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HARDY–HILBERT'S INEQUALITY AND POWER INEQUALITIES FOR BEREZIN NUMBERS OF OPERATORS

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Abstract. We give operator analogues of some classical inequalities, including Hardy and Hardy-Hilbert type inequalities for numbers. We apply these operator forms of such inequalities for proving some power inequalities for the so-called Berezin number of self-adjoint and positive operators acting on Reproducing Kernel Hilbert Spaces (RKHSs). More precisely, we prove that

$$(\mathrm{ber}\,(f\,(A)))^2 \leqslant C\mathrm{ber}\left((f\,(A))^2\right)$$

for some constants $C > 1$. We also use reproducing kernels technique to estimate $\mathrm{dist}\,(A, \mathscr{U})$, where $\mathscr{U}$ is the set of all unitary operators on a RKHS $\mathscr{H} = \mathscr{H}(\Omega)$ over some set Ω, for some operator A on $\mathscr{H}(\Omega)$.

1. Introduction

The classical Hardy inequality asserts that

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p \tag{1}$$

$p > 1$ and for any sequence $a = (a_n)_{n\geqslant 1}$ of non-negative real numbers in ℓ_p. The inequality is sharp, in the sense that the constant $\left(\frac{p}{p-1}\right)^p$ cannot be replaced by a smaller number such that the inequality remains true for all (even finite) sequences of non-negative real numbers.

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, a_n, $b_n \geqslant 0$ satisfy $0 < \sum_{n=1}^{\infty} a_n^p < +\infty$ and $0 < \sum_{n=1}^{\infty} b_n^p < +\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\sum_{n=1}^{\infty}a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}} \tag{2}$$

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and an equivalent form is

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p<\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p\sum_{n=1}^{\infty}a_n^p, \tag{3}$$

where the constant factors $\frac{\pi}{\sin\frac{\pi}{p}}$ and $\left(\frac{\pi}{\sin\frac{\pi}{p}}\right)^p$ are the best possible. Inequality (3) is called the Hardy-Hilbert inequality (see [6]), which proves to be important in many branches of mathematics including function theory, analysis and their applications (see, for example [12]). Recently many generalizations and refinements of the mentioned inequalities have been also obtained, see [2, 11, 12].

The continuous forms (the integral forms) of these inequalities are as follows:

If $p>1$ and f is a non-negative p-integrable function on $(0,+\infty)$, then

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^{x}f(t)\,dt\right)^p dx\leqslant\left(\frac{p}{p-1}\right)^p\int_0^{\infty}f^p(x)\,dx.$$

If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $f,g\geqslant 0$ with $0<\int_0^{+\infty}f^p(x)\,dx<+\infty$ and $0<\int_0^{+\infty}g^q(x)\,dx<+\infty$, then

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)\,g(y)}{x+y}dxdy<\frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(\int_0^{\infty}f^p(x)\,dx\right)^{\frac{1}{p}}\left(\int_0^{\infty}g^q(x)\,dx\right)^{\frac{1}{q}}.$$

All of these inequalities and their valuable applications in operator theory can be found, for instance, in Hansen [4], Moslehian [13], Hansen, Krulić, Pečarić and Persson [5] and Kian [9].

Note also that a little bit earlier, Hansen [4] has established an operator version of the Hardy inequality (1) in the C^*-algebra $\mathscr{B}(H)$ of all bounded linear operators on a complex Hilbert space H, in the case when $1<p\leqslant 2$ (see, also Kian [9]). An operator version of inequality (2) was also given in [5].

In this paper, we are interested in analogous inequalities for operators acting on a Reproducing Kernel Hilbert Space $\mathscr{H}=\mathscr{H}(\Omega)$ over some set Ω. We will use these inequalities in the investigation of some power inequalities for the so-called Berezin number of an operator (for the definition see below).

Before giving our results, let us introduce some necessary definitions and notations regarding the reproducing kernel Hilbert space and its operators.

