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SOME INEQUALITIES FOR FUNCTIONS HAVING AN s–CONVEX DERIVATIVE OF SUPERIOR ORDER

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Abstract. Some Hermite-Hadamard type inequalities via fractional integration are derived for superior order differentiable functions having one derivative with s-convexity of either first kind or second kind. The n-th order cumulative behavior of the function in the neighborhood of the frontier of the definition interval is studied in case of the s-convexity of second kind, by means of fractional integration. The inequalities are as best as possible from the sharpness point of view, meaning that a sharpness class of functions is identified, for each inequality, within the functions that have one derivative that is s-affine either of first kind or of second kind.

1. Introduction

The concept of convexity was extended in a wide range of directions during the last century (see [3]) due to its important applications in optimization (see [21]), geometry (see [8]) and pattern recognition (see [3], [20]). Generally, a convexity of a set takes into account some relationship between a class of geometric objects and the set itself (see [3]). Also, the convexity of a function takes into account a relationship between the graph of the function and some geometric objects of a given class (see [1], [18], [19]). This relation is defined, many times, by some functional inequation. One of the most general functional inequations, which generates a convexity concept, is in [12]. In the framework of a real or complex topological vector space X, and with T a nonempty set, if $\alpha,\beta,a,b : T \to \mathbb{R}$, a nonempty set $D \subset X$ is said to be (α,β)-convex if $\alpha(t)x+\beta(t)y \in D$, whenever $x,y \in D$ and $t \in T$. Considering $D \subset X$ a nonempty open (α,β)-convex set, a function $f : D \to \mathbb{R}$, which verifies the functional inequality

$$f(\alpha(t)x+\beta(t)y) \leqslant a(t)f(x)+b(t)f(y), \quad \forall x,y \in D, \ \ t \in T, \tag{1}$$

is called (α,β,a,b)-convex. The functions, which make sharp (1) are called (α,β,a,b)-affine.

The most important particular case of (α,β,a,b)-convexity is the classical convexity, defined as follows. Let $\mathbb{R}$ be the set of real numbers, $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$. Function f is a convex function whether it satisfies the inequality

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y) \tag{2}$$

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whenever $x,y \in I$ and $t \in [0,1]$. The functions that fulfill the equality in (2) are called affine functions and are polynomials of at most first degree. Let now $a,b \in I$ with $a < b$. Then the following inequality is known as the Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{3}$$

named after the first mathematicians publishing it, Ch. Hermite [10] and J. Hadamard [9]. This inequality is equivalent to the convexity condition. Every affine function makes it sharp.

In this paper we take into account two particular cases of (α,β,a,b)-convex functions, called s-convexities of first and of second kinds. An interesting convexity property for functions was introduced W. Orlicz in 1961 [15]. Let us denote, as usual, by $\mathbb{R}$ the set of all real numbers, $\mathbb{N}$ the set of natural numbers, $\mathbb{N}^* = \mathbb{N}\setminus 0$, $\mathbb{R}_+ = [0,+\infty)$ and if I is an interval then $L_1 I$ means the set of all Lebesque integrable functions on I. This convexity concept is defined as follows:

DEFINITION 1.1. ([15]) Let $f : \mathbb{R}_+ \to \mathbb{R}$ and $s \in [0,1]$. The function f is said to be Orlicz-convex provided that the inequality

$$f(\alpha x+\beta y) \leqslant \alpha^s f(x)+\beta^s f(y), \tag{4}$$

holds for all $x,y \in \mathbb{R}_+$, $\alpha,\beta \in [0,1]$, with $\alpha^s+\beta^s = 1$.

The functions fulfilling (4), called in many papers Orlicz-convex functions, are also often named s-convex of first kind [11] or s_1-convex [17]. Obviously, the domain $\mathbb{R}_+$ is stable under the linear combination used in (4) and each subinterval of this set has the same property. The functions that make (4) sharp are called Orlicz-affine or s-affine of first kind. It is proved in [12] that if $s \neq 1$ then the only Orlicz-affine functions are the constants.

REMARK 1.2. Let us suppose that $s \in [0,1]$. The concept introduced by the inequality (4) is equivalent to the following one (see [17]), which will be used in the second section of this paper. A function $f : I \to \mathbb{R}$ is said to be a s-convex function in the first kind (also called Orlicz-convex), if

$$f\left(tx+(1-t^s)^{\frac{1}{s}}y\right) \leqslant t^s f(x)+(1-t^s)f(y), \tag{5}$$

whenever $x,y \in I$, $t \in [0,1]$ and $s \in (0,1]$.

