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# HERMITE'S FORMULA FOR $q$–GAMMA FUNCTION

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*Abstract.* In this paper, we presented the Raabe's integral and Hermite's formula for $q$-gamma function $\Gamma_q(x)$, $0<q<1$. We deduced new proofs of the formulas $\frac{\Gamma_q'(x)}{\Gamma_q(x)}$ and $q$-Gauss's multiplication using the Hermite's formula of $\Gamma_q(x)$ and H. Jack's technique [11]. Also, we deduced new double inequality of $\Gamma_q(x)$.

## 1. Introduction

The $q$-gamma function was introduced by Thomae [20] and later by Jackson [12] (see also [18] ) by the infinite product

$$\Gamma_q(x)=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}; \qquad x\neq 0,-1,-2,..., \tag{1}$$

where $q$ is a fixed real number $0<q<1$. Here we use the following notation [8]:

$$(a;q)_0=1,$$
$$(a;q)_k=\prod_{j=0}^{k-1}(1-aq^j); \qquad k\in\mathbb{N}.$$

This function is a $q$-analogue of the gamma function since we have

$$\lim_{q\to 1}\Gamma_q(x)=\Gamma(x).$$

As same as the gamma function is fundamental in the theory of special functions, the $q$-gamma function is significant in the study of $q$-analysis, specially in the theory of the $q$-hypergeometric series [2], [8], [9], [13]. There have been a lot of literature about the $q$-gamma function, in particular its inequalities, functional equations, monotonicity and complete monotonicity properties. For more information, please refer to [1], [6], [7], [10], [14], [19] and the references therein.

Mahmoud [14] use the technique of E. Artin [3] in determining the classical gamma function by a combination of some functional equations to present the following Theorem for $\Gamma_q(x)$:

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THEOREM 1. *The $q$-gamma function $(0<q<1)$ is the only $C^2$-function $f_q(x)$ which is positive for $x>0$ and which satisfies the equations*

$$f_q(x+1) = [x]_q f_q(x), \tag{2}$$

$$f_{q^p}(x)\Gamma_q(x) = f_q(x)\Gamma_{q^p}(x), \qquad p \in \mathbb{N} \tag{3}$$

*and*

$$f_{q^2}(x/2) f_{q^2}((x+1)/2) = ([2]_q)^{1-x} \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} (1-q^2)^{1/2} f_q(x), \tag{4}$$

*where the $q$-number $[x]_q = \frac{1-q^x}{1-q}$.*

The Raabe's integral for the ordinary gamma function is given by

$$\int_x^{x+1} \log\Gamma(t)dt = x\log x - x + \frac{1}{2}\log 2\pi, \qquad \text{for } x>0 \tag{5}$$

and the Hermite's formula is given by

$$\int_0^1 (t-1/2)\frac{d}{dt}\log\Gamma(x+t)dt = \log\Gamma(x) + (1/2-x)\log x + x - \frac{1}{2}\log 2\pi, \qquad \text{for } x>0. \tag{6}$$

In 2008, an interesting extension of Raabe's integral to the $p$-adic Gamma function has appeared in [5]. Also, in 2013, Mező [15] generalized the Raabe-formula to the $q$-log gamma function by giving an integral formula for $\Gamma_q$ when $q>1$.

In this paper, we will present the Raabe's integral and the Hermite's formula for the $q$-gamma function $\Gamma_q(x)$, $0<q<1$. Using the Hermite's formula of $\Gamma_q(x)$ and the facts of the Theorem 1, we will present new proofs of a $q$-analogy of the relation

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\frac{1}{x} - \gamma - \sum_{r=1}^{\infty}\left(\frac{1}{x+r} - \frac{1}{r}\right) \tag{7}$$

and a $q$-analogy of the Gauss's multiplication formula

$$\Gamma(x/p)\Gamma((x+1)/p)\ldots\Gamma((x+p-1)/p) = \frac{(2\pi)^{\frac{p-1}{2}}}{p^{x-1/2}}\Gamma(x), \quad p\in\mathbb{N}. \tag{8}$$

In the following sequel we will consider that the function $\Gamma_q(x)$ satisfies only the conditions of Theorem 1.

