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ROUGH SINGULAR INTEGRAL OPERATORS AND ITS COMMUTATORS ON GENERALIZED WEIGHTED MORREY SPACES

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Abstract. Let $\Omega \in L_q(S^{n-1})$ be a homogeneous function of degree zero with $q>1$ and have a mean value zero on S^{n-1}. In this paper, we study the boundedness of the singular integral operators with rough kernels T_Ω and their commutators $[b,T_\Omega]$ on generalized weighted Morrey spaces $M_{p,\varphi}(w)$. We find the sufficient conditions on the pair (φ_1,φ_2) with $q' \leqslant p<\infty$, $p \neq 1$ and $w \in A_{p/q'}$ or $1<p<q$ and $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the operators T_Ω from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$ for $1<p<\infty$. We find the sufficient conditions on the pair (φ_1,φ_2) with $b \in BMO(\mathbb{R}^n)$ and $q' \leqslant p<\infty$, $p \neq 1$, $w \in A_{p/q'}$ or $1<p<q$, $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the operators $[b,T_\Omega]$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1<p<\infty$. In all cases the conditions for the boundedness of the operators T_Ω, $[b,T_\Omega]$ are given in terms of Zygmund-type integral inequalities on (φ_1,φ_2) and w, which do not assume any assumption on monotonicity of $\varphi_1(x,r)$, $\varphi_2(x,r)$ in r.

1. Introduction

It is well-known that the commutator is an important integral operator and it plays a key role in harmonic analysis. In 1965, Calderon [3, 4] studied a kind of commutators, appearing in Cauchy integral problems of Lip-line. Let K be a Calderón-Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [7] states that the commutator operator $[b,K]f = K(bf) - bKf$ is bounded on $L_p(\mathbb{R}^n)$ for $1<p<\infty$. The commutator of Calderón-Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, for example, [5, 6, 8, 10, 11]).

The classical Morrey spaces were originally introduced by Morrey in [27] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [10, 11, 14, 27]. Mizuhara [26] introduced generalized Morrey spaces. Later, Guliyev [14] defined the generalized Morrey spaces $M_{p,\varphi}$ with normalized norm. Recently, Komori and Shirai [23] considered the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal

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operator, the Calderón-Zygmund operator on these spaces. Guliyev [15] gave a concept of generalized weighted Morrey space $M_{p,\varphi}(w)$ which could be viewed as extension of both generalized Morrey space $M_{p,\varphi}$ and weighted Morrey space $L^{p,\kappa}(w)$. In [15] Guliyev also studied the boundedness of the classical operators and its commutators in these spaces $M_{p,\varphi}(w)$, see also Guliyev et al. [19, 20, 22].

Watson [30] and independently by Duoandikoetxea [9] established weighted L_p boundedness for the singular integral operators with rough kernels and their commutators.

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ the unit sphere of $\mathbb{R}^n$ $(n \geqslant 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$.

Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \tag{1.1}$$

for all $t > 0$ and $x \in \mathbb{R}^n$.

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{1.2}$$

where $x' = x/|x|$ for any $x \neq 0$.

The singular integral operator with homogeneous kernel T_Ω is defined by

$$T_\Omega(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy, \tag{1.3}$$

where Ω is homogeneous of degree zero.

Suppose that T_Ω is a singular integral operator defined by (1.3). Let Ω be a homogeneous of degree zero on $\mathbb{R}^n$. Let $T_{\Omega,\varepsilon}$ is the truncated operator of T_Ω defined by

$$T_{\Omega,\varepsilon}(f)(x) = \int_{\{y\in\mathbb{R}^n : |x-y|\geqslant\varepsilon\}} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy, \quad \varepsilon > 0. \tag{1.4}$$

Then the operator of T^*_Ω defined by

$$T^*_\Omega(f)(x) = \sup_{\varepsilon>0} \left| T_{\Omega,\varepsilon}(f)(x) \right| \tag{1.5}$$

is called the maximal singular integral operator. Therefore, it will be an interesting thing to study the property of T^*_Ω. The main purpose of this paper is to show that singular integral operators with rough kernels T_Ω are bounded from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$, $1 < p < \infty$.

The commutator of the singular integral operators with rough kernels T_Ω is defined by

$$[b, T_\Omega](f)(x) = p.v. \int_{\mathbb{R}^n} [b(x) - b(y)] \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy. \tag{1.6}$$

Let $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$. The maximal operator with rough kernel M_Ω is defined by

$$M_\Omega f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |\Omega(x-y)|\,|f(y)|dy.$$

It is obvious that when $\Omega \equiv 1$, M_Ω is the Hardy-Littlewood maximal operator M. For $b \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ the commutator of the maximal operator $M_{\Omega,b}$ is defined by

$$M_{\Omega,b} f(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |b(x)-b(y)|\,|\Omega(x-y)|\,|f(y)|dy. \tag{1.7}$$

We find the sufficient conditions on the pair (φ_1,φ_2) with $b \in BMO(\mathbb{R}^n)$ and $q' \leqslant p < \infty$, $p \neq 1$, $w \in A_{p/q'}$ or $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$ which ensures the boundedness of the commutator operators $[b,T_\Omega]$ from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$ for $1<p<\infty$. Note that, in [17] was studied the boundedness of the singular integral operators with rough kernels T_Ω and its commutators $[b,T_\Omega]$ on generalized Morrey spaces $M_{p,\varphi}$.