The convexity, in classical sense (2), is a particular case of Orlicz-convexity, obtained from (4) for $s = 1$. The affine functions are in this case the at most first degree polynomials.

DEFINITION 1.3. ([2]) A function $f : I \to (0,\infty)$ is said to be an s-convex function in the second sense, if

$$f(tx+(1-t)y) \leqslant t^s f(x)+(1-t)^s f(y), \quad \forall x,y \in I, t \in [0,1]. \tag{6}$$

The functions fulfilling (6), called in many papers Breckner-convex functions (or s-Breckner-convex functions), are also often named s-convex of second kind [11] or s_2-convex [17]. The functions that make (6) sharp are called Breckner-affine or s-affine of second kind. It is proved in [12] that if $s \neq 1$ then the only Breckner-affine functions are the constants.

DEFINITION 1.4. Let $f \in L_1[a,b]$. Then the Riemann-Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geqslant 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \quad x > a, \tag{7}$$

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \quad x < b, \tag{8}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}x^{\alpha-1}dx$ is the Gamma function.

The effort to prove similar inequalities to (1) accompanies every attempt to generalize the convexity concept, with more or less refined versions (see [6]), including the Orlicz-convexity and the Breckner-convexity (see [6], [13], [17] and [22]). Our aim is also to obtain this type of inequalities, using the Riemann-Liouville fractional integration and taking into account functions having a superior order derivative with some kind of s-convexity.

Section 2 of the present paper contains inequalities of Hermite-Hadamard type, derived via fractional integrals, for k-th order differentiable functions having an Orlicz-convex derivative of order k.

In section 3 we consider the case of k-th order differentiable functions having an Breckner-convex derivative of order k, introducing the cumulative frontier gap of order k of such a function and determining its upper bounds. The inequalities derived in this section refer to the cumulative behavior of the function and its derivatives of order at most k in around the extremities of the interval $[a,b]$.

2. Fractional Hermite-Hadamard type inequalities for functions having an Orlicz-convex derivative

Our coming results are Hermite-Hadamard inequalities for functions having s-convexity type of first kind via fractional integration.

The following auxiliary results will be used in the sequel.

LEMMA 2.1. *Let* $f : \mathbb{R}_+ \to \mathbb{R}$, $a,b \in \mathbb{R}_+$, $a < b$ *and* $s \in [0,1]$. *If* f *is differen-*

tiable of order $k \in \mathbb{N}^*$ and $\alpha > 0$ then

$$J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) + \sum_{j=1}^{k} \frac{(b-a)^{\alpha+j-1}}{\Gamma(\alpha+j)} \left[(-1)^j f^{(j-1)}(b) - f^{(j-1)}(a)\right]$$
$$= \frac{1}{\Gamma(\alpha+k)} \int_a^b \left[(b-x)^{\alpha+k-1} + (-1)^k (x-a)^{\alpha+k-1}\right] f^{(k)}(x)dx. \tag{9}$$

Proof. We compute the two fractional Riemann-Liouville integrals $J_{a^+}^{\alpha} f(b)$ and $J_{b^-}^{\alpha} f(a)$ by k successive integrations by parts, obtaining

$$J_{a^+}^{\alpha} f(b) = \sum_{j=1}^{k} \frac{(b-a)^{\alpha+j-1}}{\Gamma(\alpha+j)} f^{(j-1)}(a) + \frac{1}{\Gamma(\alpha+k)} \int_a^b (b-x)^{\alpha+k-1} f^{(k)}(x)dx,$$
$$J_{b^-}^{\alpha} f(a) = \sum_{j=1}^{k} \frac{(-1)^{j+1}(b-a)^{\alpha+j-1}}{\Gamma(\alpha+j)} f^{(j-1)}(b) + \frac{(-1)^k}{\Gamma(\alpha+k)} \int_a^b (x-a)^{\alpha+k-1} f^{(k)}(x)dx.$$

