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NEW REFINEMENTS OF GENERALIZED HÖLDER'S INEQUALITY AND THEIR APPLICATIONS

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Abstract. In this paper, we present a series of sharpened versions of generalized Hölder's inequality. As an application in information theory, we give a new refinement of Singh's inequality with respect to the 'useful' information of order α for the power distribution. The Singh's inequality include Shannon's inequality as a special case.

1. Introduction

We begin by recalling here the classical Hölder's inequality [4] as Theorem A below.

THEOREM A. *If* $\beta_j \geqslant 0$*,* $A_{rj} \geqslant 0$ $(r=1,2,\cdots,n,\ j=1,2,\cdots,m)$*, and if* $\sum_{j=1}^{m}\beta_j = 1$*, then*

$$\sum_{r=1}^{n}\prod_{j=1}^{m}A_{rj}^{\beta_j} \leqslant \prod_{j=1}^{m}\left(\sum_{r=1}^{n}A_{rj}\right)^{\beta_j}. \tag{1}$$

As is well known, Hölder's inequality plays an important and basic role in different branches of modern mathematics such as classical real and complex analysis, numerical analysis, probability and statistics, fuzzy measure theory, qualitative theory of differential equations and their applications. Due to the importance of Hölder's inequality (1), it has received considerable attention by many authors, and has motivated a large number of research papers giving it various generalizations, improvements and applications. For example, Agahi et al. [3] presented some noteworthy generalizations of the Hölder's and Minkowski's inequalities for the pseudo-integral. Liu [9] established an important Hölder-type inequality for fuzzy variables. Nikolova and Varošanec [14] obtained some new refinements of the classical Hölder inequality by using a convex function. For more detail expositions, the interested reader may consult [1, 2], [10], [15], [20, 21] and the references therein.

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As for Hölder's inequality, many generalizations have been obtained so far. Among various generalizations of (1), the exponential generalization of (1) is an important research subject. In [8], Jensen first derived an exponential extension of Hölder's inequality. Later, a lot of interesting exponential extensions of Hölder's inequality have been studied by many researchers, e.g., Carroll et al. [6], Mitrinović and Pečarić [12], Mitrinović et al. [13], Vasić and Pečarić [22], Wu and Debnath [23], and Tian [19]. The reader who wants to learn more about the exponential extension of Hölder's inequality may consult the work [13]. The most important results in the references mentioned above is derived by Vasić and Pečarić [22] as follows.

THEOREM B. *Let* $A_{rj}>0$ $(r=1,2,\cdots,n,\ j=1,2,\cdots,m)$.

(a) If $\beta_j \geqslant 0$, *and if* $\sum_{j=1}^{m}\beta_j \geqslant 1$, *then*

$$\sum_{r=1}^{n}\prod_{j=1}^{m}A_{rj}^{\beta_j} \leqslant \prod_{j=1}^{m}\left(\sum_{r=1}^{n}A_{rj}\right)^{\beta_j}. \tag{2}$$

(b) If $\beta_j \leqslant 0$ $(j=1,2,\cdots,m)$, *then*

$$\sum_{r=1}^{n}\prod_{j=1}^{m}A_{rj}^{\beta_j} \geqslant \prod_{j=1}^{m}\left(\sum_{r=1}^{n}A_{rj}\right)^{\beta_j}. \tag{3}$$

(c) If $\beta_1>0,\ \beta_j \leqslant 0$ $(j=2,3,\cdots,m)$, *and if* $\sum_{j=1}^{m}\beta_j \leqslant 1$, *then*

$$\sum_{r=1}^{n}\prod_{j=1}^{m}A_{rj}^{\beta_j} \geqslant \prod_{j=1}^{m}\left(\sum_{r=1}^{n}A_{rj}\right)^{\beta_j}. \tag{4}$$

The above inequalities are called as generalized Hölder's inequalities.

Although Hölder's inequality and generalized Hölder's inequalities play a fundamental role in many branches of mathematics and have a wide range of applications in information science. Some problems can't be precisely estimated by them. For example, if we set $m=2$, $n=N+1$, $A_{11}=0$, $A_{r1}=1$, $A_{12}=1$, $A_{r2}=0$, $r=2,3,\cdots,N+1$, and $\beta_j>0,\ \beta_1+\beta_2=1$, then from generalized Hölder's inequality (2) we just obtain $0 \leqslant N$. So it is of interest to refine Hölder's inequality and generalized Hölder's inequalities.

The main purpose of this paper is to establish some new and exquisite refinements of inequalities (2), (3), (4) and (1). Moreover, a new refinement of Singh's inequality which generalized Shannon's inequality, is given by using the obtained results.

2. New refinements of generalized Hölder's inequality

In this section we first introduce the following two lemmas, which will be used in the sequel.

LEMMA 1. [4] *Let* $A_1, A_2, \cdots, A_m$ *be real numbers, and let* m *be a natural number and* $m \geqslant 2$. *Then*

$$\sum_{1 \leqslant i < j \leqslant m} (A_i - A_j)^2 = m\left(\sum_{i=1}^{m} A_i^2\right) - \left(\sum_{i=1}^{m} A_i\right)^2. \tag{5}$$

LEMMA 2. [4] *If* $x > -1$, $\alpha > 1$ *or* $\alpha < 0$, *then*

$$(1+x)^{\alpha} \geqslant 1 + \alpha x. \tag{6}$$

The inequality is reversed for $0 < \alpha < 1$.

Next, we prove the following lemmas, which play a crucial role in proving the main results.

LEMMA 3. *Let* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m > 0$, $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1$, *let* $X_{rj} > 0$, $1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j} > 0$ $(r = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$, *and let* $m \geqslant 2$. *Then*

$$\begin{aligned}
&\prod_{j=1}^{m}\left(1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} + \sum_{r=1}^{n}\prod_{j=1}^{m} X_{rj} \\
&\leqslant \left\{1 - \frac{2}{m(m-1)}\left[m\left(\sum_{j=1}^{m}\left(\sum_{r=1}^{n} X_{rj}^{\lambda_j}\right)^2\right) - \left(\sum_{j=1}^{m}\left(\sum_{r=1}^{n} X_{rj}^{\lambda_j}\right)\right)^2\right]\right\}^{\frac{m}{2\lambda_1}}.
\end{aligned} \tag{7}$$

Proof. From the hypotheses of Lemma 3, we obtain

$$\frac{1}{(m-1)\lambda_i} > 0, \quad \frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i} \geqslant 0 \quad (1 \leqslant i < j \leqslant m),$$

and

$$\begin{aligned}
&\sum_{1 \leqslant i < j \leqslant m}\left[\frac{1}{(m-1)\lambda_i} + \frac{1}{(m-1)\lambda_i} + \frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i}\right] \\
&= \sum_{1 \leqslant i < j \leqslant m}\left[\frac{1}{(m-1)\lambda_i} + \frac{1}{(m-1)\lambda_j}\right] = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_m}.
\end{aligned} \tag{8}$$