2. Preliminaries

Next we will give the weighted boundedness of singular integral operator T_Ω with rough kernel and the corresponding maximal operator T_Ω^*. In the proof of the weighted boundedness of the Calderon-Zygmund singular integral operator, used the technique of good-λ inequality. For singular integral operators with homogeneous kernel, in the proof of their weighted boundedness essentially still use this inequality. Since the kernel of singular integral operator with rough kernel does not have any smoothness on the unit sphere, the good-λ inequality is not applicable. In order to deal with this problem, Duoandikoetxea and Rubio de Francia synthetically used the Fourier transform estimate, weighted Littlewood-Paley theory and Stein-Weiss interpolation method with change of measure, then obtained the weighted boundedness of T_Ω and T_Ω^*. In their proof, the weighted boundedness of the maximal operator T_Ω with rough kernel (for its definition, see (1.7)) is needed, while the latter itself is of great significance.

THEOREM 1. ([9]) *Suppose that Ω satisfies the conditions (1.1) and $\Omega \in L_q(S^{n-1})$, $1<q\leqslant\infty$. Then for every $q' \leqslant p < \infty$, $p \neq 1$ and $w \in A_{p/q'}$ or $1 < p \leqslant q$, $p \neq \infty$ and $w^{1-p'} \in A_{p'/q'}$, there is a constant C independent of f such that*

$$\|M_\Omega(f)\|_{L_{p,w}} \leqslant C\|f\|_{L_{p,w}}.$$

THEOREM 2. ([2]) *Suppose that Ω satisfies the conditions (1.1) and $\Omega \in L_q(S^{n-1})$, $1<q\leqslant\infty$. Let also $b \in BMO(\mathbb{R}^n)$. Then for every $q' \leqslant p < \infty$, $p \neq 1$ and $w \in A_{p/q'}$ or $1 < p \leqslant q$, $p \neq \infty$ and $w^{1-p'} \in A_{p'/q'}$, there is a constant C independent of f such that*

$$\|M_{\Omega,b}(f)\|_{L_{p,w}} \leqslant C\|f\|_{L_{p,w}}.$$

THEOREM 3. ([9, 30]) *Suppose that* Ω *satisfies the conditions* (1.1), (1.2) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$. *Then for every* $q' \leqslant p < \infty$, $p \neq 1$ *and* $w \in A_{p/q'}$ *or* $1 < p \leqslant q$, $p \neq \infty$ *and* $w^{1-p'} \in A_{p'/q'}$, *there is a constant* C *independent of* f *such that*

$$\|T_\Omega(f)\|_{L_{p,w}} \leqslant C\|f\|_{L_{p,w}}.$$

THEOREM 4. ([9, 30]) *Suppose that* Ω *satisfies the conditions* (1.1), (1.2) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$. *Let also* $b \in BMO(\mathbb{R}^n)$. *Then for every* $q' \leqslant p < \infty$, $p \neq 1$ *and* $w \in A_{p/q'}$ *or* $1 < p \leqslant q$, $p \neq \infty$ *and* $w^{1-p'} \in A_{p'/q'}$, *there is a constant* C *independent of* f *such that*

$$\big\|[b, T_\Omega](f)\big\|_{L_{p,w}} \leqslant C\|f\|_{L_{p,w}}.$$

We will use the following statements on the boundedness of the weighted Hardy operators

$$H_w g(r) := \int_r^\infty g(t)w(t)dt,\ 0 < t < \infty$$

and

$$H_w^* g(r) := \int_r^\infty \Big(1 + \ln\frac{t}{r}\Big) g(t)w(t)dt,\ 0 < t < \infty,$$

where w is a fixed function non-negative and measurable on $(0,\infty)$.

The following theorem was proved in [17, 18].

THEOREM 5. ([17, 18]) *Let* v_1, v_2 *and* w *be positive almost everywhere and measurable functions on* $(0,\infty)$. *The inequality*

$$\operatorname*{ess\,sup}_{t>0} v_2(t) H_w g(t) \leqslant C \operatorname*{ess\,sup}_{t>0} v_1(t) g(t) \tag{2.1}$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0,\infty)$ *if and only if*

$$B := \operatorname*{ess\,sup}_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ *is the best constant for* (2.1).

The following theorem was proved in [15].

THEOREM 6. ([15]) *Let* v_1, v_2 *and* w *be positive almost everywhere and measurable functions on* $(0,\infty)$. *The inequality*

$$\operatorname*{ess\,sup}_{r>0} v_2(r) H_w^* g(r) \leqslant C \operatorname*{ess\,sup}_{r>0} v_1(r) g(r) \tag{2.2}$$

holds for some $C > 0$ *for all non-negative and non-decreasing* g *on* $(0,\infty)$ *if and only if*

$$B := \sup_{r>0} v_2(r) \int_r^\infty \Big(1 + \ln\frac{t}{r}\Big) \frac{w(t)dt}{\sup_{t<s<\infty} v_1(s)} < \infty. \tag{2.3}$$

Moreover, the value $C = B$ *is the best constant for* (2.1).

REMARK 1. In (2.1)–(2.3) it is assumed that $0 \cdot \infty = 0$.

By $A \lesssim D$ we mean that $A \leqslant CD$ with some positive constant C independent of appropriate quantities. If $A \lesssim D$ and $D \lesssim A$, we write $A \approx D$ and say that A and D are equivalent.