Computing the sum of these two fractional integrals, one gets

$$J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) = \sum_{j=1}^{k} \frac{(b-a)^{\alpha+j-1}}{\Gamma(\alpha+j)} \left[f^{(j-1)}(a) + (-1)^{j+1} f^{(j-1)}(b)\right]$$
$$+ \frac{1}{\Gamma(\alpha+k)} \int_a^b \left[(b-x)^{\alpha+k-1} + (-1)^k (x-a)^{\alpha+k-1}\right] f^{(k)}(x)dx,$$

which leads to the desired equality after minor transformation. □

We denote, for simplicity, by

$$J_k(\alpha;a,b)(f) = J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a) + \sum_{j=1}^{k} \frac{(b-a)^{\alpha+j-1}}{\Gamma(\alpha+j)} \left[(-1)^j f^{(j-1)}(b) - f^{(j-1)}(a)\right],$$

all over this paper. Also, if $\alpha > 0$ we define the functions $u_k : [a,b] \to \mathbb{R}$ by

$$u_k(x) = (b-x)^{\alpha+k-1} + (-1)^k (x-a)^{\alpha+k-1}, \tag{10}$$

and remark that, according to Lemma 2.1, one has

$$J_k(\alpha;a,b)(f) = \frac{1}{\Gamma(\alpha+k)} \int_a^b u_k(x) f^{(k)}(x)dx, \tag{11}$$

for each $k \in \mathbb{N}^*$. The boundary properties of these functions are important in the proofs of the main results from this paper and we include them in the next lemma.

LEMMA 2.2. *Let $\alpha > 0$.*
(a) *If k is even then*

$$\frac{(b-a)^{\alpha+k-1}}{2^{\alpha+k-2}} \leqslant u_k(x) \leqslant (b-a)^{\alpha+k-1}. \tag{12}$$

(b) *If k is odd then*

$$-(b-a)^{\alpha+k-1} \leqslant u_k(x) \leqslant (b-a)^{\alpha+k-1}. \tag{13}$$

Proof. Since function u_k is twice differentiable for all k then one can study its monotony and its convexity or concavity by means of the sign of its derivatives.

(a) If k is even then $u_k(a) = u_k(b) = (b-a)^{\alpha+k-1}$ and

$$\begin{aligned} u_k(x) &= (b-x)^{\alpha+k-1} + (x-a)^{\alpha+k-1}, \\ u_k'(x) &= (\alpha+k-1)\left[(x-a)^{\alpha+k-2} - (b-x)^{\alpha+k-2}\right], \\ u_k''(x) &= (\alpha+k-1)(\alpha+k-2)\left[(b-x)^{\alpha+k-3} + (x-a)^{\alpha+k-3}\right]. \end{aligned}$$

Since $\alpha > 0$ and $k \geqslant 2$, then $u_k''(x) \geqslant 0$ for all $x \in [a,b] \subset [0,+\infty)$. It means that function u is convex and the root of the first derivative is a minimum point of u_k. The first derivative vanishes in $x = \frac{a+b}{2}$ and

$$u_k\left(\frac{a+b}{2}\right) = \frac{(b-a)^{\alpha+k-1}}{2^{\alpha+k-2}},$$

which is the minimum value of this function. The maximum values are taken on the frontier of the interval $[a,b]$ and, as consequence, the desired inequality holds.

(b) If k is odd then $u_k(a) = (b-a)^{\alpha+k-1}$, $u_k(b) = -(b-a)^{\alpha+k-1}$ and

$$\begin{aligned} u_k(x) &= (b-x)^{\alpha+k-1} - (x-a)^{\alpha+k-1}, \\ u_k'(x) &= -(\alpha+k-1)\left[(x-a)^{\alpha+k-2} + (b-x)^{\alpha+k-2}\right] < 0, \end{aligned}$$

for all $x \in [a,b]$, which means that function u_k is decreasing. Now the required inequality is obvious. □

THEOREM 2.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a non-negative function, and the numbers $s \in (0,1]$ and $\alpha > 0$. Suppose that $[a,b] \subset \mathbb{R}_+$ is an interval such that there is a number $k \in \mathbb{N}^*$ with the property that the derivative $f^{(k)}$ exists, is s-convex of first kind on $[a,b]$ and $f^{(k)} \in L_1[a,b]$. Then the following inequality holds:*

$$J_k(\alpha;a,b)(f) \leqslant \frac{a(b-a)^{\alpha+k-1}}{2s\Gamma(\alpha+k)}\left[f^{(k)}(a) + f^{(k)}(b)\right]. \tag{14}$$

The inequality (14) is sharp.