## 2. Raabe's integral and Hermite's formula for $\Gamma_q(x)$

For $x>0$, consider the function

$$G_q(x) = \int_x^{x+1} \log\Gamma_q(t)dt = \int_0^1 \log\Gamma_q(x+t)dt, \tag{9}$$

then

$$G_q(x) = \frac{-1}{\log q} Li_2(q^x) - x\log(1-q) + c_q, \tag{10}$$

where the polylogarithm function $Li_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$. As $x \to 0^+$, we get

$$c_q = \int_0^1 \log \Gamma_q(t)dt + \frac{\pi^2}{6\log q}. \tag{11}$$

By integrating the relation (4) from $0$ to $1$, we obtain

$$2\left[c_{q^2} - \log(q^2,q^2)_\infty - \frac{1}{2}\log(1-q^2)\right] = c_q - \log(q,q)_\infty - \frac{1}{2}\log(1-q). \tag{12}$$

Now if $f(q) = c_q - \log(q,q)_\infty - \frac{1}{2}\log(1-q)$, we get

$$2^n f(q^{2^n}) = f(q), \qquad n \in \mathbb{N}.$$

If we put $q^{2^n} = w$, we obtain

$$f(q)\ln q = f(w)\ln w$$

and hence

$$f(q) = \frac{\alpha}{\ln q}, \qquad \alpha \in R.$$

Then

$$c_q = \frac{\alpha}{\ln q} + \log\left((q,q)_\infty \sqrt{1-q}\right)$$

and

$$G_q(0) = \frac{\alpha}{\ln q} + \delta_q, \tag{13}$$

where

$$\delta_q = \log\left(e^{-\frac{\pi^2}{6\log q}}(q,q)_\infty\sqrt{1-q}\right).$$

Using the relation [17]

$$\delta_q = \lim_{n\to\infty}(\delta_{n,q}),$$

where

$$\delta_{n,q} = \log\left(\frac{[n]_q!}{[n]_q^n\sqrt{[n]_q}}\right) - \frac{Li_2(1-q^n)}{\log q}$$

and $[n]_q! = [1]_q![2]_q!\ldots[n]_q!$, $n \in N$. Then

$$\lim_{q\to 1}\delta_{n,q} = \ln\left(\frac{n!e^n}{n^n\sqrt{n}}\right)$$

and using Stirling's formula

$$n! \sim \sqrt{2n\pi}\,(n/e)^n,$$

we have

$$\lim_{n\to\infty}\lim_{q\to 1}\delta_{n,q} = \ln\sqrt{2\pi}.$$

Hence

$$\lim_{q\to 1}\delta_q = \ln\sqrt{2\pi}. \tag{14}$$

and using the relation (5), we get

$$\lim_{q\to 1} G_q(0) = \ln\sqrt{2\pi}. \tag{15}$$

Now using the relation (13) as $q \to 1$ with the relations (14) and (15), we have $\alpha = 0$ and

$$c_q = \log\left[(q,q)_\infty\sqrt{1-q}\right]. \tag{16}$$

Hence we obtain the following result:

THEOREM 2. (Raabe's integral for $\Gamma_q(x)$) *For $x > 0$ and $0 < q < 1$,*

$$G_q(x) = \int_x^{x+1}\log\Gamma_q(t)\,dt = \log\left[(q,q)_\infty(1-q)^{1/2-x}\right] - \frac{1}{\log q}Li_2(q^x). \tag{17}$$

Now for $x > 0$, consider the function

$$R_q(x) = \int_0^1 (t-1/2)\frac{d}{dt}\log\Gamma_q(x+t)dt. \tag{18}$$

On integrating by parts, then

$$R_q(x) = \frac{1}{2}\log[x]_q + \log\Gamma_q(x) - G_q(x). \tag{19}$$

Hence we obtain the following result:

THEOREM 3. (Hermite's formula for $\Gamma_q(x)$) *For $x > 0$ and $0 < q < 1$,*

$$\begin{aligned} R_q(x) &= \int_0^1 (t-1/2)\frac{d}{dt}\log\Gamma_q(x+t)dt \\ &= \frac{1}{2}\log[x]_q + \log\Gamma_q(x) - \log\left[(q,q)_\infty(1-q)^{1/2-x}\right] + \frac{1}{\log q}Li_2(q^x). \end{aligned} \tag{20}$$