3. Generalized weighted Morrey spaces

The classical Morrey spaces $M_{p,\lambda}$ were originally introduced by Morrey in [27] to study the local behavior of solutions to second order elliptic partial differential equations. For the properties and applications of classical Morrey spaces, we refer the readers to [12, 24].

We denote by $M_{p,\lambda} \equiv M_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $1 \leqslant p < \infty$ and $0 \leqslant \lambda \leqslant n$.

Note that $M_{p,0} = L_p(\mathbb{R}^n)$ and $M_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $M_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

We recall that a weight function w is in the Muckenhoupt class A_p [28], $1 < p < \infty$, if

$$\begin{aligned} [w]_{A_p} :&= \sup_B [w]_{A_p(B)} \\ &= \sup_B \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\right)^{p-1}, \end{aligned} \tag{3.1}$$

where the sup is taken with respect to all the balls B and $\frac{1}{p} + \frac{1}{p'} = 1$. Note that, for all balls B using Hölder's inequality, we have that

$$[w]_{A_p(B)}^{1/p} = |B|^{-1}\|w\|_{L_1(B)}^{1/p}\|w^{-1/p}\|_{L_{p'}(B)} \geqslant 1. \tag{3.2}$$

For $p = 1$, the class A_1 is defined by the condition $Mw(x) \leqslant Cw(x)$ with $[w]_{A_1} = \sup_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)}$, and for $p = \infty$ $A_\infty = \bigcup_{1\leqslant p<\infty} A_p$ and $[w]_{A_\infty} = \inf_{1\leqslant p<\infty} [w]_{A_p}$.

REMARK 2. It is known that

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}(B)}^{q'/p'} = |B|^{-1}\|w^{1-p'}\|_{L_1(B)}^{q'/p'}\|w^{q'/p}\|_{L_{(p'/q')'}(B)}.$$

Moreover, we can write $w^{1-p'} \in A_{p'/q'} \Rightarrow w^{1-p'} \in A_{p'}$ because of $w^{1-p'} \in A_{p'/q'} \subset A_{p'}$. Therefore, we get

$$\begin{aligned} w^{1-p'} \in A_{p'/q'} &\Rightarrow w^{1-p'} \in A_{p'} \\ &\Rightarrow [w^{1-p'}]_{A_{p'}(B)}^{1/p'} = |B|^{-1}\|w^{1-p'}\|_{L_1(B)}^{1/p'}\|w^{1/p}\|_{L_p(B)}. \end{aligned} \tag{3.3}$$

But the opposite is not true.

REMARK 3. Let's write $w^{1-p'} \in A_{p'/q'}$ and used the definitions A_p classes we get the following

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{\frac{q(p-1)}{p(q-1)}} = |B|^{-1} \|w^{1-p'}\|_{L_1(B)}^{\frac{q(p-1)}{p(q-1)}} \|w^{q'/p}\|_{L_{(p'/q')'}(B)}$$

$$\Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{1/p'} = |B|^{-\frac{q-1}{q}} \|w^{1-p'}\|_{L_1(B)}^{1/p'} \|w\|_{L_{\frac{q}{q-p}}(B)}^{1/p}, \tag{3.4}$$

where the following equalities are provided.

$$1-p' = -\frac{p'}{p}, \quad \frac{q'}{p} = \frac{q}{p(q-1)}, \quad \frac{q'}{p'} = \frac{q(p-1)}{p(q-1)}, \quad \left(\frac{q}{p}\right)' = \frac{q}{q-p}, \quad \left(\frac{p'}{q'}\right)' = \frac{p(q-1)}{q-p}.$$

Then from eq. (3.3) and eq. (3.4) we have

$$w^{1-p'} \in A_{p'/q'} \Rightarrow [w^{1-p'}]_{A_{p'/q'}}^{1/p'}$$

$$= |B|^{\frac{1}{q}} [w^{1-p'}]_{A_{p'}(B)}^{1/p'} \|w^{1/p}\|_{L_p(B)}^{-1} \|w\|_{L_{\frac{q}{q-p}}(B)}^{1/p}. \tag{3.5}$$

DEFINITION 1. ([14]) Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and $1 \leqslant p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} |B(x,r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where $WL_p(B(x,r))$ denotes the weak L_p-space consisting of all measurable functions f for which

$$\|f\|_{WL_p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{R}^n)} < \infty.$$

Also the spaces $L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and $WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ endowed with the natural topology are defined as the sets of all functions f such that $f\chi_B \in L_p(\mathbb{R}^n)$ and $f\chi_B \in WL_p(\mathbb{R}^n)$ for all balls $B \subset \mathbb{R}^n$, respectively.

According to this definition, we recover the space $M_{p,\lambda}$ under the choice $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},$$

$$WM_{p,\lambda} = WM_{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

We define the generalized weighed Morrey spaces as follows.

DEFINITION 2. ([15]) Let $1 \leqslant p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))},$$

where $L_{p,w}(B(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,w}(\mathbb{R}^n)} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL^{\rm loc}_{p,w}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{WL_{p,w}(B(x,r))} < \infty,$$

where $WL_{p,w}(B(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_{p,w}(\mathbb{R}^n)} = \sup_{t>0} t\left(\int_{\{y\in B(x,r):\, |f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

REMARK 4.