Proof. In order to prove (14) we change the variable $x = t^{\frac{1}{s}}a + (1-t)^{\frac{1}{s}}b$. Then $x(0) = b$, $x(1) = a$ and $dx = \frac{1}{s}\left[t^{\frac{1-s}{s}}a - (1-t)^{\frac{1-s}{s}}b\right]dt$, which leads to

$$J_k(\alpha;a,b)(f) = \frac{1}{s\Gamma(\alpha+k)}\int_0^1 u_k\left(t^{\frac{1}{s}}a + (1-t)^{\frac{1}{s}}b\right) f^{(k)}\left(t^{\frac{1}{s}}a + (1-t)^{\frac{1}{s}}b\right) v(t)dt,$$

with $v:[0,1]\to\mathbb{R}$ defined by

$$v(t)=t^{\frac{1-s}{s}}a-(1-t)^{\frac{1-s}{s}}b.$$

Function v is differentiable on $(0,1)$ and

$$v'(t)=\frac{1-s}{s}\left[t^{\frac{1-2s}{s}}a+(1-t)^{\frac{1-2s}{s}}b\right]\geqslant 0$$

for all $t\in(0,1)$, which means that v is increasing on $[0,1]$. As consequence,

$$v(0)=-b\leqslant v(t)\leqslant v(1)=a, \tag{15}$$

for all $t\in(0,1)$. This implies that

$$\begin{aligned}&\frac{-b}{s\Gamma(\alpha+k)}\int_0^1 u_k\left(t^{\frac{1}{s}}a+(1-t)^{\frac{1}{s}}b\right)f^{(k)}\left(t^{\frac{1}{s}}a+(1-t)^{\frac{1}{s}}b\right)dt\\ &\leqslant J_k(\alpha;a,b)(f)\\ &\leqslant\frac{a}{s\Gamma(\alpha+k)}\int_0^1 u_k\left(t^{\frac{1}{s}}a+(1-t)^{\frac{1}{s}}b\right)f^{(k)}\left(t^{\frac{1}{s}}a+(1-t)^{\frac{1}{s}}b\right)dt,\end{aligned} \tag{16}$$

for all k. Now we analyze the upper bound of (11) in both cases, k even and k odd. We realize, due to Lemma 2.2, that in both cases one has $u_k(x)\leqslant(b-a)^{\alpha+k-1}$, which implies

$$J_k(\alpha;a,b)(f)\leqslant\frac{a(b-a)^{\alpha+k-1}}{s\Gamma(\alpha+k)}\int_0^1 f^{(k)}\left(t^{\frac{1}{s}}a+(1-t)^{\frac{1}{s}}b\right)dt.$$

Since f^k is Orlicz-convex, this is upper bounded by

$$J_k(\alpha;a,b)(f)\leqslant\frac{a(b-a)^{\alpha+k-1}}{s\Gamma(\alpha+k)}\int_0^1\left(tf^{(k)}(a)+(1-t)f^{(k)}(b)\right)dt,$$

and computing the integral one gets the required result.

In order to prove the sharpness of (14), we consider the particular case $k=1$. The functions that make sharp (14) are supposed to be among the differentiable functions having an s-affine of first kind first derivative. According to Corollary 3.3 from [12], $|f'|$ should be constant. Let us suppose that $|f'|\equiv 0$, which means that function f is constant, i.e. $f\equiv c$, for some $c\in\mathbb{R}$. In this case, by direct computation one obtains

$$J^{\alpha}_{a^+}f(b)=J^{\alpha}_{b^-}f(a)=\frac{c(b-a)^{\alpha}}{\Gamma(\alpha+1)}$$

and as consequence $J_k(\alpha;a,b)(f)=0$. The right side of (14) also vanishes in this case, which completely proves the sharpness. $\square$

THEOREM 2.4. *Let $f:\mathbb{R}_+\to\mathbb{R}$ be a non-negative function, and the numbers $s\in(0,1]$ and $\alpha>0$. Suppose that $[a,b]\subset\mathbb{R}_+$ is an interval such that there is a* *number $k \in \mathbb{N}^*$ with the property that the derivative $f^{(k)}$ exists, is s-convex of first kind on $[a,b]$ and $f^{(k)} \in L_1[a,b]$. Then the following inequality holds:*

(a) *If k is even then*

$$\frac{(b-a)^{\alpha+k}}{2^{\alpha+k-2}\Gamma(\alpha+k)} f^{(k)}\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leqslant J_k(\alpha;a,b)(f). \tag{17}$$

(b) *If k is odd then*

$$\frac{-a(b-a)^{\alpha+k-1}}{2^{\alpha+k-1}s\Gamma(\alpha+k)}\left[f^{(k)}(a)+f^{(k)}(b)\right] \leqslant J_k(\alpha;a,b)(f). \tag{18}$$

Both inequalities are sharp.