Recall that a reproducing kernel Hilbert space (shortly, RKHS) is a Hilbert space $\mathscr{H}=\mathscr{H}(\Omega)$ of complex-valued functions on a (nonempty) set Ω, which has the property that the point evaluation map $f\to f(\lambda)$ is continuous on $\mathscr{H}$ for all $\lambda\in\Omega$. Then the Riesz representation theorem guarantees that for every $\lambda\in\Omega$ there is a unique element $k_{\mathscr{H},\lambda}\in\mathscr{H}$ such that $f(\lambda)=\left\langle f,k_{\mathscr{H},\lambda}\right\rangle_{\mathscr{H}}$ for all $f\in\mathscr{H}$. The function $k_{\mathscr{H},\lambda}$

is called the reproducing kernel of $\mathscr{H}$. It is well known that (see [1] and [16, 17]) the reproducing kernel $k_{\mathscr{H},\lambda}$ of $\mathscr{H}$ is represented by

$$k_{\mathscr{H},\lambda} = \sum_{n=1}^{\infty} \overline{e_n(\lambda)} e_n(z)$$

for any orthonormal basis $(e_n)_{n\geqslant 1}$ of $\mathscr{H}$. (For example, since $\{z^n\}_{n\geqslant 0}$ is an orthonormal basis in the Hardy-Hilbert space $H^2 = H^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$, the reproducing kernel of H^2 is the function $k_{H,\lambda}(z) = \sum_{n=1}^{\infty} \overline{\lambda^n} z^n = \left(1 - \overline{\lambda} z\right)^{-1}$, $\lambda \in \mathbb{D}$.)

Let $\widehat{k}_{\mathscr{H},\lambda} := \frac{k_{\mathscr{H},\lambda}}{\|k_{\mathscr{H},\lambda}\|_{\mathscr{H}}}$ be the normalized reproducing kernel of $\mathscr{H}$. The Berezin sembol $\widetilde{A}$ of a bounded linear operator A on $\mathscr{H}$ is the following bounded function (see, for instance, Nordgren and Rosenthal [14]):

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_{\mathscr{H},\lambda}, \widehat{k}_{\mathscr{H},\lambda} \right\rangle, \ \lambda \in \Omega.$$

Often the behavior of the Berezin symbol of an operator provides important information about the operator, for example, it is known that (see, Zhu [19]) $A = 0$ if and only if $\widetilde{A}(\lambda) = 0$ for all $\lambda \in \Omega$, which shows that the Berezin symbol defines the operator uniquely.

The Berezin set and the Berezin number of an operator A are defined by (see Karaev [7, 8])

$$\operatorname{Ber}(A) = \operatorname{Range}\left(\widetilde{A}\right) = \left\{\widetilde{A}(\lambda) : \lambda \in \Omega\right\}$$

and

$$\operatorname{ber}(A) = \sup\{|\mu| : \mu \in \operatorname{Range}(A)\},$$

respectively. Clearly, $\operatorname{Ber}(A) \subset W(A) = \{\langle Ax, x\rangle : \|x\|_{\mathscr{H}} = 1\}$ (numerical range) and $\operatorname{ber}(A) \leqslant w(A) = \sup\{|\langle Ax, x\rangle| : \|x\|_{\mathscr{H}} = 1\}$ (numerical radius).

2. Hardy-Hilbert inequality and Berezin number

In this section, we prove an analog of inequality (2) for self-adjoint operators on a RKHS $\mathscr{H}(\Omega)$ and apply it for estimating Berezin number of its powers.

Recall that it is well known that (see, for example, Halmos [3])

$$w(A^n) \leqslant (w(A))^n$$

for all integers $n \geqslant 1$. However, it seems that the similar inequality

$$\operatorname{ber}(A^n) \leqslant (\operatorname{ber}(A))^n$$

$n \geqslant 2$, is not known in the literature. The second open question is the following: does there exist a constant $C = C(n) > 1$ such that the power inequality for the Berezin number

$$(\operatorname{ber}(A))^n \leqslant C(\operatorname{ber}(A^n)), \ n > 1, \tag{4}$$

holds?

Here we partially solve the latter question by proving inequality (4) for $n=2$ and for some positive operators A on $\mathscr{H}(\Omega)$. For this, we essentially use some arguments from [9].