(1) If $w \equiv 1$, then $M_{p,\varphi}(1) = M_{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,\kappa}(v,w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0<\lambda<n$, then $M_{p,\varphi}(w) = L_{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM_{p,\varphi}(w) = WL_{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(5) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

Suppose that T_Ω represents a linear or a sublinear operator, such that that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin supp f$

$$|T_\Omega f(x)| \leqslant c_0 \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n}\, |f(y)|dy, \tag{3.6}$$

where c_0 is independent of f and x.

For a function b, suppose that the commutator operator $T_{\Omega,b}$ represents a linear or a sublinear operator, such that for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin supp f$

$$|T_{\Omega,b} f(x)| \leqslant c_0 \int_{\mathbb{R}^n} |b(x)-b(y)|\, \frac{|\Omega(x-y)|}{|x-y|^n}\, |f(y)|dy, \tag{3.7}$$

where c_0 is independent of f and x.

We point out that the condition (3.6) in the case $\Omega \equiv 1$ was first introduced by Soria and Weiss in [29]. The condition (3.6) are satisfied by many interesting operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson's maximal operator, Hardy–Littlewood maximal operator, C. Fefferman's singular multipliers, R. Fefferman's singular integrals, Ricci-Stein's oscillatory singular integrals, the Bochner–Riesz means and so on (see [25], [29] for details).

The following statement, was proved in [22], see also [15, 19].

THEOREM 7. *Let* $1 \leqslant p < \infty$, $w \in A_p$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leqslant C \varphi_2(x,r), \tag{3.8}$$

where C *does not depend on* x *and* r. *Let* $T \equiv T_1$ *be a sublinear operator satisfying condition* (3.6) *with* $\Omega \equiv 1$ *bounded on* $L_{p,w}(\mathbb{R}^n)$ *for* $p > 1$, *and bounded from* $L_{1,w}(\mathbb{R}^n)$ *to* $WL_{1,w}(\mathbb{R}^n)$. *Then the operator* T *is bounded from* $M_{p,\varphi_1}(w)$ *to* $M_{p,\varphi_2}(w)$ *for* $p > 1$ *and from* $M_{1,\varphi_1}(w)$ *to* $WM_{1,\varphi_2}(w)$.

The following statement, was proved in [19], see also [15].

THEOREM 8. *Let* $1 < p < \infty$, $w \in A_p$, $b \in BMO(\mathbb{R}^n)$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leqslant C \varphi_2(x,r), \tag{3.9}$$

where C *does not depend on* x *and* r. *Let* $T_b \equiv T_{1,b}$ *be a sublinear commutator operator satisfying condition* (3.7) *with* $\Omega \equiv 1$ *bounded on* $L_{p,w}(\mathbb{R}^n)$. *Then the operator* T_b *is bounded from* $M_{p,\varphi_1}(w)$ *to* $M_{p,\varphi_2}(w)$.

Note that, in the case $w = 1$ Theorem 7 was proved in [16] and for the operators M and K in [1].

4. Singular integral operator with rough kernels T_Ω in the spaces $M_{p,\varphi}(w)$

In the following lemma we get local estimate (see, for example, [13, 14] in the case $w = 1$ and [15] in the case $w \in A_p$) for the operator T_Ω.

LEMMA 1. *Suppose that* Ω *be satisfies the conditions* (1.1), (1.2) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$.

If $q' \leqslant p < \infty$, $p \neq 1$ *and* $w \in A_{p/q'}$, *then the inequality*

$$\|T_\Omega(f)\|_{L_{p,w}(B(x_0,r))} \lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^\infty \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

If $1 < p \leqslant q,\ p \neq \infty$ *and* $w^{1-p'} \in A_{p'/q'}$, *then the inequality*

$$\|T_\Omega(f)\|_{L_{p,w}(B(x_0,r))} \lesssim \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x_0,r))} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))} \|w\|^{-1/p}_{L_{\frac{q}{q-p}}(B(x_0,t))} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L^{\rm loc}_{p,w}(\mathbb{R}^n)$.

Proof. Let Ω be satisfies the conditions (1.1), (1.2) and $\Omega \in L_q(S^{n-1}),\ 1 < q \leqslant \infty$. Note that

$$\begin{aligned}\|\Omega(x-\cdot)\|_{L_q(B(x_0,t))} &= \left(\int_{B(x-x_0,t)} |\Omega(y)|^q dy\right)^{\frac{1}{q}} \\ &\leqslant \left(\int_{B(0,t+|x-x_0|)} |\Omega(y)|^q dy\right)^{\frac{1}{q}} \qquad (4.1)\\ &= \left(\int_0^{t+|x-x_0|} r^{n-1} dr \int_{S^{n-1}} |\Omega(y')|^q d\sigma(y')\right)^{\frac{1}{q}} \\ &= c_0\, \|\Omega\|_{L_q(S^{n-1})}\, |B(0,t+|x-x_0|)|^{\frac{1}{q}},\end{aligned}$$

where $c_0 = (nv_n)^{-1/q}$ and $v_n = |B(0,1)|$.