Proof. We begin by taking into account (11) and we analyze each case.

(a) If k is even then according to Lemma 2.2 one has

$$u_k \geqslant \frac{(b-a)^{\alpha+k-1}}{2^{\alpha+k-2}}.$$

Then, by (11) and the inequality of Theorem 181 from [6] pp. 281 (first proved in [5]),

$$f\left(\frac{a+b}{2^{\frac{1}{s}}}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx.$$

which is valid without the monotony condition from [6], one obtains

$$\begin{aligned} J_k(\alpha;a,b)(f) &\geqslant \frac{(b-a)^{\alpha+k-1}}{2^{\alpha+k-2}\Gamma(\alpha+k)}\int_a^b f^{(k)}(x)dx \\ &\geqslant \frac{(b-a)^{\alpha+k}}{2^{\alpha+k-2}\Gamma(\alpha+k)} f^{(k)}\left(\frac{a+b}{2^{\frac{1}{s}}}\right), \end{aligned}$$

which is the desired inequality.

(b) If k is odd then $u_k \geqslant -(b-a)^{\alpha+k-1}$ and

$$J_k(\alpha;a,b)(f) \geqslant \frac{-(b-a)^{\alpha+k-1}}{2^{\alpha+k-2}\Gamma(\alpha+k)}\int_a^b f^{(k)}(x)dx. \tag{19}$$

On another hand, since $f^{(k)}$ is s-convex of first kind, one can compute, substituting in the same manner as in the proof of the previous theorem and using the properties of the same function v:

$$\begin{aligned} \int_a^b f^{(k)}(x)dx &= \frac{1}{s}\int_0^1 f^{(k)}\left(t^{\frac{1}{s}}a+(1-t)^{\frac{1}{s}}b\right)v(t)dt \\ &\leqslant \frac{a}{s}\int_0^1 f^{(k)}\left(t^{\frac{1}{s}}a+(1-t)^{\frac{1}{s}}b\right)dt \\ &\leqslant \frac{a}{s}\int_0^1 \left[t^{\frac{1}{s}}f^{(k)}(a)+(1-t)^{\frac{1}{s}}f^{(k)}(b)\right]dt \\ &= \frac{a}{2s}\left[f^{(k)}(a)+f^{(k)}(b)\right]. \end{aligned}$$

Introducing this evaluation in (3) one obtains the required result. The sharpness conclusion is provided by the same example as in the proof of the previous theorem. □

3. Cumulative k-th order frontier gap of functions having an s-Breckner-convex derivative

In this section we prove our main results within the class of Breckner's s-convex functions. They are sharp inequalities for superior order differentiable function on an interval $[a,b] \subseteq I \subset \mathbb{R}$, which have at least one derivative belonging to the class of s-convex functions of second kind.

First of all let us remind that for the real or complex numbers a,b,c, other than $0,-1,-2,\ldots$, the hypergeometric series is defined by

$$ {}_2F_1[a,b,c;z] = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m}\frac{z^m}{m!}. $$

Here $(\phi)_m$ is the Pochhammer symbol, which is defined by

$$ (\phi)_m = \begin{cases} 1 & m = 0, \\ \phi(\phi+1)\ldots(\phi+m-1), & m > 0. \end{cases} $$

The integral form of the hypergeometric function is

$$ {}_2F_1(a,b;c;z) = \frac{1}{\mathrm{B}(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt $$

for $|z| < 1$, $c > b > 0$. Here

$$ \mathrm{B}(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, $$

is the beta function.

All over this section we suppose that $\alpha > 0$.

LEMMA 3.1. *Let us suppose that $k \in \mathbb{N}^*$, the function $f : I \to \mathbb{R}$ is of k-th order differentiable, $a,b \in I$, $a < b$ and $n \in \mathbb{N}^*$. Then*

$$ \begin{aligned} &\mathscr{H}(k,n,\alpha,a,b)(f) \\ &= \int_0^1 (1-t)^{\alpha+k-1} \left[f^{(k)}\left(\frac{n+t}{n+1}a + \frac{1-t}{n+1}b\right) + f^{(k)}\left(\frac{1-t}{n+1}a + \frac{n+t}{n+1}b\right)\right] dt, \end{aligned} \tag{20} $$

where

$$ \begin{aligned} &\mathscr{H}(k,n,\alpha,a,b)(f) \\ &= \left(\frac{n+1}{b-a}\right)^{k+\alpha} \Gamma(\alpha+k) \left[J^{\alpha}_{\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)^+} f(b) + (-1)^k J^{\alpha}_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-} f(a) \right] \end{aligned} $$

$$-\sum_{j=1}^{k}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k-j+1)}\left(\frac{n+1}{b-a}\right)^{j}$$
$$\times\left[f^{(k-j)}\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)+(-1)^{j}f^{(k-j)}\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\right]. \quad (21)$$