In what follows, we will denote by J the interval contained in the positive semi-axis $(0,+\infty)$. Also, we will sometimes use the notation λ for the scalar operator λI.

LEMMA 1. *Let f,g be continuous functions defined on an interval J and $f,g \geqslant 0$. If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\frac{1}{2}\widetilde{f(A)g(A)}(\lambda)+\frac{1}{3}\widetilde{f(A)}(\lambda)\widetilde{g(B)}(\mu)+\frac{1}{3}\widetilde{f(A)}(\mu)\widetilde{g(B)}(\lambda)+\frac{1}{4}\widetilde{f(B)g(B)}(\mu)$$
$$\leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left[\left(f^p(A)+f^p(B)\right)^{1/p}\left(g^q(A)+g^q(B)\right)^{1/q}\right]^{\sim}(\lambda)$$

for any self-adjoint operators $A,B\in\mathscr{B}(\mathscr{H})$ with spectra contained in J and all $\lambda,\mu\in\Omega$.

Proof. In inequality (2), put $a_n=0$, $b_n=0$ for all $n\geqslant 3$. Since $a_n,b_n\geqslant 0$, we have from (2) that

$$\frac{a_1b_1}{2}+\frac{a_1b_2}{3}+\frac{a_2b_1}{3}+\frac{a_2b_2}{4}\leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(a_1^p+a_2^p\right)^{1/p}\left(b_1^q+b_2^q\right)^{1/q}. \tag{5}$$

Let $x,y\in J$. By considering that $f,g\geqslant 0$ and putting $a_1=f(x)$, $a_2=f(y)$, $b_1=g(x)$ and $b_2=g(y)$ in (5), we obtain

$$\frac{f(x)g(x)}{2}+\frac{f(x)g(y)}{3}+\frac{f(y)g(x)}{3}+\frac{f(y)g(y)}{4}$$
$$\leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(f^p(x)+f^p(y)\right)^{1/p}\left(g^q(x)+g^q(y)\right)^{1/q} \tag{6}$$

for all $x,y\in J$. Since a self-adjoint operator A admits functional calculus with respect to the class of continuous functions, it follows from (6) that

$$\frac{f(A)g(A)}{2}+\frac{f(A)g(y)}{3}+\frac{f(y)g(A)}{3}+\frac{f(y)g(y)}{4}$$
$$\leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left(f^p(A)+f^p(y)\right)^{1/p}\left(g^q(A)+g^q(y)\right)^{1/q},$$

whence

$$\frac{1}{2}\left\langle f(A)g(A)\hat{k}_{\mathscr{H},\lambda},\hat{k}_{\mathscr{H},\lambda}\right\rangle+\frac{1}{3}g(y)\left\langle f(A)\hat{k}_{\mathscr{H},\lambda},\hat{k}_{\mathscr{H},\lambda}\right\rangle$$
$$+\frac{1}{3}f(y)\left\langle g(A)\hat{k}_{\mathscr{H},\lambda},\hat{k}_{\mathscr{H},\lambda}\right\rangle+\frac{f(y)g(y)}{4}$$
$$\leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left\langle\left(f^p(A)+f^p(y)\right)^{1/p}\left(g^q(A)+g^q(y)\right)^{1/q}\hat{k}_{\mathscr{H},\lambda},\hat{k}_{\mathscr{H},\lambda}\right\rangle$$

for all $\lambda \in \Omega$ and $y \in J$.

Again, applying the functional calculus to the self-adjoint operator B, we have

$$\begin{aligned}&\frac{1}{2}\widetilde{(fg)(A)}(\lambda)+\frac{1}{3}\widetilde{f(A)}(\lambda)\,g(B)+\frac{1}{3}\widetilde{g(A)}(\lambda)\,f(B)+\frac{1}{4}(fg)(B)\\&\leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left[\left(f^{p}(A)+f^{p}(B)\right)^{1/p}\left(g^{q}(A)+g^{q}(B)\right)^{1/q}\right]^{\sim}(\lambda).\end{aligned}$$