For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0,2r)$. We represent f as

$$f = f_1 + f_2,\ \ f_1(y) = f(y)\chi_{2B}(y),\ \ f_2(y) = f(y)\chi_{\complement_{(2B)}}(y),\ \ r > 0 \qquad (4.2)$$

and have

$$\|T_\Omega(f)\|_{L_{p,w}(B)} \leqslant \|T_\Omega(f_1)\|_{L_{p,w}(B)} + \|T_\Omega(f_2)\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n),\ T_\Omega(f_1) \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of T_Ω in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_{p/q'}$ and $q' \leqslant p < \infty,\ p \neq 1$ (see Theorem 3) it follows that

$$\begin{aligned}\|T_\Omega(f_1)\|_{L_{p,w}(B)} &\leqslant \|T_\Omega(f_1)\|_{L_{p,w}(\mathbb{R}^n)} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\approx \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|f\|_{L_{p,w}(2B)}.\end{aligned}$$

It's clear that $x \in B,\ y \in {}^{\complement}(2B)$ implies $\frac{1}{2}|x_0-y| \leqslant |x-y| \leqslant \frac{3}{2}|x_0-y|$. Then by the Minkowski inequality and conditions on Ω, we get

$$T_\Omega(f_2(x)) \lesssim \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n} dy.$$

By Fubini's theorem we have

$$\int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}dy \approx \int_{\complement_{(2B)}} |\Omega(x-y)|\,|f(y)| \int_{|x-y|}^{\infty} \frac{dt}{t^{n+1}}dy$$
$$= \int_{2r}^{\infty}\int_{2r\leqslant|x_0-y|<t} |\Omega(x-y)|\,|f(y)|dy\frac{dt}{t^{n+1}}$$
$$\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)} |\Omega(x-y)|\,|f(y)|dy\frac{dt}{t^{n+1}} .$$

By applying Hölder's inequality for $q' \leqslant p < \infty,\ p \neq 1$ and $w \in A_{p/q'}$, we get

$$\int_{\complement_{(2B)}} \frac{|\Omega(x-y)|\,|f(y)|}{|x_0-y|^n}dy \lesssim \int_{2r}^{\infty} \|\Omega(x-\cdot)\|_{L_q(B(x_0,t))}\,\|f\|_{L_{q'}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,\|w^{-q'/p}\|^{\frac{1}{q'}}_{L_{(p/q')'}(B(x_0,t))}\,|B(0,t+|x-x_0|)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,|B(x_0,t)|^{\frac{1}{q'}}\,|B(0,t)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}}$$
$$\approx \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \tag{4.3}$$

Moreover, for all $q' \leqslant p < \infty,\ p \neq 1$ the inequality

$$\|T_\Omega\,(f_2)\,\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}.$$

is valid. Thus

$$\|T_\Omega(f)\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\Big(\|f\|_{L_{p,w}(2B)} + w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}\Big).$$

On the other hand,

$$\|f\|_{L_{p,w}(2B)} \approx |B|\|f\|_{L_{p,w}(2B)} \int_{2r}^{\infty} \frac{dt}{t^{n+1}}$$
$$\lesssim |B| \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim w(B)^{\frac{1}{p}}\,\|w^{-1/p}\|_{L_{p'}(B)} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,\|w^{-1/p}\|_{L_{p'}(B(x_0,t))}\,\frac{dt}{t^{n+1}}$$
$$\lesssim [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\,w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \tag{4.4}$$

Thus

$$\begin{aligned}
&\|T_\Omega(f)\|_{L_{p,w}(B)}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}}\, w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}\\
&\lesssim w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}.
\end{aligned}$$

Let also $1<p\leqslant q,\ p\neq\infty$ and $w^{1-p'}\in A_{p'/q'}$. Since $f_1\in L_{p,w}(\mathbb{R}^n)$, $T_\Omega(f_1)\in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of T_Ω in $L_{p,w}(\mathbb{R}^n)$ for $w^{1-p'}\in A_{p'/q'}$ and $1<p<q$ (see Theorem 3) it follows that

$$\begin{aligned}
\|T_\Omega(f_1)\|_{L_{p,w}(B)} \leqslant \|T_\Omega(f_1)\|_{L_{p,w}(\mathbb{R}^n)} &\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}}\, \|f_1\|_{L_{p,w}(\mathbb{R}^n)}\\
&\approx \|\Omega\|_{L_q(S^{n-1})}\,[w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}}\, \|f\|_{L_{p,w}(2B)}.
\end{aligned}$$

If $1<p\leqslant q,\ p\neq\infty$ and $w^{1-p'}\in A_{p'/q'}$, then Minkowski theorem and Hölder inequality,

$$\begin{aligned}
\|T_\Omega(f_2)\|_{L_{p,w}(B)} &\leqslant \left(\int_B\left(\int_{2r}^{\infty}\int_{B(x_0,t)}|\Omega(x-y)||f(y)|dy\,\frac{dt}{t^{n+1}}\right)^p w(x)dx\right)^{\frac{1}{p}}\\
&\leqslant \int_{2r}^{\infty}\int_{B(x_0,t)}\|\Omega(\cdot-y)\|_{L_{p,w}(B)}|f(y)|\,dy\,\frac{dt}{t^{n+1}}\\
&\lesssim \int_{2r}^{\infty}\int_{B(x_0,t)}\|\Omega(\cdot-y)\|_{L_q(B)}\,\|w\|_{L_{(q/p)'}(B)}^{\frac{1}{p}}\,|f(y)|\,dy\,\frac{dt}{t^{n+1}}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\,\|w\|_{L_{(q/p)'}(B)}^{\frac{1}{p}}\int_{2r}^{\infty}\int_{B(x_0,t)}|B(0,r+|x_0-y|)|^{\frac{1}{q}}\,|f(y)|\,dy\,\frac{dt}{t^{n+1}}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\,\|w\|_{L_{(q/p)'}(B)}^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L_1(B(x_0,t))}\,|B(0,r+t)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\,\|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,\|w^{-p'/p}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}}\,|B(x_0,t)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\,|B|^{\frac{1}{q}}\,\|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,\|w^{1-p'}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}}\,|B(x_0,t)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}}
\end{aligned}$$