Then, in virtue of the generalized Hölder's inequality (2) we have

$$\begin{aligned}
&\prod_{1 \leqslant i < j \leqslant m}\left[1 - \left(\sum_{r=1}^{n} X_{ri}^{\lambda_i} - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\right)^2\right]^{\frac{1}{(m-1)\lambda_i}} \\
&= \prod_{1 \leqslant i < j \leqslant m}\left\{\left[\sum_{r=1}^{n} X_{ri}^{\lambda_i} + \left(1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\right)\right]^{\frac{1}{(m-1)\lambda_i}}\left[\sum_{r=1}^{n} X_{rj}^{\lambda_j} + \left(1 - \sum_{r=1}^{n} X_{ri}^{\lambda_i}\right)\right]^{\frac{1}{(m-1)\lambda_i}}\right.
\end{aligned}$$

$$\times \left[\sum_{r=1}^{n} X_{rj}^{\lambda_j} + \Big(1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\Big)\right]^{\frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i}} \Bigg\}$$

$$\geqslant \prod_{1\leqslant i<j\leqslant m} \left[\big(X_{1i}^{\lambda_i}\big)^{\frac{1}{(m-1)\lambda_i}} \big(X_{1j}^{\lambda_j}\big)^{\frac{1}{(m-1)\lambda_i}} \big(X_{1j}^{\lambda_j}\big)^{\frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i}} \right]$$

$$+ \prod_{1\leqslant i<j\leqslant m} \left[\big(X_{2i}^{\lambda_i}\big)^{\frac{1}{(m-1)\lambda_i}} \big(X_{2j}^{\lambda_j}\big)^{\frac{1}{(m-1)\lambda_i}} \big(X_{2j}^{\lambda_j}\big)^{\frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i}} \right]$$

$$+ \cdots$$

$$+ \prod_{1\leqslant i<j\leqslant m} \left[\big(X_{ni}^{\lambda_i}\big)^{\frac{1}{(m-1)\lambda_i}} \big(X_{nj}^{\lambda_j}\big)^{\frac{1}{(m-1)\lambda_i}} \big(X_{nj}^{\lambda_j}\big)^{\frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i}} \right]$$

$$+ \prod_{1\leqslant i<j\leqslant m} \left[\big(1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\big)^{\frac{1}{(m-1)\lambda_i}} \big(1 - \sum_{r=1}^{n} X_{ri}^{\lambda_i}\big)^{\frac{1}{(m-1)\lambda_i}} \big(1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\big)^{\frac{1}{(m-1)\lambda_j} - \frac{1}{(m-1)\lambda_i}} \right]$$

$$= \prod_{1\leqslant i<j\leqslant m} X_{1i}^{\frac{1}{m-1}} X_{1j}^{\frac{1}{m-1}} + \prod_{1\leqslant i<j\leqslant m} X_{2i}^{\frac{1}{m-1}} X_{2j}^{\frac{1}{m-1}} + \ldots + \prod_{1\leqslant i<j\leqslant m} X_{ni}^{\frac{1}{m-1}} X_{nj}^{\frac{1}{m-1}}$$

$$+ \prod_{1\leqslant i<j\leqslant m} \left[\big(1 - \sum_{r=1}^{n} X_{ri}^{\lambda_i}\big)^{\frac{1}{(m-1)\lambda_i}} \big(1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\big)^{\frac{1}{(m-1)\lambda_j}} \right]$$

$$= \prod_{j=1}^{m} X_{1j} + \prod_{j=1}^{m} X_{1j} + \ldots + \prod_{j=1}^{m} X_{1j} + \prod_{j=1}^{m} \big(1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\big)^{\frac{1}{\lambda_j}}$$

$$= \sum_{r=1}^{n} \prod_{j=1}^{m} X_{rj} + \prod_{j=1}^{m} \big(1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\big)^{\frac{1}{\lambda_j}}. \tag{9}$$

Noting the fact that there are $\frac{m(m-1)}{2}$ product terms in the expression $\prod_{1\leqslant i<j\leqslant m} \big[1 - \big(\sum_{r=1}^{n} X_{ri}^{\lambda_i} - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\big)^2\big]$, and applying the arithmetic-geometric mean's inequality, we find

$$\prod_{1\leqslant i<j\leqslant m} \left[1 - \Big(\sum_{r=1}^{n} X_{ri}^{\lambda_i} - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\Big)^2\right]$$

$$\leqslant \left\{ \frac{2}{m(m-1)} \sum_{1\leqslant i<j\leqslant m} \left[1 - \Big(\sum_{r=1}^{n} X_{ri}^{\lambda_i} - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\Big)^2\right] \right\}^{\frac{m(m-1)}{2}}$$

$$= \left[1 - \frac{2}{m(m-1)} \sum_{1\leqslant i<j\leqslant m} \Big(\sum_{r=1}^{n} X_{ri}^{\lambda_i} - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\Big)^2\right]^{\frac{m(m-1)}{2}}, \tag{10}$$

so that

$$\prod_{1\leqslant i<j\leqslant m} \left[1 - \Big(\sum_{r=1}^{n} X_{ri}^{\lambda_i} - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\Big)^2\right]^{\frac{1}{(m-1)\lambda_i}}$$

$$\leqslant \left\{ \prod_{1\leqslant i<j\leqslant m} \left[1 - \Big(\sum_{r=1}^{n} X_{ri}^{\lambda_i} - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\Big)^2\right] \right\}^{\frac{1}{(m-1)\lambda_1}}$$

$$\leqslant \left[1-\frac{2}{m(m-1)}\sum_{1\leqslant i<j\leqslant m}\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]^{\frac{m}{2\lambda_1}}. \tag{11}$$

On the other hand, from Lemma 1 we have

$$\begin{aligned}&\left[1-\frac{2}{m(m-1)}\sum_{1\leqslant i<j\leqslant m}\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]^{\frac{m}{2\lambda_1}}\\&=\left\{1-\frac{2}{m(m-1)}\left[m\left(\sum_{j=1}^{m}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right)-\left(\sum_{j=1}^{m}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)\right)^2\right]\right\}^{\frac{m}{2\lambda_1}}. \end{aligned}\tag{12}$$

Consequently, from inequalities (9), (11) and (12), we have the desired result. □

LEMMA 4. *Let* $\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_m<0$, *let* $X_{rj}>0$, $1-\sum_{r=1}^{n}X_{rj}^{\lambda_j}>0$ $(r=1,2,\cdots,n,\ j=1,2,\cdots,m)$, *and let* $m\geqslant 2$. *Then*

$$\begin{aligned}&\prod_{j=1}^{m}\left(1-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\sum_{r=1}^{n}\prod_{j=1}^{m}X_{rj}\\&\geqslant\left\{1-\frac{2}{m(m-1)}\left[m\left(\sum_{j=1}^{m}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right)-\left(\sum_{j=1}^{m}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)\right)^2\right]\right\}^{\frac{m}{2\lambda_1}}. \end{aligned}\tag{13}$$