THEOREM 4. *For $x > 0$ and $0 < q < 1$,*

$$\frac{d^2}{dx^2}\log\Gamma_q(x) > 0. \tag{21}$$

*Proof.* Using equation (2), we get

$$\frac{d^2}{dx^2}\log\Gamma_q(x) > \frac{d^2}{dx^2}\log\Gamma_q(x+1)$$

and hence

$$\frac{d^2}{dx^2}\log\Gamma_q(x) > \frac{d^2}{dx^2}\log\Gamma_q(x+n); \qquad n\in N.$$

Then

$$\frac{d^2}{dx^2}\log\Gamma_q(x) > h(q), \tag{22}$$

where

$$h(q) = \lim_{t\to\infty}\frac{d^2}{dt^2}\log\Gamma_q(t).$$

Using equation (4), we have

$$\begin{aligned}\lim_{t\to\infty}\frac{d^2}{dt^2}\log\Gamma_q(t) &= \lim_{t\to\infty}\frac{d^2}{dt^2}\log\Gamma_{q^2}(t/2)+\lim_{t\to\infty}\frac{d^2}{dt^2}\log\Gamma_{q^2}((t+1)/2)\\ &= \frac{1}{2}\lim_{t\to\infty}\frac{d^2}{dt^2}\log\Gamma_{q^2}(t) = \ldots = \frac{1}{2^n}\lim_{t\to\infty}\frac{d^2}{dt^2}\log\Gamma_{q^{2^n}}(t).\end{aligned}$$

Then

$$h(q) = \frac{1}{2^n}h(q^{2^n}).$$

If we put $q^{2^n} = w$, we obtain

$$\frac{h(q)}{\ln q} = \frac{h(w)}{\ln w}$$

and hence

$$h(q) = \beta\ln q, \qquad \beta\in R.$$

Now

$$\frac{d^2}{dx^2}G_q(x) = \frac{-q^x\ln q}{1-q^x} = \int_0^1\frac{d^2}{dx^2}\ln\Gamma_q(x+t)dt$$

and using that the $q$-gamma function ($0<q<1$) is positive and $C^2$-function for $x>0$, we have

$$\lim_{x\to\infty}\frac{-q^x\ln q}{1-q^x} = \int_0^1\lim_{x\to\infty}\frac{d^2}{dx^2}\ln\Gamma_q(x+t)dt.$$

Then

$$h(q) = \int_0^1 h(q)dt = \int_0^1\lim_{x\to\infty}\frac{d^2}{dx^2}\ln\Gamma_q(x+t)dt = 0$$

and using the inequality (22), we obtain the inequality (21). □

Now we will define a $(q,n)$-analog of Euler's constant by:

$$\gamma_{q,n} = -\log[n]_q - \frac{\log q}{1-q}\sum_{i=1}^{n}\frac{q^i}{[i]_q}; \qquad n = 1,2,3,\ldots, \tag{23}$$

which give us the $q$-analog of Euler's constant

$$\gamma_q = \lim_{n\to\infty} \gamma_{q,n} = \log(1-q) - \frac{\log q}{1-q} \sum_{i=1}^{\infty} \frac{q^i}{[i]_q}, \tag{24}$$

given by Bradley [4], where $\gamma = \lim_{q\to 1} \gamma_q$ is the ordinary Euler's constant.

THEOREM 5. *For $x > 0$ and $0 < q < 1$,*

$$\frac{\Gamma_q'(x)}{\Gamma_q(x)} = -\gamma_q + \frac{q^x \log q}{1-q^x} + \log q \sum_{i=1}^{\infty} \left[ \frac{q^{i+x}}{1-q^{i+x}} - \frac{q^i}{1-q^i} \right]. \tag{25}$$

*Proof.* Using the Hermite's formula (20), we get

$$\frac{\Gamma_q'(x)}{\Gamma_q(x)} = \frac{q^x \log q}{2(1-q^x)} + \log[x]_q + \int_0^1 (t-1/2) \frac{d^2}{dt^2} \log \Gamma_q(x+t)\, dt$$

and replace $x$ by $x+n+1$ to obtain

$$\begin{aligned} \frac{\Gamma_q'(x+n+1)}{\Gamma_q(x+n+1)} = & \frac{q^{x+n+1} \log q}{2(1-q^{x+n+1})} + \log[x+n+1]_q \\ & + \int_0^1 (t-1/2) \frac{d^2}{dt^2} \log \Gamma_q(x+n+t+1)\, dt \end{aligned} \tag{26}$$