Proof. Let us compute

$$\int_0^1(1-t)^{\alpha+k-1}\left[f^{(k)}\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f^{(k)}\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]dt$$
$$=\int_0^1(1-t)^{\alpha+k-1}f^{(k)}\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt$$
$$+\int_0^1(1-t)^{\alpha+k-1}f^{(k)}\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)dt=I_1+I_2.$$

Compute I_1. Integrating by parts k times successively one obtains:

$$I_1=\sum_{j=1}^{k}\frac{(-1)^{j-1}}{\alpha+k}\left(\frac{n+1}{b-a}\right)^{j}f^{(k-j)}\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\prod_{p=0}^{j-1}(\alpha+k-p)$$
$$+\frac{(-1)^{k}}{\alpha+k}\left(\frac{n+1}{b-a}\right)^{k}\left[\prod_{p=0}^{k}(\alpha+k-p)\right]\int_0^1(1-t)^{\alpha-1}f\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt$$
$$=\sum_{j=1}^{k}\frac{(-1)^{j-1}}{\alpha+k}\left(\frac{n+1}{b-a}\right)^{j}f^{(k-j)}\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\prod_{p=0}^{j-1}(\alpha+k-p)$$
$$+(-1)^{k}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\left(\frac{n+1}{b-a}\right)^{k}\int_0^1(1-t)^{\alpha-1}f\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt$$
$$=\sum_{j=1}^{k}(-1)^{j-1}\left(\frac{n+1}{b-a}\right)^{j}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k-j+1)}f^{(k-j)}\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)$$
$$+(-1)^{k}\Gamma(\alpha+k)\left(\frac{n+1}{b-a}\right)^{\alpha+k}J^{\alpha}_{(\frac{n}{n+1}a+\frac{1}{n+1}b)-}f(a).$$

Compute I_2. Again integrating by parts k times successively one gets:

$$I_2=-\sum_{j=1}^{k}\frac{\prod_{p=0}^{j-1}(\alpha+k-p)}{\alpha+k}\left(\frac{n+1}{b-a}\right)^{j}f^{(k-j)}\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)$$
$$+\frac{\prod_{p=0}^{k}(\alpha+k-p)}{\alpha+k}\left(\frac{n+1}{b-a}\right)^{k}\int_0^1(1-t)^{\alpha-1}f\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)dt$$
$$=-\sum_{j=1}^{k}\left(\frac{n+1}{b-a}\right)^{j}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k-j+1)}f^{(k-j)}\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)$$
$$+\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}\left(\frac{n+1}{b-a}\right)^{k}\int_0^1(1-t)^{\alpha-1}f\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)dt$$

$$= -\sum_{j=1}^{k}\left(\frac{n+1}{b-a}\right)^{j}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k-j+1)}f^{(k-j)}\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)$$
$$+\Gamma(\alpha+k)\left(\frac{n+1}{b-a}\right)^{\alpha+k}J^{\alpha}_{(\frac{1}{n+1}a+\frac{n}{n+1}b)^{+}}f(b).$$

Now, by *computing the sum* of I_1 and I_2, one gets:

$$I_1+I_2=\sum_{j=1}^{k}\frac{\Gamma(\alpha+k)}{\Gamma(\alpha+k-j+1)}\left(\frac{n+1}{b-a}\right)^{j}\left[(-1)^{j-1}f^{(k-j)}\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)\right.$$
$$\left.-f^{(k-j)}\left(\frac{1}{n+1}a+\frac{n}{n+1}b\right)\right]$$
$$+\Gamma(\alpha+k)\left(\frac{n+1}{b-a}\right)^{\alpha+k}\left[J^{\alpha}_{(\frac{1}{n+1}a+\frac{n}{n+1}b)^{+}}f(b)+(-1)^{k}J^{\alpha}_{(\frac{n}{n+1}a+\frac{1}{n+1}b)^{-}}f(a)\right].$$

After conveniently arranging this sum one obtains the required result. □

REMARK 3.2. The number $\mathscr{H}(k,n,\alpha,a,b)(f)$ is called the cumulative k-th order frontier gap of function f. For $n=1$ and $k=2$ Lemma 3.1 becomes Lemma 3.1 [13]. For $k=2$ the Lemma 3.1 becomes Lemma 1.3 [14].