Proof. By the same method as in Lemma 3, using generalized Holder's inequality (3) and Lemma 1, we can obtain the desired inequality (13). □

LEMMA 5. *Let* $\lambda_m>0$, $\lambda_1\leqslant\lambda_2\leqslant\ldots\leqslant\lambda_{m-1}<0$, $\sum_{j=1}^{m}\frac{1}{\lambda_j}\leqslant 1$, *and let* $X_{ij}>0$, $1-\sum_{i=1}^{n}X_{ij}^{\lambda_j}>0$ $(i=1,2,\cdots,n,\ j=1,2,\cdots,m)$. *If* $m>2$, *then*

$$\begin{aligned}&\prod_{j=1}^{m}\left(1-\sum_{i=1}^{n}X_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\sum_{i=1}^{n}\prod_{j=1}^{m}X_{ij}\\&\geqslant\left\{1-\frac{2}{(m-1)(m-2)}\left[(m-1)\left(\sum_{j=1}^{m-1}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right)-\left(\sum_{j=1}^{m-1}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)\right)^2\right]\right\}^{\frac{m-1}{2\lambda_1}}; \end{aligned}\tag{14}$$

If $m=2$, *then*

$$\begin{aligned}&\prod_{j=1}^{2}\left(1-\sum_{i=1}^{n}X_{ij}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\sum_{i=1}^{n}\prod_{j=1}^{2}X_{ij}\\&\geqslant\left\{1-\left[2\left(\sum_{j=1}^{2}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right)-\left(\sum_{j=1}^{2}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)\right)^2\right]\right\}^{\frac{1}{\lambda_1}}. \end{aligned}\tag{15}$$

Proof. We first consider the case where $m>2$. From the assumptions in Lemma 5, we find

$$\frac{1}{(m-2)\lambda_i}<0,\quad \frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}\leqslant 0\quad (1\leqslant i<j\leqslant m-1),$$

and

$$\begin{aligned}&\sum_{1\leqslant i<j\leqslant m-1}\left[\frac{1}{(m-2)\lambda_i}+\frac{1}{(m-2)\lambda_i}+\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}\right]\\&=\sum_{1\leqslant i<j\leqslant m-1}\left[\frac{1}{(m-2)\lambda_i}+\frac{1}{(m-2)\lambda_j}\right]=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\cdots+\frac{1}{\lambda_{m-1}}.\end{aligned}\tag{16}$$

Then, by using inequality (4) we get

$$\begin{aligned}&\prod_{1\leqslant i<j\leqslant m-1}\left[1-\Big(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\Big)^2\right]^{\frac{1}{(m-2)\lambda_i}}\\
&=\left[\sum_{r=1}^{n}X_{rm}^{\lambda_m}+\big(1-\sum_{r=1}^{n}X_{rm}^{\lambda_m}\big)\right]^{\frac{1}{\lambda_m}}\prod_{1\leqslant i<j\leqslant m-1}\left\{\left[\sum_{r=1}^{n}X_{ri}^{\lambda_i}+\big(1-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\big)\right]^{\frac{1}{(m-2)\lambda_i}}\right.\\
&\quad\left.\times\left[\sum_{r=1}^{n}X_{rj}^{\lambda_j}+\big(1-\sum_{r=1}^{n}X_{ri}^{\lambda_i}\big)\right]^{\frac{1}{(m-2)\lambda_i}}\left[\sum_{r=1}^{n}X_{rj}^{\lambda_j}+\big(1-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\big)\right]^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\right\}\\
&\leqslant X_{1m}^{\frac{\lambda_m}{\lambda_m}}\prod_{1\leqslant i<j\leqslant m-1}\left[\big(X_{1i}^{\lambda_i}\big)^{\frac{1}{(m-2)\lambda_i}}\big(X_{1j}^{\lambda_j}\big)^{\frac{1}{(m-2)\lambda_i}}\big(X_{1j}^{\lambda_j}\big)^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\right]\\
&\quad+X_{2m}^{\frac{\lambda_m}{\lambda_m}}\prod_{1\leqslant i<j\leqslant m-1}\left[\big(X_{2i}^{\lambda_i}\big)^{\frac{1}{(m-2)\lambda_i}}\big(X_{2j}^{\lambda_j}\big)^{\frac{1}{(m-2)\lambda_i}}\big(X_{2j}^{\lambda_j}\big)^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\right]\\
&\quad+\cdots\\
&\quad+X_{nm}^{\frac{\lambda_m}{\lambda_m}}\prod_{1\leqslant i<j\leqslant m-1}\left[\big(X_{ni}^{\lambda_i}\big)^{\frac{1}{(m-2)\lambda_i}}\big(X_{nj}^{\lambda_j}\big)^{\frac{1}{(m-2)\lambda_i}}\big(X_{nj}^{\lambda_j}\big)^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\right]\\
&\quad+\big(1-X_{1m}^{\lambda_m}\big)^{\frac{1}{\lambda_m}}\prod_{1\leqslant i<j\leqslant m-1}\left[\big(1-X_{1j}^{\lambda_j}\big)^{\frac{1}{(m-2)\lambda_i}}\big(1-X_{1i}^{\lambda_i}\big)^{\frac{1}{(m-2)\lambda_i}}\right.\\
&\quad\left.\times\big(1-X_{2j}^{\lambda_j}\big)^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\right]\\
&\quad+\big(1-X_{2m}^{\lambda_m}\big)^{\frac{1}{\lambda_m}}\prod_{1\leqslant i<j\leqslant m-1}\left[\big(1-X_{2j}^{\lambda_j}\big)^{\frac{1}{(m-2)\lambda_i}}\big(1-X_{2i}^{\lambda_i}\big)^{\frac{1}{(m-2)\lambda_i}}\right.\\
&\quad\left.\times\big(1-X_{2j}^{\lambda_j}\big)^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\right]\\
&\quad+\cdots\end{aligned}$$

$$
\begin{aligned}
&+\left(1-X_{nm}^{\lambda_m}\right)^{\frac{1}{\lambda_m}} \prod_{1\leqslant i<j\leqslant m-1}\Bigg[\left(1-X_{nj}^{\lambda_j}\right)^{\frac{1}{(m-2)\lambda_i}}\left(1-X_{ni}^{\lambda_i}\right)^{\frac{1}{(m-2)\lambda_i}}\\
&\times\left(1-X_{nj}^{\lambda_j}\right)^{\frac{1}{(m-2)\lambda_j}-\frac{1}{(m-2)\lambda_i}}\Bigg]\\
=&\sum_{r=1}^{n}\left(X_{rm}\prod_{1\leqslant i<j\leqslant m-1}X_{ri}^{\frac{1}{m-2}}X_{rj}^{\frac{1}{m-2}}\right)\\
&+\sum_{r=1}^{n}\left\{\left(1-X_{rm}^{\lambda_m}\right)^{\frac{1}{\lambda_m}}\prod_{1\leqslant i<j\leqslant m-1}\left[\left(1-X_{ri}^{\lambda_i}\right)^{\frac{1}{(m-2)\lambda_i}}\left(1-X_{rj}^{\lambda_j}\right)^{\frac{1}{(m-2)\lambda_j}}\right]\right\}\\
=&\sum_{r=1}^{n}\prod_{j=1}^{m}X_{rj}+\sum_{r=1}^{n}\prod_{j=1}^{m}\left(1-X_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}. \qquad (17)
\end{aligned}
$$