This shows that

$$\begin{aligned}&\frac{1}{2}\widetilde{f(A)g(A)}(\lambda)+\frac{1}{3}\widetilde{f(A)}(\lambda)\,\widetilde{g(B)}(\mu)+\frac{1}{3}\widetilde{f(B)}(\mu)\,\widetilde{g(A)}(\lambda)\\&\quad+\frac{1}{4}\widetilde{f(B)g(B)}(\mu)\\&\quad\leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}\left[\left(f^{p}(A)+f^{p}(B)\right)^{1/p}\left(g^{q}(A)+g^{q}(B)\right)^{1/q}\right]^{\sim}(\lambda).\end{aligned}\tag{7}$$

for any self-adjoint operators $A,B\in\mathscr{H}(\Omega)$ with spectra contained in J and all $\lambda,\mu\in\Omega$. This proves the lemma. □

COROLLARY 1. $(\operatorname{ber}(f(A)))^{2}\leqslant\left(\frac{3\pi}{\sin\left(\frac{\pi}{p}\right)}-\frac{9}{8}\right)\operatorname{ber}\left(f^{2}(A)\right)$ *for any positive operator* $A\in\mathscr{B}(\mathscr{H})$ *with spectrum contained in* J.

Proof. Indeed, replacing B by A, μ by λ in Lemma 1, and using the fact that $\frac{1}{p}+\frac{1}{q}=1$, we get from (7) that

$$\frac{2}{3}\left(\widetilde{f(A)}(\lambda)\right)^{2}\leqslant\left[\frac{2\pi}{\sin(\pi/p)}-\frac{3}{4}\right]\widetilde{f^{2}(A)}(\lambda)$$

and hence

$$\left(\widetilde{f(A)}(\lambda)\right)^{2}\leqslant\left[\frac{3\pi}{\sin(\pi/p)}-\frac{9}{8}\right]\widetilde{f^{2}(A)}(\lambda)$$

for all $\lambda\in\Omega$. Since $\left(\widetilde{f(A)}(\lambda)\right)^{2}\geqslant 0$ and $\widetilde{f^{2}(A)}(\lambda)\geqslant 0$, this inequality implies that

$$\left(\sup_{\lambda\in\Omega}\widetilde{f(A)}(\lambda)\right)^{2}\leqslant\left[\frac{3\pi}{\sin(\pi/p)}-\frac{9}{8}\right]\sup_{\lambda\in\Omega}\widetilde{f^{2}(A)}(\lambda),$$

or equivalently,

$$(\operatorname{ber}(f(A)))^{2}\leqslant\left[\frac{3\pi}{\sin(\pi/p)}-\frac{9}{8}\right]\operatorname{ber}\left(f^{2}(A)\right)$$

for any positive operator A on $\mathscr{H}(\Omega)$ with spectrum $\sigma(A)\subset J$. This proves the corollary. □

Note that, in general, $\widetilde{AB}\neq\widetilde{A}\widetilde{B}$; for more details, see Kılıç [10].

COROLLARY 2. $(\operatorname{ber}(f(A)))^2 \leqslant \left(3\pi - \frac{9}{8}\right) \operatorname{ber}\left((f(A))^2\right)$ *for any positive operator A with $\sigma(A) \subset J$.*

Now by considering an equivalent form

$$\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\frac{a_m}{m+n}\right)^p < \left(\frac{\pi}{\sin(\pi/p)}\right)^p \sum_{n=1}^{\infty} a_n^p$$

of the Hardy-Hilbert inequality (2), we prove an inverse inequality for the Berezin number of positive operators. The main result of the paper is the inequality given in the next theorem, which gives a sharper estimate than Corollary 2.

THEOREM 1. *Let f be a non-negative continuous function defined on an interval J. Then*

$$(\operatorname{ber}(f(A)))^2 \leqslant \frac{16\pi^2 - 77}{72} \operatorname{ber}\left(f^2(A)\right)$$

for any positive operator $A \in \mathscr{B}(\mathscr{H}(\Omega))$ with spectrum contained in J.