THEOREM 3.3. *Let us suppose that $k\in\mathbb{N}^*$ and $f:I\to\mathbb{R}$ is of k-th order differentiable, $a,b\in I$, $a<b$ and $n\in\mathbb{N}^*$. If $|f^{(k)}|$ is s-Breckner convex function, then*

$$(n+1)^{s}\left|\mathscr{H}(k,n,\alpha,a,b)(f)\right|\leqslant\left[\mathscr{E}(\alpha,k,s,t)+\frac{1}{k+s+\alpha}\right]\left[|f^{(k)}(a)|+|f^{(k)}(b)|\right], \tag{22}$$

where

$$\mathscr{E}(\alpha,k,s,t)=\int_0^1(1-t)^{\alpha+k-1}(n+t)^{s}dt=\frac{n^{s}\,{}_2F_1[1,-s,1+\alpha+k,-\frac{1}{n}]}{\alpha+k}. \tag{23}$$

The inequality (22) is sharp.

Proof. By Lemma 3.1 and the fact that $|f^{(k)}|$ is s-Breckner convex function, one has

$$|\mathscr{H}(k,n,\alpha,a,b)(f)|$$
$$=\left|\int_0^1(1-t)^{\alpha+k-1}\left[f^{(k)}\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)+f^{(k)}\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right]dt\right|$$
$$\leqslant\left|\int_0^1(1-t)^{\alpha+k-1}f^{(k)}\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)dt\right|$$
$$+\left|\int_0^1(1-t)^{\alpha+k-1}f^{(k)}\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)dt\right|$$

$$\leqslant \int_0^1 (1-t)^{\alpha+k-1}\left[\left(\frac{n+t}{n+1}\right)^s |f^{(k)}(a)| + \left(\frac{1-t}{n+1}\right)^s |f^{(k)}(b)|\right] dt$$
$$+ \int_0^1 (1-t)^{\alpha+k-1}\left[\left(\frac{1-t}{n+1}\right)^s |f^{(k)}(a)| + \left(\frac{n+t}{n+1}\right)^s |f^{(k)}(b)|\right] dt$$
$$= \frac{1}{(n+1)^s}\left[\mathscr{E}(\alpha,k,s,t) + \frac{1}{k+s+\alpha}\right]\left[|f^{(k)}(a)| + |f^{(k)}(b)|\right].$$

This completes the proof of inequality (22).

For proving the sharpness we analyze the proof of inequality (22) and realize that the main inequality used in order to get the required upper bound is (6). This is sharp for s-affine functions of second kind, that are constants if $s \neq 1$ and at most first degree polynomials whether $s=1$. Let us consider the case $s \neq 1$ and $k=2$. Let us suppose that $|f''|$ is a s-convex function of second kind. Then, according to Corollary 3.3 from [12], $|f''|$ should be constant and that implies that $f''(x) = \pm c$, with $c \in \mathbb{R}$. But f'' is a derivative, which means that it has Darboux property, transforming an interval into another interval. As consequence, $f'' \equiv c$. Let us consider, for simplicity, $f'' \equiv 0$. It implies that $f' \equiv p$ and $f(x) = px + q$ for all $x \in [a,b]$, with $p,q \in \mathbb{R}$. In this case,

$$J^{\alpha}_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^-} f(a) = \frac{1}{\Gamma(\alpha+1)}\left(\frac{b-a}{n+1}\right)^{\alpha}\left(p\frac{na+b}{n+1}+q\right) - p\left(\frac{b-a}{n+1}\right)^{\alpha+1},$$

$$J^{\alpha}_{\left(\frac{n}{n+1}a+\frac{1}{n+1}b\right)^+} f(b) = \frac{1}{\Gamma(\alpha+1)}\left(\frac{b-a}{n+1}\right)^{\alpha}\left(p\frac{a+nb}{n+1}+q\right) + p\left(\frac{b-a}{n+1}\right)^{\alpha+1},$$

which implies, by direct computation, that

$$\mathscr{H}(k,n,\alpha,a,b)(f) = 0.$$

On another hand, the right side of inequality (22) also equals to zero, which proves the sharpness of this inequality. The integral $\mathscr{E}(\alpha,k,s,t)$ is computed by Maple. □