is obtained. By applying (3.3) for $\|w^{1-p'}\|_{L_1(B(x_0,t))}^{\frac{1}{p'}}$ and (3.5) for $\|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}}$ we have the following inequality

$$\begin{aligned}
&\|T_\Omega(f_2)\|_{L_{p,w}(B)}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\,[w^{1-p'}]_{A_{\frac{p'}{q'}}}^{\frac{1}{p'}}\,\|w\|_{L_{\frac{q}{q-p}}(B)}^{\frac{1}{p}}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\,\|w\|_{L_{\frac{q}{q-p}}(B(x_0,t))}^{-\frac{1}{p}}\,\frac{dt}{t}
\end{aligned}$$

is valid. Thus

$$\|T_\Omega(f)\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})}[w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{q'}}}\Big(\|f\|_{L_{p,w}(2B)}+\|w\|_{L^{\frac{1}{p}}_{\frac{q}{q-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x_0,t))}\frac{dt}{t}\Big).$$

On the other hand,

$$\|f\|_{L_{p,w}(2B)} \approx |B|\|f\|_{L_{p,w}(2B)}\int_{2r}^{\infty}\frac{dt}{t^{n+1}}$$
$$\lesssim |B|\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\frac{dt}{t^{n+1}}$$
$$= [w^{1-p'}]^{-\frac{1}{p'}}_{A_{p'}(B)}|B|^{\frac{1}{q}}\|w^{1-p'}\|^{\frac{1}{p'}}_{L_1(B)}\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,\omega}(B(x_0,t))}\frac{dt}{t^{n+1}}$$
$$\lesssim [w^{1-p'}]^{-\frac{1}{p'}}_{A_{p'}(B)}\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,\omega}(B(x_0,t))}|B(x_0,t)|^{\frac{1}{q}}\|w^{1-p'}\|^{\frac{1}{p'}}_{L_1(B(x_0,t))}\frac{dt}{t^{n+1}}$$
$$\lesssim \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,\omega}(B(x_0,t))}\|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x_0,t))}\frac{dt}{t}.$$

Thus

$$\|T_\Omega(f)\|_{L_{p,w}(B)}$$
$$\lesssim \|\Omega\|_{L_q(S^{n-1})}[w^{1-p'}]^{\frac{1}{p'}}_{A_{\frac{p'}{q'}}}\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)}\int_{2r}^{\infty}\|f\|_{L_{p,w}(B(x_0,t))}\|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x_0,t))}\frac{dt}{t}.$$

Thus we complete the proof of Lemma 1. □

THEOREM 9. *Suppose that Ω be satisfies the conditions (1.1), (1.2) and $\Omega \in L_q(S^{n-1})$, $1<q\leqslant\infty$. Let also, for $q'<p<\infty$, $w\in A_{p/q'}$ the pair (φ_1,φ_2) satisfies the condition (3.8) and for $1<p<q$, $w^{1-p'}\in A_{p'/q'}$ the pair (φ_1,φ_2) satisfies the condition*

$$\int_r^{\infty}\frac{\operatorname*{ess\,inf}_{t<\tau<\infty}\varphi_1(x,\tau)\|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,\tau))}}{\|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,t))}}\frac{dt}{t}\leqslant C\,\varphi_2(x,r)\frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}}, \tag{4.5}$$

where C does not depend on x and r.

Then the operator T_Ω is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover

$$\|T_\Omega(f)\|_{M_{p,\varphi_2}(w)}\lesssim\|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. When $q' < p < \infty$, $w \in A_{p/q'}$, by Lemma 1 and Theorem 5 with $\nu_2(r) = \varphi_2(x,r)^{-1}$, $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = w(B(x,r))^{-\frac{1}{p}} r^{-1}$ we have

$$\begin{aligned}\|T_\Omega(f)\|_{M_{p,\varphi_2}(w)} &= \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|\mu_\Omega(f)\|_{L_{p,w}(B(x,r))} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))} \\ &= \|f\|_{M_{p,\varphi_1}(w)}.\end{aligned}$$

For the case of $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$, by Lemma 1 and Theorem 5 with $\nu_2(r) = \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}$, $\nu_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))} r^{-1}$ we have

$$\begin{aligned}&\|T_\Omega(f)\|_{M_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|\mu_\Omega(f)\|_{L_{p,w}(B(x,r))} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \int_r^\infty \|f\|_{L_{p,w}(B(x_0,t))} \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x_0,t))} \frac{dt}{t} \\ &\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))} \\ &= \|f\|_{M_{p,\varphi_1}(w)}. \quad \square\end{aligned}$$

5. Commutator of singular integral operator with rough kernels $[b, T_\Omega]$ in the spaces $M_{p,\varphi}(w)$

REMARK 5. ([21])

(1) The John-Nirenberg inequality: There are constants C_1, $C_2 > 0$, such that for all $b \in BMO(\mathbb{R}^n)$ and $\beta > 0$

$$|\{x \in B : |b(x) - b_B| > \beta\}| \leqslant C_1 |B| e^{-C_2\beta/\|b\|_*}, \quad \forall B \subset \mathbb{R}^n.$$

(2) The John-Nirenberg inequality implies that

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} |b(y) - b_{B(x,r)}|^p dy \right)^{\frac{1}{p}} \tag{5.1}$$

for $1 < p < \infty$.