Proof. For $p = 2$, we have from (3) that

$$\left(\frac{a_1}{2} + \frac{a_2}{3}\right)^2 + \left(\frac{a_1}{3} + \frac{a_2}{4}\right)^2 < \pi^2\left(a_1^2 + a_2^2\right). \tag{7}$$

Let $x, y \in J$. Since f is a continuous positive function on J, by putting $a_1 = f(x)$ and $a_2 = f(y)$ in (7), we get

$$\left(\frac{f(x)}{2} + \frac{f(y)}{3}\right)^2 + \left(\frac{f(x)}{3} + \frac{f(y)}{4}\right)^2 < \pi^2\left(f^2(x) + f^2(y)\right),$$

and hence

$$\frac{13}{36} f^2(x) + \frac{1}{2} f(x) f(y) + \frac{25}{144} f^2(y) < \pi^2\left(f^2(x) + f^2(y)\right). \tag{8}$$

So, as in the proof of Lemma 1, introducing the functional calculus in equality (8), we obtain

$$\frac{13}{36} f^2(A) + \frac{1}{2} f(y) f(A) + \frac{25}{144} f^2(y) < \pi^2\left(f^2(A) + f^2(y)\right),$$

and therefore

$$\begin{aligned}&\frac{13}{36}\left\langle f^2(A)\hat{k}_{\mathscr{H},\lambda}, \hat{k}_{\mathscr{H},\lambda}\right\rangle + \frac{1}{2} f(y)\left\langle f(A)\hat{k}_{\mathscr{H},\lambda}, \hat{k}_{\mathscr{H},\lambda}\right\rangle + \frac{25}{144} f^2(y)\\ &< \pi^2\left(\left\langle f^2(A)\hat{k}_{\mathscr{H},\lambda}, \hat{k}_{\mathscr{H},\lambda}\right\rangle + f^2(y)\right)\end{aligned}$$

for all $\lambda \in \Omega$. Applying the functional calculus once more to A, we get from the latter equality that

$$\frac{13}{36}\widetilde{f^2(A)}(\lambda) + \frac{1}{2}\widetilde{f(A)}(\lambda) f(A) + \frac{25}{144} f^2(A) < \pi^2\left(\widetilde{f^2(A)}(\lambda) + f^2(A)\right).$$

From this we have

$$\frac{13}{36}\widetilde{f^2(A)}(\lambda)+\frac{1}{2}\left(\widetilde{f(A)}(\lambda)\right)^2+\frac{25}{144}\widetilde{f^2(A)}(\lambda)<2\pi^2\widetilde{f^2(A)}(\lambda)$$

for all $\lambda \in \Omega$, and hence

$$\left(\widetilde{f(A)}(\lambda)\right)^2<\frac{16\pi^2-77}{72}\left(\widetilde{f(A)^2}(\lambda)\right)$$

for all $\lambda \in \Omega$. This implies that

$$(\operatorname{ber}(f(A)))^2<\frac{16\pi^2-77}{72}\operatorname{ber}\left((f(A))^2\right)\ \left(\approx 1.121\operatorname{ber}\left(f^2(A)\right)\right).$$

This proves the theorem. □

3. A dist-estimate for the set of unitary operators

It is well known that (see Stampfli [18]), unitary operators on a Hilbert space H can be characterized by invertible contractions with contractive inverses, i.e., in terms of operators A with $\|A\|\leqslant 1$ and $\left\|A^{-1}\right\|\leqslant 1$.

In general, if $A\in\mathscr{B}(H)$ is invertible then (see Rogers [15])

$$\operatorname{dist}(A,\mathscr{U})=\max\left\{\|A\|-1,1-\left\|A^{-1}\right\|^{-1}\right\},$$

where $\mathscr{U}:=\{U\in\mathscr{B}(H):U \text{ unitary}\}$ is the set of all unitary operators on H. Also, it follows from the proof of this formula that if $A\in\mathscr{B}(H)$ is an invertible operator satisfying $\|A\|\leqslant r$ and $\left\|A^{-1}\right\|\leqslant r$ for some $r\geqslant 1$, then there exists a unitary operator $U\in\mathscr{B}(H)$ such that

$$\|A-U\|\leqslant r-1. \tag{9}$$

Here, in case of $H=\mathscr{H}(\Omega)$, we give an estimate for $\operatorname{dist}(A,\mathscr{U})$ which is similar to (9).

