).$$

This completes the proof. $\square$

Like the case $\gamma \geqslant 1$, consider the changes: $p \longrightarrow -p$ and $q \longrightarrow -q$. We can prove that the reverse inequality of (7.1) holds for the case $0 < \gamma < 1$, provided $-\infty < p, q < 0$, $\dfrac{\alpha+1}{q} \leqslant \dfrac{\beta+1}{p}$, (7.7) holds, and $(1/\Phi)^\varepsilon \in CV^+(I)$ for all sufficiently small $\varepsilon > 0$. Here,

$$C_{p,q} = \begin{cases} \left(\frac{1}{e|A|}\right)^{1/p-1/q} (-q)^{1/q} \left(\sup_{q(\gamma-1)<\varepsilon<s} \frac{\gamma^{-1/\varepsilon}\left(1+\frac{\varepsilon}{p}-\frac{\varepsilon}{q}\right)^{1/q-1/p} \varepsilon^{-1/q} \mathscr{B}^{1/q}_{-p/\varepsilon, -q/\varepsilon}(*,*)}{\left(1+\frac{\varepsilon(\beta+1)}{p}+\frac{\varepsilon\theta}{p}\right)^{-1/\varepsilon-1/p}} \right) & (q \leqslant p); \\ \left(\frac{|A|(p-q)}{(\alpha+1)p-(\beta+1)q}\right)^{1/q-1/p} (-q)^{1/q} \left(\sup_{q(\gamma-1)<\varepsilon<s} \frac{\gamma^{-1/\varepsilon} \varepsilon^{-1/q} \mathscr{B}^{1/q}_{-p/\varepsilon, -q/\varepsilon}(*,*)}{\left(1+\frac{\varepsilon(\beta+1)}{p}+\frac{\varepsilon\theta}{p}\right)^{-1/\varepsilon-1/q}} \right) & (p < q) \end{cases}$$

and s is the largest positive constant subject to the condition (7.6).

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REFERENCES

[1] K. F. ANDERSEN, H. P. HEINIG, *Weighted norm inequalities for certain integral operators*, Siam J. Math. Anal. **14**, no. 4, 834–844 (1983).

[2] L. CARLESON, *A proof of an inequality of Carleman*, Proc. Amer. Math. Soc. **5**, 932–933 (1954).

[3] C.-P. CHEN, J.-W. LAN, D.-C. LUOR, *The best constants for multidimensional modular inequalities over spherical cones*, Linear and Multilinear Algebra, **62**, Issue 5, pp. 683–713 (2014). DOI:10.1080/03081087.2013.777438.

[4] M. CHRIST, L. GRAFAKOS *Best constants for two nonconvolution inequalities*, Proc. Amer. Math. Soc. **123**, no. 6, 1687–1693 (1995).

[5] A. ČIŽMEŠIJA, J. PEČARIĆ, *Some new generalisations of inequalities of Hardy and Levin-Cochran-Lee*, Bull. Austral. Math. Soc. **63**, 105–113 (2001).
[6] A. ČIŽMEŠIJA, J. PEČARIĆ, I. PERIĆ, *Mixed means and inequalities of Hardy and Levin-Cochran-Lee type for multidimensional balls*, Proc. Amer. Math. Soc., **128**, no. 9, 2543–2552 (2000).
[7] A. ČIŽMEŠIJA, J. PEČARIĆ, L.-E. PERSSON, *On strengthened Hardy and Pólya-Knopp's inequalities*, J. Approx. Theory **125**, 74–84 (2003).
[8] J. A. COCHRAN, C.-S. LEE, *Inequalities related to Hardy's and Heinig's*, Math. Proc. Cambridge Philos. Soc. **96**, 1–7 (1984).
[9] P. DRÁBEK, H. P. HEINIG, A. KUNFER, *Higher dimensional Hardy inequality*, Intenat. Ser. Numer. Math. **123**, 3–16 (1997).
[10] B. GUPTA, P. JAIN, L. E. PERSSON, A. WEDESTIG, *Weighted geometric mean inequalities over cones in* $\mathbb{R}^N$, J. Inequal. Pure Appl. Math. **4**, Issue 4, Article 68 (2003).
[11] G. H. HARDY, J. E. LITTLEWOOD, G. PÓLYA, *Inequalities*, 2nd edition, Cambridge University Press, Cambridge (1967).
[12] H. P. HEINIG, *Weighted norm inequalities for certain integral operators II*, Proc. Amer. Math. Soc. **95**, no. 3, 387–395 (1985).
[13] P. JAIN, L.-E. PERSSON, A. WEDESTIG, *Carleman-Knopp type inequalities via Hardy inequalities*, Math. Inequal. Appl. **4(3)**, 343–355 (2001).
[14] A. KUFNER, L.-E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific Publishing Co., Singapore, New Jersey, London, Hong Kong (2003).
[15] V. I. LEVIN, S. B. STEČKIN, *Inequalities*, Amer. Math. Soc. Transl. (2), **14**, 1–29 (1960).
[16] N. LEVINSON, *Generalizations of an inequality of Hardy*, Duke Math. J. **31**, 389–394 (1964).
[17] E. R. LOVE *Inequalities related to those of Hardy and of Cochran and Lee*, Math. Proc. Cambridge Philos. Soc. **99**, 395–408 (1986).
[18] V. M. MANAKOV, *On the best constant in weighted inequalities for Riemann-Liouville integrals*, Bull. London Math. Soc. **24**, 442–448 (1992).
[19] M. NASSYROVA, L.-E. PERSSON, V. D. STEPANOV, *On weighted inequalities with geometric mean operator generated by the Hardy-type integral transform*, J. Inequal. Pure Appl. Math. **3**, Issue 4, Article 48 (electronic)(2002).
[20] L.-E. PERSSON, V. D. STEPANOV, *Weighted integral inequalities with the geometric mean operator*, J. Inequal. Appl. **7**, No. 5, 727–746 (2002).
[21] D. V. PROKHOROV *Weighted Hardy's inequalities for negative indices*, Publications Matematiques, **48**, 423–443 (2004).
[22] G. SINNAMON, *One-dimensional Hardy-type inequalities in many dimensions*, Proc. Roy. Soc. Edinburgh, **128A**, 833–848 (1998).
[23] A. WEDESTIG, *Weighted Inequalities of Hardy-type and their Limiting Iinequalities*, PhD thesis 2003:17, Luleå University of Technology, Luleå 2003.
[24] R. L. WHEEDEN, A. ZYGMUND, *Measure and Integral*, Marcel Dekker Inc., New York (1977).

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Chang-Pao Chen
Center for General Education, Hsuan Chuang University
Hsinchu, Taiwan 30092, Republic of China
e-mail: cpchen@wmail.hcu.edu.tw

Jin-Wen Lan
Municipal Jianguo High School
Taipei, Taiwan 10066, Republic of China
e-mail: d937210@oz.nthu.edu.tw

Dah-Chin Luor
Department of Applied Mathematics, I-Shou University
Dashu District, Kaohsiung city
Taiwan 84001, Republic of China
e-mail: dclour@isu.edu.tw