Noting that there are $\frac{(m-1)(m-2)}{2}$ product terms in the expression $\prod_{1\leqslant i<j\leqslant m-1}\left[1-\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]$, we then deduce from the arithmetic-geometric mean's inequality that

$$
\begin{aligned}
&\prod_{1\leqslant i<j\leqslant m-1}\left[1-\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]\\
&\leqslant\left\{\frac{2}{(m-1)(m-2)}\sum_{1\leqslant i<j\leqslant m-1}\left[1-\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]\right\}^{\frac{(m-1)(m-2)}{2}}\\
&=\left[1-\frac{2}{(m-1)(m-2)}\sum_{1\leqslant i<j\leqslant m}\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]^{\frac{(m-1)(m-2)}{2}}, \qquad (18)
\end{aligned}
$$

so that

$$
\begin{aligned}
&\prod_{1\leqslant i<j\leqslant m-1}\left[1-\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]^{\frac{1}{(m-2)\lambda_i}}\\
&\geqslant\left\{\prod_{1\leqslant i<j\leqslant m-1}\left[1-\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]\right\}^{\frac{1}{(m-2)\lambda_1}}\\
&\geqslant\left[1-\frac{2}{(m-1)(m-2)}\sum_{1\leqslant i<j\leqslant m-1}\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]^{\frac{m-1}{2\lambda_1}}. \qquad (19)
\end{aligned}
$$

On the other hand, from Lemma 1 we have

$$
\begin{aligned}
&\left[1-\frac{2}{(m-1)(m-2)}\sum_{1\leqslant i<j\leqslant m-1}\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]^{\frac{m-1}{2\lambda_1}}\\
&=\left\{1-\frac{2}{(m-1)(m-2}\left[(m-1)\left(\sum_{j=1}^{m-1}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right)-\left(\sum_{j=1}^{m-1}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)\right)^2\right]\right\}^{\frac{m-1}{2\lambda_1}}. \qquad (20)
\end{aligned}
$$

Consequently, from (17), (19) and (20), we immediately obtain the inequality (14).

Next, suppose that $m=2$. By the same method as in Lemma 3, applying generalized Hölder's inequality (4) and Lemma 1, we can deduce the desired inequality (15). □

LEMMA 6. *Let* $\lambda_1,\lambda_2,\ldots,\lambda_m>0$, $\sum_{j=1}^{m}\frac{1}{\lambda_j}\geqslant 1$, *let* $X_{rj}>0$, $1-\sum_{r=1}^{n}X_{rj}^{\lambda_j}>0$ $(r=1,2,\cdots,n,\ j=1,2,\cdots,m)$, *and let* $m\geqslant 2$. *Then*

$$\prod_{j=1}^{m}\left(1-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\sum_{r=1}^{n}\prod_{j=1}^{m}X_{rj}$$
$$\leqslant\left[1-\frac{2}{m(m-1)}\sum_{1\leqslant i<j\leqslant m}\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]^{\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}}. \tag{21}$$

Proof. After simply rearranging, we write by $\lambda_{j_1}\geqslant\lambda_{j_2}\geqslant\ldots\geqslant\lambda_{j_m}$ the component of $\lambda_1,\lambda_2,\ldots,\lambda_m$ in decreasing order, where $j_1,j_2,\ldots,j_m$ is a permutation of $1,2,\ldots,m$.

Then, from Lemma 3 and Lemma 1 we get

$$\prod_{j=1}^{m}\left(1-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\sum_{r=1}^{n}\prod_{j=1}^{m}X_{rj}$$
$$=\left(1-\sum_{r=1}^{n}X_{rj_1}^{\lambda_{j_1}}\right)^{\frac{1}{\lambda_{j_1}}}\left(1-\sum_{r=1}^{n}X_{rj_2}^{\lambda_{j_2}}\right)^{\frac{1}{\lambda_{j_2}}}\ldots\left(1-\sum_{r=1}^{n}X_{rj_m}^{\lambda_{j_m}}\right)^{\frac{1}{\lambda_{j_m}}}+\sum_{r=1}^{n}X_{rj_1}X_{rj_2}\ldots X_{rj_m}$$
$$\leqslant\left\{1-\frac{2}{m(m-1)}\left[m\left(\sum_{k=1}^{m}\left(\sum_{r=1}^{n}X_{rj_k}^{\lambda_{j_k}}\right)^2\right)-\left(\sum_{k=1}^{m}\left(\sum_{r=1}^{n}X_{rj_k}^{\lambda_{j_k}}\right)\right)^2\right]\right\}^{\frac{m}{2\lambda_{j_1}}}$$
$$=\left\{1-\frac{2}{m(m-1)}\left[m\left(\sum_{j=1}^{m}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right)-\left(\sum_{j=1}^{m}\left(\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)\right)^2\right]\right\}^{\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}}$$
$$=\left[1-\frac{2}{m(m-1)}\sum_{1\leqslant i<j\leqslant m}\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]^{\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}}. \tag{22}$$

The proof of Lemma 6 is completed. □

By the same method as in Lemma 6, we obtain the following two lemmas.

LEMMA 7. *Let* $\lambda_1,\lambda_2,\ldots,\lambda_m<0$, *let* $X_{rj}>0$, $1-\sum_{r=1}^{n}X_{rj}^{\lambda_j}>0$ $(r=1,2,\cdots,n,\ j=1,2,\cdots,m)$, *and let* $m\geqslant 2$. *Then*

$$\prod_{j=1}^{m}\left(1-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}+\sum_{r=1}^{n}\prod_{j=1}^{m}X_{rj}$$
$$\geqslant\left[1-\frac{2}{m(m-1)}\sum_{1\leqslant i<j\leqslant m}\left(\sum_{r=1}^{n}X_{ri}^{\lambda_i}-\sum_{r=1}^{n}X_{rj}^{\lambda_j}\right)^2\right]^{\frac{m}{2\min\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}}. \tag{23}$$

LEMMA 8. *Let* $\lambda_m > 0,\ \lambda_1, \lambda_2, \ldots, \lambda_{m-1} < 0,\ \sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$, *let* $X_{rj} > 1,\ 1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j} > 0\ (r = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m-1)$, *and let* $0 < X_{rm} < 1,\ 1 - \sum_{r=1}^{n} X_{rm}^{\lambda_m} > 0$. *If* $m > 2$, *then*

$$\prod_{j=1}^{m} \left(1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} + \sum_{r=1}^{n} \prod_{j=1}^{m} X_{rj}$$
$$\geqslant \left[1 - \frac{2}{(m-1)(m-2)} \sum_{1 \leqslant i < j \leqslant m-1} \left(\sum_{r=1}^{n} X_{ri}^{\lambda_i} - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\right)^2\right]^{\frac{m-1}{2\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}}}; \qquad (24)$$

If $m = 2$, *then*

$$\prod_{j=1}^{2} \left(1 - \sum_{r=1}^{n} X_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}} + \sum_{r=1}^{n} \prod_{j=1}^{2} X_{rj} \geqslant \left[1 - \left(\sum_{r=1}^{n} X_{r1}^{\lambda_1} - \sum_{r=1}^{n} X_{r2}^{\lambda_2}\right)^2\right]^{\frac{1}{\lambda_1}}. \qquad (25)$$

Finally, we present some new refinements of inequalities (2), (3) and (4).