(3) Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that

$$\left|b_{B(x,r)} - b_{B(x,t)}\right| \leqslant C\|b\|_* \ln \frac{t}{r} \text{ for } 0 < 2r < t, \tag{5.2}$$

where C is independent of b, x, r and t.

In the following lemma we get local estimate (see, for example, [15]) for the commutator operator $T_{\Omega,b}$.

LEMMA 2. *Suppose that* Ω *be satisfies the conditions* (1.1), (1.2) *and* $\Omega \in L_q(S^{n-1})$, $1 < q \leqslant \infty$.

If $q' < p < \infty$ *and* $w \in A_{p/q'}$, *then the inequality*

$$\|T_{\Omega,b}(f)\|_{L_{p,w}(B(x_0,r))}$$
$$\lesssim \|b\|_* w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L^{\text{loc}}_{p,w}(\mathbb{R}^n)$.

If $1 < p < q$ *and* $w^{1-p'} \in A_{p'/q'}$, *then the inequality*

$$\|T_{\Omega,b}(f)\|_{L_{p,w}(B(x_0,r))}$$
$$\lesssim \|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x_0,r))} \int_{2r}^{\infty} \left(1 + \ln \frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \|w\|^{-1/p}_{L_{\frac{q}{q-p}}(B(x_0,t))} \frac{dt}{t}$$

holds for any ball $B(x_0,r)$, *and for all* $f \in L^{\text{loc}}_{p,w}(\mathbb{R}^n)$.

Proof. Let $p \in (1,\infty)$ and $b \in BMO(\mathbb{R}^n)$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2B = B(x_0,2r)$. We represent f as (4.2) and have

$$\|T_{\Omega,b}(f)\|_{L_{p,w}(B)} \leqslant \|T_{\Omega,b}(f_1)\|_{L_{p,w}(B)} + \|T_{\Omega,b}(f_2)\|_{L_{p,w}(B)}.$$

Since $f_1 \in L_{p,w}(\mathbb{R}^n)$, $T_{\Omega,b}(f_1) \in L_{p,w}(\mathbb{R}^n)$ and from the boundedness of $T_{\Omega,b}$ in $L_{p,w}(\mathbb{R}^n)$ for $w \in A_{p/q'}$ and $q' < p < \infty$ (see Theorem 4) it follows that

$$\begin{aligned}\|T_{\Omega,b}(f_1)\|_{L_{p,w}(B)} &\leqslant \|T_{\Omega,b}(f_1)\|_{L_{p,w}(\mathbb{R}^n)} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|b\|_* \|f_1\|_{L_{p,w}(\mathbb{R}^n)} \\ &\approx \|\Omega\|_{L_q(S^{n-1})} [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|b\|_* \|f\|_{L_{p,w}(2B)}.\end{aligned}$$

For $x \in B$ we have

$$T_{\Omega,b}(f_2(x)) \lesssim \int_{\complement_{(2B)}} |b(y) - b(x)||\Omega(x-y)| \frac{|f(y)|}{|x_0 - y|^n} dy.$$

Then

$$
\begin{aligned}
&\|T_{\Omega,b}(f_2)\|_{L_{p,w}(B)}\\
&\lesssim \left(\int_B\left(\int_{\complement_{(2B)}}|b(y)-b(x)||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p w(x)dx\right)^{\frac{1}{p}}\\
&\lesssim \left(\int_B\left(\int_{\complement_{(2B)}}|b(y)-b_{B,w}||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p w(x)dx\right)^{\frac{1}{p}}\\
&\quad+\left(\int_B\left(\int_{\complement_{(2B)}}|b(x)-b_{B,w}||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\right)^p w(x)dx\right)^{\frac{1}{p}}\\
&=I_1+I_2.
\end{aligned}
$$

Let us estimate I_1.

$$
\begin{aligned}
I_1&=w(B)^{\frac{1}{p}}\int_{\complement_{(2B)}}|b(y)-b_{B,w}||\Omega(x-y)|\frac{|f(y)|}{|x_0-y|^n}\,dy\\
&\approx w(B)^{\frac{1}{p}}\int_{\complement_{(2B)}}|b(y)-b_{B,w}||\Omega(x-y)||f(y)|\int_{|x_0-y|}^{\infty}\frac{dt}{t^{n+1}}\,dy\\
&\approx w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\int_{2r\leqslant|x_0-y|\leqslant t}|b(y)-b_{B,w}||\Omega(x-y)||f(y)|\,dy\frac{dt}{t^{n+1}}\\
&\lesssim w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\int_{B(x_0,t)}|b(y)-b_{B,w}||\Omega(x-y)||f(y)|dy\frac{dt}{t^{n+1}}.
\end{aligned}
$$