Now using the relation (2), we have

$$\Gamma_q(x+n+1) = [x+n]_q [x+n-1]_q \ldots [x+1]_q [x]_q \Gamma_q(x)$$

then

$$\frac{\Gamma_q'(x)}{\Gamma_q(x)} = \sum_{i=0}^{n} \frac{q^{x+i} \log q}{1-q^{x+i}} + \frac{\Gamma_q'(x+n+1)}{\Gamma_q(x+n+1)}.$$

Using the relation (23), we get

$$\frac{\Gamma_q'(x)}{\Gamma_q(x)} = -\gamma_{q,n} - \log[n]_q + \frac{q^x \log q}{1-q^x} + \log q \sum_{i=1}^{\infty} \left[ \frac{q^{i+x}}{1-q^{i+x}} - \frac{q^i}{1-q^i} \right] + \frac{\Gamma_q'(x+n+1)}{\Gamma_q(x+n+1)}. \tag{27}$$

The relations (26) and (27) give us the following formula

$$\begin{aligned} \frac{\Gamma_q'(x)}{\Gamma_q(x)} = & -\gamma_{q,n} - \log[n]_q + \frac{q^x \log q}{1-q^x} + \log q \sum_{i=1}^{\infty} \left[ \frac{q^{i+x}}{1-q^{i+x}} - \frac{q^i}{1-q^i} \right] + \frac{q^{x+n+1} \log q}{2(1-q^{x+n+1})} \\ & + \log[x+n+1]_q + \int_0^1 (t-1/2) \frac{d^2}{dt^2} \log \Gamma_q(x+n+t+1)\, dt. \end{aligned} \tag{28}$$

As $n \to \infty$, we obtain

$$\begin{aligned} \frac{\Gamma_q'(x)}{\Gamma_q(x)} = & -\gamma_q + \frac{q^x \log q}{1-q^x} + \log q \sum_{i=1}^{\infty} \left[ \frac{q^{i+x}}{1-q^{i+x}} - \frac{q^i}{1-q^i} \right] \\ & + \lim_{n\to\infty} \int_0^1 (t-1/2) \frac{d^2}{dt^2} \log \Gamma_q(x+n+t+1)\, dt. \end{aligned} \tag{29}$$

But for $0 \leqslant t \leqslant 1$ and using $\frac{d^2}{dt^2}\log\Gamma_q(x+n+t+1) \geqslant 0$ , we get

$$\begin{aligned}\left|\int_0^1 (t-1/2)\frac{d^2}{dt^2}\log\Gamma_q(x+n+t+1)\,dt\right| &\leqslant \frac{1}{2}\int_0^1 \frac{d^2}{dt^2}\log\Gamma_q(x+n+t+1)\,dt\\ &\leqslant \frac{1}{2}\frac{d}{dx}\int_0^1 \frac{d}{dt}\log\Gamma_q(x+n+t+1)\,dt\\ &\leqslant \frac{q^{x+n+1}\log q}{2(q-1)} \to 0 \ \ as \ \ n\to\infty.\end{aligned}$$

Then

$$\frac{\Gamma_q'(x)}{\Gamma_q(x)} = -\gamma_q + \frac{q^x\log q}{1-q^x} + \log q\sum_{i=1}^{\infty}\left[\frac{q^{i+x}}{1-q^{i+x}} - \frac{q^i}{1-q^i}\right]. \quad \square$$

COROLLARY 1. *For* $0<q<1$*,*

$$\Gamma_q'(1) = -\gamma_q. \tag{30}$$

COROLLARY 2. *For* $x>0$ *and* $0<q<1$*,*

$$\frac{d^2}{dx^2}\log\Gamma_q(x) = (\log q)^2\sum_{i=0}^{\infty}\frac{q^{x+i}}{(1-q^{x+i})^2}. \tag{31}$$

When $q\to 1$, we obtain the following relations for the ordinary gamma function:

$$\frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{i=1}^{\infty}\left[\frac{1}{i} - \frac{1}{x+i}\right],$$

$$\Gamma'(1) = -\gamma$$

and

$$\frac{d^2}{dx^2}\log\Gamma(x) = \sum_{i=0}^{\infty}\frac{1}{(x+i)^2}.$$

THEOREM 6. *For* $x>0$ *and* $0<q<1$*,*

$$\frac{q^x(1+q^x)(\log q)^3}{384(1-q^x)^3} < R_q(x) + \frac{q^x\log q}{12[x]_q} < 0 \tag{32}$$