THEOREM 1. *Let* $A_{rj} \geqslant 0\ (r = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m)$, *and let* l *be any given natural number* $(1 \leqslant l \leqslant n)$.

(a) If $\lambda_1, \lambda_2, \ldots, \lambda_m > 0,\ \sum_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1,\ m \geqslant 2$, *then*

$$\sum_{r=1}^{n} \prod_{j=1}^{m} A_{rj} \leqslant \left[\prod_{j=1}^{m} \left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]$$
$$\times \left[1 - \frac{2}{m(m-1)} \sum_{1 \leqslant i < j \leqslant m} \left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}} - \frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2\right]^{\frac{m}{2\max\{\lambda_1, \lambda_2, \ldots, \lambda_m\}}}.$$
$$\leqslant \left[\prod_{j=1}^{m} \left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]. \qquad (26)$$

(b) If $\lambda_1, \lambda_2, \ldots, \lambda_m < 0,\ m \geqslant 2$, *then*

$$\sum_{r=1}^{n} \prod_{j=1}^{m} A_{rj} \geqslant \left[\prod_{j=1}^{m} \left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]$$
$$\times \left[1 - \frac{2}{m(m-1)} \sum_{1 \leqslant i < j \leqslant m} \left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}} - \frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2\right]^{\frac{m}{2\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}}}$$
$$\geqslant \left[\prod_{j=1}^{m} \left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]. \qquad (27)$$

(c) Let $\lambda_m > 0,\ \lambda_1, \lambda_2, \ldots, \lambda_{m-1} < 0,\ \sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$. *If* $m > 2$, *then*

$$\begin{aligned}\sum_{r=1}^{n}\prod_{j=1}^{m} A_{rj} \geqslant & \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \\ & \times\left[1-\frac{2}{(m-1)(m-2)}\sum_{1\leqslant i<j\leqslant m-1}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}}-\frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2\right]^{\frac{m-1}{2\min\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}} \\ & \geqslant \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]; \end{aligned} \tag{28}$$

If $m = 2$, *then*

$$\begin{aligned}\sum_{r=1}^{n}\prod_{j=1}^{2} A_{rj} \geqslant & \left[\prod_{j=1}^{2}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]\left[1-\left(\frac{A_{l1}^{\lambda_1}}{\sum_{k=1}^{n} A_{k1}^{\lambda_1}}-\frac{A_{l2}^{\lambda_2}}{\sum_{k=1}^{n} A_{k2}^{\lambda_2}}\right)^2\right]^{\frac{1}{\lambda_1}} \\ & \geqslant \left[\prod_{j=1}^{2}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]. \end{aligned} \tag{29}$$

Proof. (a). Consider the following substitution:

$$X_{rj} = \frac{A_{rj}}{\left(\sum_{k=1}^{n} A_{kj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}} \quad (r = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m). \tag{30}$$

It is easy to see that, for any given natural number $l\ (1 \leqslant l \leqslant n)$, the following inequalities hold

$$X_{rj} > 0, \quad 1 - \sum_{1\leqslant r\leqslant n, r\neq l} X_{rj}^{\lambda_j} > 0.$$

Consequently, by using the substitution (30) and inequality (21), we have

$$\begin{aligned}&\prod_{j=1}^{m}\left[1-\sum_{1\leqslant r\leqslant n, r\neq l}\left(\frac{A_{rj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)\right]^{\frac{1}{\lambda_j}} + \sum_{1\leqslant r\leqslant n, r\neq l}\left[\prod_{j=1}^{m}\frac{A_{rj}}{\left(\sum_{k=1}^{n} A_{kj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}\right] \\ &\leqslant \left\{1-\frac{2}{m(m-1)}\sum_{1\leqslant i<j\leqslant m}\left[\sum_{1\leqslant r\leqslant n,\ r\neq l}\left(\frac{A_{ri}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}}\right)\right.\right. \\ &\quad \left.\left. -\sum_{1\leqslant r\leqslant n, r\neq l}\left(\frac{A_{rj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)\right]^2\right\}^{\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}}, \end{aligned} \tag{31}$$

and thus we have

$$\frac{\prod_{j=1}^{m} A_{lj}}{\prod_{j=1}^{m}\left(\sum_{k=1}^{n} A_{kj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}} + \frac{\sum_{1\leqslant r\leqslant n, r\neq l}\prod_{j=1}^{m} A_{rj}}{\prod_{j=1}^{m}\left(\sum_{k=1}^{n} A_{kj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}$$
$$\leqslant \left[1-\frac{2}{m(m-1)}\sum_{1\leqslant i<j\leqslant m}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}}-\frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2\right]^{\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}}, \quad (32)$$

that is

$$\frac{\sum_{r=1}^{n}\left(\prod_{j=1}^{m} A_{rj}\right)}{\prod_{j=1}^{m}\left(\sum_{k=1}^{n} A_{kj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}}$$
$$\leqslant \left[1-\frac{2}{m(m-1)}\sum_{1\leqslant i<j\leqslant m}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}}-\frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2\right]^{\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}}. \quad (33)$$

So, we have the desired inequality (26). The proof of inequalities (27), (28) and (29) are similar to the one of inequality (26) and we omit it. The proof of Theorem 1 is completed. □

REMARK 1. If $l=1$, $m=2$, $n=N+1$, $A_{11}=0$, $A_{r1}=1$, $A_{12}=1$, $A_{r2}=0$, $r=2,3,\cdots,N+1$, and if $\lambda_1,\lambda_2>0$, $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=1$, then from inequality (26) we obtain $0\leqslant 0$.