Set $m=p/q'>1$. Since $w\in A_m$, from (3.3), we know $w^{1-m'}\in A_{m'}$. Applying Hölder's inequality and by (5.2), we get

$$
\begin{aligned}
I_1&\lesssim \|b\|_* w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|\Omega(x-\cdot)\|_{L_q(B(x_0,t))}\,\||b(y)-b_{B,w}|f\|_{L_{q'}(B(x_0,t))}\,\frac{dt}{t^{n+1}}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\,w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\|b-b_{B,w}\|_{L_{m'q',w^{1-m'}}(B(x_0,t))}\,\|f\|_{L_{p,w}(B(x_0,t))}\times\\
&\qquad\times|B(x_0,t+|x-x_0|)|^{\frac{1}{q}}\,\frac{dt}{t^{n+1}}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}\|b\|_* w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)(w^{1-m'}(B(x_0,t)))^{\frac{1}{m'q'}}\times\\
&\qquad\times\|f\|_{L_{p,w}(B(x_0,t))}\,|B(x_0,t)|\,\frac{dt}{t^{n+1}}\\
&\lesssim \|\Omega\|_{L_q(S^{n-1})}[w]_{A_{\frac{p}{q'}}}^{\frac{1}{p}}\,\|b\|_* w(B)^{\frac{1}{p}}\int_{2r}^{\infty}\left(1+\ln\frac{t}{r}\right)\|f\|_{L_{p,w}(B(x_0,t))}\,w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}.
\end{aligned}
$$

In order to estimate I_2 note that

$$I_2 = \left(\int_B |b(x)-b_{B,w}|^p w(x)dx\right)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}\,dy.$$

By (4.3) and (5.2), we get

$$\begin{aligned} I_2 &\lesssim \|b\|_* w(B)^{\frac{1}{p}} \int_{\complement_{(2B)}} \frac{|\Omega(x-y)||f(y)|}{|x_0-y|^n}\,dy \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|b\|_* w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \|f\|_{L_{p,w}(B(x,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \end{aligned}$$

Summing up I_1 and I_2, for all $p\in(1,\infty)$ we get

$$\begin{aligned} &\|T_{\Omega,b}(f_2)\|_{L_{p,w}(B)} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}[w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}} \|b\|_* w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \end{aligned}$$

Thus

$$\begin{aligned} \|T_{\Omega,b}(f)\|_{L_{p,w}(B)} \lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\, \|b\|_* \Big(\|f\|_{L_{p,w}(2B)} \\ + w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))} w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}\Big). \end{aligned}$$

On the other hand, by (4.4) we get

$$\begin{aligned} &\|T_{\Omega,b}(f)\|_{L_{p,w}(B)} \\ &\lesssim \|\Omega\|_{L_q(S^{n-1})}\, [w]^{\frac{1}{p}}_{A_{\frac{p}{q'}}}\, \|b\|_* w(B)^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t} \\ &\lesssim \|b\|_* w(B(x_0,r))^{\frac{1}{p}} \int_{2r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_{p,w}(B(x_0,t))}\, w(B(x_0,t))^{-\frac{1}{p}}\,\frac{dt}{t}. \end{aligned}$$

With similar techniques for $1<p<q$, $w^{1-p'}\in A_{p'/q'}$ can be achieved and the proof is finished. □

THEOREM 10. *Suppose that* Ω *be satisfies the conditions* (1.1), (1.2) *and* $\Omega\in L_q(S^{n-1})$, $1<q\leqslant\infty$. *Let* $b\in BMO(\mathbb{R}^n)$. *Let also, for* $q'<p<\infty$, $w\in A_{p/q'}$ *the pair* (φ_1,φ_2) *satisfies the condition* (3.9) *and for* $1<p<q$, $w^{1-p'}\in A_{p'/q'}$ *the pair* (φ_1,φ_2) *satisfies the condition*

$$\int_r^{\infty}\Big(1+\ln\frac{t}{r}\Big)\frac{\operatorname*{ess\,inf}_{t<\tau<\infty}\varphi_1(x,\tau)\|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,\tau))}}{\|w\|^{1/p}_{L_{\frac{q}{q-p}}(B(x,t))}}\,\frac{dt}{t} \leqslant C\,\varphi_2(x,r)\,\frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}}, \tag{5.3}$$

where C does not depend on x and r.

Then the operator $T_{\Omega,b}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$.

$$\|T_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Proof. When $q' < p < \infty$, $w \in A_{p/q'}$, by Lemma 2 and Theorem 6 with $v_2(r) = \varphi_2(x,r)^{-1}$, $v_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = w(B(x,r))^{-\frac{1}{p}} r^{-1}$ we have

$$\begin{aligned}
&\|T_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|\mu_{\Omega,b}(f)\|_{L_{p,w}(B(x,r))} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} \int_r^\infty \left(1+\ln\frac{t}{r}\right) \|f\|_{L_{p,w}(B(x,t))} w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))} \\
&= \|b\|_* \|f\|_{M_{p,\varphi_1}(w)}.
\end{aligned}$$

For the case of $1 < p < q$, $w^{1-p'} \in A_{p'/q'}$, by Lemma 1 and Theorem 6 with $v_2(r) = \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))}$, $v_1(r) = \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_{p,w}(B(x,r))}$ and $w(r) = \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x,r))} r^{-1}$ we have

$$\begin{aligned}
&\|T_{\Omega,b}(f)\|_{M_{p,\varphi_2}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|\mu_\Omega(f)\|_{L_{p,w}(B(x,r))} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_2(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|w\|^{\frac{1}{p}}_{L_{\frac{q}{q-p}}(B)} \\
&\quad \times \int_r^\infty \left(1+\ln\frac{t}{r}\right) \|f\|_{L_{p,w}(B(x_0,t))} \|w\|^{-\frac{1}{p}}_{L_{\frac{q}{q-p}}(B(x_0,t))} \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{R}^n, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(B(x,r))} \\
&= \|f\|_{M_{p,\varphi_1}(w)}. \quad \square
\end{aligned}$$

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