THEOREM 2. *Let $r\geqslant 1$ and $A:\mathscr{H}(\Omega)\rightarrow\mathscr{H}(\Omega)$ be an operator such that there exists an operator $X:=X_A$ satisfying*

1) $XA=AX=rI$ (or equivalently, $X\left(\frac{1}{r}A\right)=\left(\frac{1}{r}A\right)X=I$)

and

2)

$$\sup_{\lambda\in\Omega}\left(\left\|A^*\widehat{k}_{\mathscr{H},\lambda}\right\|^2+\left\|X\widehat{k}_{\mathscr{H},\lambda}\right\|^2\right)\leqslant 2r. \tag{10}$$

Then $\operatorname{dist}(A,\mathscr{U})\leqslant\sqrt{r}-1$.

Proof. Since $AX = XA = rI$, we have for all $\lambda \in \Omega$ that:

$$\begin{aligned}
\left\| (A^* - X)\widehat{k}_{\mathscr{H},\lambda} \right\|^2 &= \left\langle (A^* - X)\widehat{k}_{\mathscr{H},\lambda}, (A^* - X)\widehat{k}_{\mathscr{H},\lambda} \right\rangle \\
&= \left\| A^*\widehat{k}_{\mathscr{H},\lambda} \right\|^2 + \left\| X\widehat{k}_{\mathscr{H},\lambda} \right\|^2 - \left\langle A^*\widehat{k}_{\mathscr{H},\lambda}, X\widehat{k}_{\mathscr{H},\lambda} \right\rangle \\
&\quad - \left\langle X\widehat{k}_{\mathscr{H},\lambda}, A^*\widehat{k}_{\mathscr{H},\lambda} \right\rangle \\
&= \left\| A^*\widehat{k}_{\mathscr{H},\lambda} \right\|^2 + \left\| X\widehat{k}_{\mathscr{H},\lambda} \right\|^2 - \left\langle \widehat{k}_{\mathscr{H},\lambda}, AX\widehat{k}_{\mathscr{H},\lambda} \right\rangle \\
&\quad - \left\langle AX\widehat{k}_{\mathscr{H},\lambda}, \widehat{k}_{\mathscr{H},\lambda} \right\rangle \\
&= \left\| A^*\widehat{k}_{\mathscr{H},\lambda} \right\|^2 + \left\| X\widehat{k}_{\mathscr{H},\lambda} \right\|^2 - 2r \leqslant 0 \text{ (by condition } (10)),
\end{aligned}$$

which implies that $(A^* - X)\frac{k_{\mathscr{H},\lambda}}{\|k_{\mathscr{H},\lambda}\|} = 0$; that is $(A^* - X)k_{\mathscr{H},\lambda} = 0$ for all $\lambda \in \Omega$ (because $\|k_{\mathscr{H},\lambda}\| \neq 0$ for all $\lambda \in \Omega$), and since $span\{k_{\mathscr{H},\lambda} : \lambda \in \Omega\} = \mathscr{H}(\Omega)$, we conclude that $X = A^*$. Therefore, $AA^* = A^*A = rI$, and hence $\left(\frac{A}{\sqrt{r}}\right)^* \frac{A}{\sqrt{r}} = \frac{A}{\sqrt{r}} \left(\frac{A}{\sqrt{r}}\right)^* = I$, which implies that $\frac{A}{\sqrt{r}}$ is unitary. Then we obtain

$$\left\| A - \frac{A}{\sqrt{r}} \right\| = \left(\sqrt{r} - 1\right) \left\| \frac{A}{\sqrt{r}} \right\| = \sqrt{r} - 1,$$

which implies that $\operatorname{dist}(A, \mathscr{U}) \leqslant \left\| A - \frac{A}{\sqrt{r}} \right\| = \sqrt{r} - 1$, that is $\operatorname{dist}(A, \mathscr{U}) \leqslant \sqrt{r} - 1$. This proves the theorem. □

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