*Proof.* Integrating parts of (18), we get

$$\begin{aligned} R_q(x) &= \frac{1}{2}\int_0^1 (t-t^2)\frac{d^2}{dt^2}\log\Gamma_q(x+t)\,dt \\ &= \frac{1}{12}\int_0^1 \frac{d^2}{dt^2}\log\Gamma_q(x+t)\,dt + \int_0^1\left(\frac{-1}{12}+\frac{t}{2}-\frac{t^2}{2}\right)\frac{d^2}{dt^2}\log\Gamma_q(x+t)\,dt \\ &= \frac{1}{12}\frac{d^2}{dx^2}\int_0^1 \log\Gamma_q(x+t)\,dt + \int_0^1\left(\frac{t^3}{6}-\frac{t^2}{4}+\frac{t}{12}\right)\frac{d^3}{dt^3}\log\Gamma_q(x+t)\,dt \\ &= \frac{1}{12}\frac{d^2}{dx^2}G_q(x) - \frac{1}{24}\int_0^1 t^2(1-t)^2\frac{d^4}{dt^4}\log\Gamma_q(x+t)\,dt \\ &= \frac{-q^x\log q}{12[x]_q} - \frac{1}{24}\int_0^1 t^2(1-t)^2\frac{d^4}{dt^4}\log\Gamma_q(x+t)\,dt. \end{aligned}$$

But $R_q(x) > 0$ and $\frac{d^4}{dt^4}\log\Gamma_q(x+t) = (\log q)^4\sum_{i=0}^{\infty}\frac{q^{x+t+i}\left[1+4q^{x+t+i}+q^{2(x+t+i)}\right]}{(1-q^{x+i})^4}$, then

$$R_q(x) < \frac{-q^x\log q}{12[x]_q}. \tag{33}$$

Also, for $0 \leqslant t \leqslant 1$, $t^2(1-t)^2 \leqslant (\frac{1}{2})^2(1-\frac{1}{2})^2 = \frac{1}{16}$ and

$$\begin{aligned} \int_0^1 t^2(1-t)^2\frac{d^4}{dt^4}\log\Gamma(x+t)\,dt &< \frac{1}{16}\int_0^1 \frac{d^4}{dt^4}\log\Gamma(x+t)\,dt \\ &< \frac{1}{16}\frac{d^4}{dx^4}G_q(x) \\ &< \frac{1}{16}\frac{q^x(1+q^x)(\log q)^3}{(q^x-1)^3}. \end{aligned}$$

Then

$$R_q(x) > \frac{q^x(1+q^x)(\log q)^3}{384(1-q^x)^3} - \frac{q^x\log q}{12[x]_q}. \quad \square \tag{34}$$

COROLLARY 3. *For $x > 0$ and $0 < q < 1$,*

$$\frac{(q;q)_\infty(1-q)^{1-x}}{\sqrt{1-q^x}}e^{\frac{q^x(1+q^x)(\log q)^3}{384(1-q^x)^3}-\frac{q^x\log q}{12[x]_q}-\frac{Li_2(q^x)}{\log q}} < \Gamma_q(x) < \frac{(q;q)_\infty(1-q)^{1-x}}{\sqrt{1-q^x}}e^{-\frac{q^x\log q}{12[x]_q}-\frac{Li_2(q^x)}{\log q}}. \tag{35}$$

COROLLARY 4. *For $0 < q < 1$,*

$$\lim_{x\to\infty} R_q(x) = 0. \tag{36}$$

## 3. $q$-Gauss's multiplication formula

For $x > 0$ and integers $n > 0$, consider the function

$$M_q(x) = \Gamma_{q^n}\left(\frac{x}{n}\right)\Gamma_{q^n}\left(\frac{x+1}{n}\right)\ldots\Gamma_{q^n}\left(\frac{x+n-1}{n}\right). \tag{37}$$