THEOREM 2. *Let $A_{rj}\geqslant 0$ $(r=1,2,\cdots,n,\ j=1,2,\cdots,m)$, and let l be any given natural number $(1\leqslant l\leqslant n)$.*

(a) If $\lambda_1,\lambda_2,\ldots,\lambda_m>0$, $\sum_{j=1}^{m}\frac{1}{\lambda_j}\geqslant 1$, $m\geqslant 2$, then

$$\sum_{r=1}^{n}\prod_{j=1}^{m} A_{rj} \leqslant \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]$$
$$\times\left[1-\frac{1}{(m-1)\max\{\lambda_1,\lambda_2,\ldots,\lambda_m,\frac{m}{2}\}}\sum_{1\leqslant i<j\leqslant m}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}}-\frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2\right]$$
$$\leqslant \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]. \quad (34)$$

(b) If $\lambda_1,\lambda_2,\ldots,\lambda_m<0$, $m\geqslant 2$, then

$$\sum_{r=1}^{n}\prod_{j=1}^{m} A_{rj} \geqslant \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]$$
$$\times\left[1-\frac{1}{(m-1)\min\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}\sum_{1\leqslant i<j\leqslant m}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}}-\frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2\right]$$
$$\geqslant \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]. \quad (35)$$

(c) Let $\lambda_m > 0,\ \lambda_1, \lambda_2, \ldots, \lambda_{m-1} < 0,\ \sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$. *If* $m > 2$, *then*

$$\sum_{r=1}^{n}\prod_{j=1}^{m} A_{rj} \geqslant \left[\prod_{j=1}^{m}\Big(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\Big)^{\frac{1}{\lambda_j}}\right] \times \left[1 - \frac{1}{(m-2)\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}} \sum_{1\leqslant i<j\leqslant m-1}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}} - \frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2\right] \geqslant \left[\prod_{j=1}^{m}\Big(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\Big)^{\frac{1}{\lambda_j}}\right]; \tag{36}$$

If $m = 2$, *then*

$$\sum_{r=1}^{n}\prod_{j=1}^{2} A_{rj} \geqslant \left[\prod_{j=1}^{2}\Big(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\Big)^{\frac{1}{\lambda_j}}\right]\left[1 - \frac{1}{\lambda_1}\left(\frac{A_{l1}^{\lambda_1}}{\sum_{k=1}^{n} A_{k1}^{\lambda_1}} - \frac{A_{l2}^{\lambda_2}}{\sum_{k=1}^{n} A_{k2}^{\lambda_2}}\right)^2\right] \geqslant \left[\prod_{j=1}^{2}\Big(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\Big)^{\frac{1}{\lambda_j}}\right]. \tag{37}$$

Proof. Case (a). When $\max\{\lambda_1, \lambda_2, \ldots, \lambda_m\} > \frac{m}{2}$, which implies $\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}} \in (0,1)$. From Lemma 2 and Theorem 1, we obtain

$$\left[1 - \frac{2}{m(m-1)} \sum_{1\leqslant i<j\leqslant m}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}} - \frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2\right]^{\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}} \leqslant 1 - \frac{1}{(m-1)\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}} \sum_{1\leqslant i<j\leqslant m}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}} - \frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2 = 1 - \frac{1}{(m-1)\max\{\lambda_1,\lambda_2,\ldots,\lambda_m,\frac{m}{2}\}} \sum_{1\leqslant i<j\leqslant m}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}} - \frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2. \tag{38}$$

When $\max\{\lambda_1, \lambda_2, \ldots, \lambda_m\} \leqslant \frac{m}{2}$, then this implies $\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}} \geqslant 1$. Noting that the function $f(x) = a^x\ (0 < a < 1)$ is strictly decreasing on $(-\infty, +\infty)$, then we have

$$\left[1 - \frac{2}{m(m-1)} \sum_{1\leqslant i<j\leqslant m}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}} - \frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2\right]^{\frac{m}{2\max\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}} \leqslant 1 - \frac{2}{m(m-1)} \sum_{1\leqslant i<j\leqslant m}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}} - \frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2 = 1 - \frac{1}{(m-1)\max\{\lambda_1,\lambda_2,\ldots,\lambda_m,\frac{m}{2}\}} \sum_{1\leqslant i<j\leqslant m}\left(\frac{A_{li}^{\lambda_i}}{\sum_{k=1}^{n} A_{ki}^{\lambda_i}} - \frac{A_{lj}^{\lambda_j}}{\sum_{k=1}^{n} A_{kj}^{\lambda_j}}\right)^2. \tag{39}$$

Combining inequalities (38) and (39) leads to the desired inequalty (34). From Lemma 2, inequalities (35), (36) and (37) are valid. The proof of Theorem 2 is completed. □

From Theorem 2, the following refinements of (2), (3) and (4) hold.

COROLLARY 1. *Let* $A_{rj} \geqslant 0$ $(r = 1,2,\cdots,n,\ j = 1,2,\cdots,m)$, *and let* l *be any given natural number* $(1 \leqslant l \leqslant n)$.

(a) If $\lambda_1, \lambda_2, \ldots, \lambda_m > 0$, $\sum_{j=1}^{m} \frac{1}{\lambda_j} \geqslant 1$, $m \geqslant 2$, *then*

$$\begin{aligned}\sum_{r=1}^{n}\prod_{j=1}^{m} A_{rj} \leqslant & \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \\ & \times\left[1-\frac{1}{(m-1)\max\{\lambda_1,\lambda_2,\ldots,\lambda_m,\frac{m}{2}\}}\left(\frac{A_{l1}^{\lambda_1}}{\sum_{k=1}^{n} A_{k1}^{\lambda_1}}-\frac{A_{l2}^{\lambda_2}}{\sum_{k=1}^{n} A_{k2}^{\lambda_2}}\right)^2\right] \\ \leqslant & \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right], \end{aligned} \tag{40}$$

and

$$\begin{aligned}\sum_{r=1}^{n}\prod_{j=1}^{m} A_{rj} \leqslant & \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \\ & \times\left[1-\frac{1}{(m-1)\max\{\lambda_1,\lambda_2,\ldots,\lambda_m,\frac{m}{2}\}}\left(\frac{A_{l1}^{\lambda_1}}{\sum_{k=1}^{n} A_{k1}^{\lambda_1}}-\frac{A_{lm}^{\lambda_m}}{\sum_{k=1}^{n} A_{km}^{\lambda_m}}\right)^2\right] \\ \leqslant & \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]. \end{aligned} \tag{41}$$

(b) If $\lambda_1, \lambda_2, \ldots, \lambda_m < 0$, $m \geqslant 2$, *then*

$$\begin{aligned}\sum_{r=1}^{n}\prod_{j=1}^{m} A_{rj} \geqslant & \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \\ & \times\left[1-\frac{1}{(m-1)\min\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}\left(\frac{A_{l1}^{\lambda_1}}{\sum_{k=1}^{n} A_{k1}^{\lambda_1}}-\frac{A_{l2}^{\lambda_2}}{\sum_{k=1}^{n} A_{k2}^{\lambda_2}}\right)^2\right] \\ \geqslant & \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right], \end{aligned} \tag{42}$$

and

$$\begin{aligned}\sum_{r=1}^{n}\prod_{j=1}^{m} A_{rj} \geqslant & \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right] \\ & \times\left[1-\frac{1}{(m-1)\min\{\lambda_1,\lambda_2,\ldots,\lambda_m\}}\left(\frac{A_{l1}^{\lambda_1}}{\sum_{k=1}^{n} A_{k1}^{\lambda_1}}-\frac{A_{lm}^{\lambda_m}}{\sum_{k=1}^{n} A_{km}^{\lambda_m}}\right)^2\right] \\ \geqslant & \left[\prod_{j=1}^{m}\left(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\right)^{\frac{1}{\lambda_j}}\right]. \end{aligned} \tag{43}$$