Then

$$\begin{aligned}
\frac{d^2}{dx^2}\log M_q(x) &= \sum_{i=0}^{n-1}\frac{d^2}{dx^2}\log\Gamma_{q^n}\left(\frac{x+i}{n}\right)\\
&= (\log q)^2\sum_{i=0}^{n-1}\sum_{k=0}^{\infty}\frac{q^{nk+x+i}}{1-q^{nk+x+i}}\\
&= (\log q)^2\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\frac{(1-q^{n+nr})q^{(r+1)(x+nk)}}{1-q^{r+1}}\\
&= (\log q)^2\sum_{r=0}^{\infty}\frac{q^{x(r+1)}}{1-q^{r+1}}\\
&= (\log q)^2\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}q^{(r+1)(x+p)}\\
&= (\log q)^2\sum_{p=0}^{\infty}\frac{q^{x+p}}{1-q^{x+p}}
\end{aligned}$$

and hence

$$\frac{d^2}{dx^2}\log M_q(x) = \frac{d^2}{dx^2}\log\Gamma_q(x). \tag{38}$$

By solving the equation (38), we obtain

$$M_q(x) = be^{cx}\Gamma_q(x), \tag{39}$$

where $b$ and $c$ depend on $n$. Using the relation (2) for $\Gamma_q(x)$, we get

$$M_q(x+1) = \left[\frac{x}{n}\right]_{q^n} M_q(x) = \frac{[x]_q}{[n]_q}M_q(x).$$

Hence

$$M_q(x) = be^{c(x+1)}[n]_q\Gamma_q(x) \tag{40}$$

and the comparison between (39) and (40) give us that

$$c = -\log[n]_q.$$

Now

$$M_q(x) = b[n]_q^{-x}\Gamma_q(x)$$

then

$$\log b = x\log[n]_q + \sum_{r=0}^{n-1} \log\Gamma_{q^n}\left(\frac{x+r}{n}\right) - \log\Gamma_q(x) \tag{41}$$

Using the relation (20), we have

$$\log\Gamma_q(x) = \frac{-1}{2}\log[x]_q + \log(q,q)_\infty + (1/2-x)\log(1-q) - \frac{1}{\log q}Li_2(q^x) + R_q(x). \tag{42}$$

Also,

$$\begin{aligned}\sum_{r=0}^{n-1}\log\Gamma_{q^n}\left(\frac{x+r}{q^n}\right) &= \frac{-1}{2}\sum_{r=0}^{n-1}\log\left[\frac{x+r}{n}\right]_{q^n} + \sum_{r=0}^{n-1}\left(\frac{1}{2}-\frac{x+r}{n}\right)\log(1-q^n)\\ &\quad + n\log(q^n,q^n)_\infty - \frac{1}{n\log q}\sum_{r=0}^{n-1}Li_2(q^{x+r}) + \sum_{r=0}^{n-1}R_{q^n}\left(\frac{x+r}{n}\right)\\ &= \frac{-1}{2}\sum_{r=0}^{n-1}\log\left[\frac{x+r}{n}\right]_{q^n} + n\log(q^n,q^n)_\infty + \left(\frac{1}{2}-x\right)\log(1-q^n)\\ &\quad - \frac{1}{n\log q}\sum_{r=0}^{n-1}Li_2(q^{x+r}) + \sum_{r=0}^{n-1}R_{q^n}\left(\frac{x+r}{n}\right)\end{aligned}$$

Then we get

$$\begin{aligned}\log b = &\frac{-1}{2}\sum_{r=0}^{n-1}\log\left[\frac{x+r}{n}\right]_{q^n} + n\log(q^n,q^n)_\infty - \frac{1}{n\log q}\sum_{r=0}^{n-1}Li_2(q^{x+r}) + \sum_{r=0}^{n-1}R_{q^n}\left(\frac{x+r}{n}\right)\\ &+\frac{1}{2}\log[x]_q - \log(q,q)_\infty + \frac{Li_2(q^x)}{\log q} - R_q(x) + \frac{1}{2}\log[n]_q\end{aligned} \tag{43}$$

When $x \to \infty$, we obtain

$$\log b = \log\left[\frac{(q^n,q^n)_\infty^n (1-q)^{\frac{n-1}{2}}[n]_q}{(q,q)_\infty}\right]. \tag{44}$$

Then we get the following result:

THEOREM 7. ($q$-Gauss's multiplication formula) *For $x>0,\ 0<q<1$ and integers $n>0$*

$$\Gamma_{q^n}\left(\frac{x}{n}\right)\Gamma_{q^n}\left(\frac{x+1}{n}\right)\ldots\Gamma_{q^n}\left(\frac{x+n-1}{n}\right) = \frac{(q^n,q^n)_\infty^n(1-q)^{\frac{n-1}{2}}[n]_q^{1-x}}{(q,q)_\infty}\Gamma_q(x). \tag{45}$$

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