(c) Let $\lambda_m > 0,\ \lambda_1, \lambda_2, \ldots, \lambda_{m-1} < 0,\ \sum_{j=1}^{m} \frac{1}{\lambda_j} \leqslant 1$. *If* $m > 2$, *then*

$$\sum_{r=1}^{n} \prod_{j=1}^{m} A_{rj} \geqslant \left[\prod_{j=1}^{m} \Big(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\Big)^{\frac{1}{\lambda_j}}\right] \times \left[1 - \frac{1}{(m-2)\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}} \left(\frac{A_{l1}^{\lambda_1}}{\sum_{k=1}^{n} A_{k1}^{\lambda_1}} - \frac{A_{l2}^{\lambda_2}}{\sum_{k=1}^{n} A_{k2}^{\lambda_2}}\right)^2\right] \geqslant \left[\prod_{j=1}^{m} \Big(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\Big)^{\frac{1}{\lambda_j}}\right], \tag{44}$$

and

$$\sum_{r=1}^{n} \prod_{j=1}^{m} A_{rj} \geqslant \left[\prod_{j=1}^{m} \Big(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\Big)^{\frac{1}{\lambda_j}}\right] \times \left[1 - \frac{1}{(m-2)\min\{\lambda_1, \lambda_2, \ldots, \lambda_m\}} \left(\frac{A_{l1}^{\lambda_1}}{\sum_{k=1}^{n} A_{k1}^{\lambda_1}} - \frac{A_{l(m-1)}^{\lambda_{(m-1)}}}{\sum_{k=1}^{n} A_{k(m-1)}^{\lambda_{(m-1)}}}\right)^2\right] \geqslant \left[\prod_{j=1}^{m} \Big(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\Big)^{\frac{1}{\lambda_j}}\right]; \tag{45}$$

If $m = 2$, *then*

$$\sum_{r=1}^{n} \prod_{j=1}^{2} A_{rj} \geqslant \left[\prod_{j=1}^{2} \Big(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\Big)^{\frac{1}{\lambda_j}}\right] \left[1 - \frac{1}{\lambda_1} \left(\frac{A_{l1}^{\lambda_1}}{\sum_{k=1}^{n} A_{k1}^{\lambda_1}} - \frac{A_{l2}^{\lambda_2}}{\sum_{k=1}^{n} A_{k2}^{\lambda_2}}\right)^2\right] \geqslant \left[\prod_{j=1}^{2} \Big(\sum_{r=1}^{n} A_{rj}^{\lambda_j}\Big)^{\frac{1}{\lambda_j}}\right]. \tag{46}$$

3. Application

In this section, we present a new refinement of Singh's inequality with respect to information of order α for the power distribution and exponential mean length.

Let χ be the utility information scheme

$$\chi = \begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ p_1^{\beta} & p_2^{\beta} & p_3^{\beta} & \cdots & p_n^{\beta} \\ u_1 & u_2 & u_3 & \cdots & u_n \end{bmatrix}$$

where $X = (x_1, x_2, x_3, \cdots, x_n)$ is the alphabet; $P^{\beta} = (p_1^{\beta}, p_2^{\beta}, p_3^{\beta}, \cdots, p_n^{\beta})$ is the power probability distribution; $U = (u_1, u_2, u_3, \cdots, u_n)$ is the utility distribution $u_r > 0$ for all $r = 1, 2, 3, \cdots, n$; $\beta \neq 1,\ \beta > 0,\ \sum_{r=1}^{n} p_r^{\beta} = 1$.

In [18], Singh et al. defined the 'useful' information of order α for the power distribution P^{β} as

$$\frac{1}{1-\alpha}\log\sum_{r=1}^{n}\left(\frac{p_r^{\beta\alpha}u_r}{\sum_{i=1}^{n}p_i^{\beta}u_i}\right), \tag{47}$$

and defined the exponential 'useful' mean lengths of codewords weighted with the function of power probabilities and utilities as

$$\frac{\alpha}{1-\alpha}\sum_{r=1}^{n}p_r^{\beta}\left(\frac{u_r}{\sum_{i=1}^{n}p_i^{\beta}u_i}\right)^{\frac{1}{\alpha}}D^{\frac{(1-\alpha)l_r}{\alpha}}. \tag{48}$$

It is obvious that when $\beta=1$, $u_r=1$ for all $r=1,2,\ldots,n$, (47) is a generalization of Renyi's [16] entropy of order α. It is also very clear that when $\beta=1$, $\alpha\to 1$, and $u_r=1$ for all $r=1,2,\ldots,n$, (47) reduce to Shannon entropy [17].

Moreover, for every uniquely decipherable code, Singh et al. [18] obtained

$$\frac{\alpha}{1-\alpha}\log_D\left(\sum_{r=1}^{n}\frac{p_r^{\beta}u_r^{\frac{1}{\alpha}}D^{\frac{l_r(1-\alpha)}{\alpha}}}{(\sum_{i=1}^{n}p_i^{\beta}u_i)^{\frac{1}{\alpha}}}\right)\geqslant\frac{\log_2\sum_{r=1}^{n}\left(\frac{p_r^{\beta\alpha}u_r}{\sum_{i=1}^{n}p_i^{\beta}u_i}\right)}{(1-\alpha)\log_2 D}, \tag{49}$$

where $\alpha>0$, $\alpha\neq 1$, $D\geqslant 2$, l_r, integers, $p_r\geqslant 0$, $r=1,2,3,\cdots,n$ and $\sum_{r=1}^{n}D^{-l_r}\leqslant 1$. The inequality is called as Singh's inequality.

A fundamental result related to the notion of the Shannon entropy is the following inequality

$$\sum_{r=1}^{n}p_r\log\frac{1}{p_r}\leqslant\sum_{r=1}^{n}p_r\log\frac{1}{q_r}, \tag{50}$$

which is valid for all positive real numbers p_r and q_r with $\sum_{r=1}^{n}p_r=1$, $\sum_{r=1}^{n}q_r=1$. This result, sometimes called the fundamental lemma of information theory, has extensive applications (see, for example, [11]). Obviously, the inequality (49), which gives the relation between (47) and (48), generalized Shannon's inequality (50).

Now, from Corollary 1 we present a refinement of Singh's inequality (49).

THEOREM 3. *Let $\alpha>0$, $\beta>0$, $\alpha,\ \beta\neq 1$, $p_r\geqslant 0$, $r=1,2,3,\cdots,n$ and $\sum_{r=1}^{n}p_r^{\beta}=1$, let D $(D\geqslant 2)$ is the size of the code alphabet. If N_r, $r=1,2,3,\cdots,n$ are the lengths of the codewords satisfying the Kraft inequality*

$$\sum_{r=1}^{n}D^{-N_r}\leqslant 1.$$

Then for every uniquely decipherable code, the 'useful' α-average length of codewords satisfies

$$\frac{\alpha}{1-\alpha}\log_D\left(\sum_{r=1}^{n}\frac{p_r^{\beta}u_r^{\frac{1}{\alpha}}D^{\frac{N_r(1-\alpha)}{\alpha}}}{(\sum_{i=1}^{n}p_i^{\beta}u_i)^{\frac{1}{\alpha}}}\right)\geqslant\frac{\log_2\sum_{r=1}^{n}(\frac{p_r^{\beta\alpha}u_r}{\sum_{i=1}^{n}p_i^{\beta}u_i})}{(1-\alpha)\log_2 D}+\log_D(1+\omega t^2), \tag{51}$$

where

$$t=\frac{p_1^{\beta}u_1^{\frac{1}{\alpha}}D^{\frac{N_1(1-\alpha)}{\alpha}}}{\sum_{r=1}^{n}p_r^{\beta}u_r^{\frac{1}{\alpha}}D^{\frac{N_r(1-\alpha)}{\alpha}}}-\frac{u_1p_1^{\alpha\beta}}{\sum_{r=1}^{n}u_rp_r^{\alpha\beta}},$$

$$\omega=\begin{cases}\frac{1}{\alpha-1} & (\alpha>1)\\ \frac{\alpha}{1-\alpha} & (0<\alpha<1).\end{cases}$$

Proof. We consider two cases: $\alpha>1$ and $0<\alpha<1$. For $\alpha>1$, by using Corollary 1 with a substitution

$$\lambda_1=\frac{\alpha-1}{\alpha},\quad \lambda_2=1-\alpha,\quad A_{r1}=p_r^{\frac{\alpha\beta}{\alpha-1}}\left(\frac{u_r}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)^{\frac{1}{\alpha-1}}D^{-N_r},$$

$$A_{r2}=p_r^{\frac{\alpha\beta}{1-\alpha}}\left(\frac{u_r}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)^{\frac{1}{1-\alpha}}$$

in (41), and setting $m=2$. we have

$$\begin{aligned}\sum_{r=1}^{n}D^{-N_r}\geqslant&\left[\sum_{r=1}^{n}p_r^{\beta}\left(\frac{u_r}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)^{\frac{1}{\alpha}}D^{-\frac{(\alpha-1)N_r}{\alpha}}\right]^{\frac{\alpha}{\alpha-1}}\left[\sum_{r=1}^{n}p_r^{\alpha\beta}\left(\frac{u_r}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)\right]^{\frac{1}{1-\alpha}}\\&\times\left\{1-\frac{1}{1-\alpha}\left[p_1^{\beta}\left(\frac{u_1}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)^{\frac{1}{\alpha}}D^{-\frac{(\alpha-1)N_1}{\alpha}}\Big/\sum_{r=1}^{n}p_r^{\beta}\left(\frac{u_r}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)^{\frac{1}{\alpha}}D^{-\frac{(\alpha-1)N_r}{\alpha}}\right.\right.\\&\left.\left.-\left(\frac{u_1p_1^{\alpha\beta}}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)\Big/\sum_{r=1}^{n}p_r^{\alpha\beta}\left(\frac{u_r}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)\right]^2\right\}.\end{aligned}\tag{52}$$

In view of $\sum_{r=1}^{n}D^{-N_r}\leqslant 1$, and carrying detailed computing, we have

$$\begin{aligned}&\left[\sum_{r=1}^{n}p_r^{\beta}\left(\frac{u_r}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)^{\frac{1}{\alpha}}D^{-\frac{(\alpha-1)N_r}{\alpha}}\right]^{\frac{\alpha}{1-\alpha}}\\&\geqslant\left[\sum_{r=1}^{n}p_r^{\alpha\beta}\left(\frac{u_r}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)\right]^{\frac{1}{1-\alpha}}\left[1-\frac{1}{1-\alpha}\left(\frac{p_1^{\beta}u_1^{\frac{1}{\alpha}}D^{-\frac{(\alpha-1)N_1}{\alpha}}}{\sum_{r=1}^{n}p_r^{\beta}u_r^{\frac{1}{\alpha}}D^{-\frac{(\alpha-1)N_r}{\alpha}}}-\frac{u_1p_1^{\alpha\beta}}{\sum_{r=1}^{n}u_rp_r^{\alpha\beta}}\right)^2\right].\end{aligned}\tag{53}$$

Hence we have the desired inequality (51) in the case of $\alpha>1$. Moreover, for the case $0<\alpha<1$, by the same way, using Corollary 1 with a substitution

$$\lambda_1=\frac{\alpha-1}{\alpha},\quad \lambda_2=1-\alpha,\quad A_{r1}=p_r^{\frac{\alpha\beta}{\alpha-1}}\left(\frac{u_r}{\sum_{i=1}^{n}u_ip_i^{\beta}}\right)^{\frac{1}{\alpha-1}}D^{-N_r},$$

$$A_{r2} = p_r^{\frac{\alpha\beta}{1-\alpha}} \left(\frac{u_r}{\sum_{i=1}^n u_i p_i^\beta} \right)^{\frac{1}{1-\alpha}}$$

in (46) and setting $m = 2$, we can obtain the desired inequality (51). So, the proof of Theorem 3 is completed. □

REMARK 2. If we set $\beta = 1$, $u_r = 1$ $(r = 1,2,\dots,n)$ in (51), then a new refinement of Campbell's inequality ([5], Lemma) holds.

$$\frac{\alpha}{1-\alpha} \log_D \left(\sum_{r=1}^n p_r D^{\frac{N_r(1-\alpha)}{\alpha}} \right) \geqslant \frac{1}{1-\alpha} \log_D \left(\sum_{r=1}^n p_r^\alpha \right) + \log_D(1+\omega_1 t_1), \tag{54}$$

where

$$t_1 = \frac{p_1 D^{\frac{N_1(1-\alpha)}{\alpha}}}{\sum_{r=1}^n p_r D^{\frac{N_r(1-\alpha)}{\alpha}}} - \frac{p_1^\alpha}{\sum_{r=1}^n p_r^\alpha},$$

$$\omega_1 = \begin{cases} \frac{1}{\alpha-1} & (\alpha > 1) \\ \frac{\alpha}{1-\alpha} & (0 < \alpha < 1). \end{cases}$$

And then, for $\alpha \to 1$, by using L'Hospital theorem and inequality (54) we obtain the refinement of the Feinstein's inequality [7] as follows.

$$\sum_{r=1}^n N_r p_r \geqslant -\sum_{r=1}^n p_r \log_D p_r + \log_D(1+\theta^*), \tag{55}$$

where

$$\theta^* = \begin{cases} -p_1 \left(\ln D - \sum_{r=1}^n p_r N_r \ln D \right) & (\alpha > 1) \\ p_1 \left(\ln D - \sum_{r=1}^n p_r N_r \ln D \right) & (0 < \alpha < 1). \end{cases}$$

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