THEOREM 3.4. *Let us suppose that $k \in \mathbb{N}^*$ and $f : I \to \mathbb{R}$ is of k-th order differentiable, $a,b \in I$, $a<b$ and $n \in \mathbb{N}^*$. If $|f^{(k)}|^q$ is s-Breckner convex function where $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, then*

$$(n+1)^{\frac{s}{q}}\,[p(\alpha+k-1)+1]^{\frac{1}{p}}\,(s+1)^{\frac{1}{q}}\,|\mathscr{H}(k,n,\alpha,a,b)(f)|$$
$$\leqslant \left(\{(n+1)^{s+1}-n^{s+1}\}|f^{(k)}(a)| + |f^{(k)}(b)|\right)^{\frac{1}{q}}$$
$$+ \left(|f^{(k)}(a)| + \{(n+1)^{s+1}-n^{s+1}\}|f^{(k)}(b)|\right)^{\frac{1}{q}}. \tag{24}$$

The inequality (24) is sharp.

Proof. By Lemma 3.1, the s-Breckner convexity of $|f^{(k)}|$ and Holder's inequality, one gets

$$
\begin{aligned}
&|\mathscr{H}(k,n,\alpha,a,b)(f)| \\
&\leqslant \left(\int_0^1 (1-t)^{p(\alpha+k-1)}dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f^{(k)}\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 (1-t)^{p(\alpha+k-1)}dt\right)^{\frac{1}{p}} \left(\int_0^1 \left|f^{(k)}\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_0^1 (1-t)^{p(\alpha+k-1)}dt\right)^{\frac{1}{p}} \left(\int_0^1 \left\{\left(\frac{n+t}{n+1}\right)^s |f^{(k)}(a)|^q + \left(\frac{1-t}{n+1}\right)^s |f^{(k)}(b)|^q\right\}dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 (1-t)^{p(\alpha+k-1)}dt\right)^{\frac{1}{p}} \left(\int_0^1 \left\{\left(\frac{1-t}{n+1}\right)^s |f^{(k)}(a)|^q + \left(\frac{n+t}{n+1}\right)^s |f^{(k)}(b)|^q\right\}dt\right)^{\frac{1}{q}} \\
&= \frac{1}{(n+1)^{\frac{s}{q}}} \left(\frac{1}{p(\alpha+k-1)+1}\right)^{\frac{1}{p}} \left(\frac{1}{s+1}\right)^{\frac{1}{q}} \\
&\quad \times \Bigg[\left(\{(n+1)^{s+1}-n^{s+1}\}|f^{(k)}(a)| + |f^{(k)}(b)|\right)^{\frac{1}{q}} \\
&\qquad + \left(|f^{(k)}(a)| + \{(n+1)^{s+1}-n^{s+1}\}|f^{(k)}(b)|\right)^{\frac{1}{q}}\Bigg].
\end{aligned}
$$

This completes the proof of inequality (24). This inequality is sharp for first degree polynomials. The proof is similar to the proof of the sharpness of (22). □

THEOREM 3.5. *Let us suppose that $k \in \mathbb{N}^*$ and $f : I \to \mathbb{R}$ is of k-th order differentiable, $a,b \in I$, $a < b$ and $n \in \mathbb{N}^*$. If $|f^{(k)}|^q$ is s-Breckner convex function where $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, then*

$$
\begin{aligned}
&(n+1)^{\frac{s}{q}}\,|\mathscr{H}(k,n,\alpha,a,b)(f)| \\
&\leqslant \left(\mathscr{E}(\alpha,k,s,q,t)|f^{(k)}(a)| + \frac{1}{q(\alpha+k-1)+s+1}|f^{(k)}(b)|\right)^{\frac{1}{q}} \\
&\quad + \left(\frac{1}{q(\alpha+k-1)+s+1}|f^{(k)}(a)| + \mathscr{E}(\alpha,k,s,q,t)|f^{(k)}(b)|\right)^{\frac{1}{q}}, \qquad (25)
\end{aligned}
$$

where

$$
\mathscr{E}(\alpha,k,s,q,t) = \int_0^1 (1-t)^{q(\alpha+k-1)}(n+t)^s dt = \frac{n^s\,{}_2F_1[1,-s,2+(1-\alpha+k)q,-\frac{1}{n}]}{1+(-1+\alpha+k)q}. \qquad (26)
$$

The inequality (25) is sharp.

Proof. Using Lemma 3.1, the fact that $|f^{(k)}|$ is a s-Breckner convex function and the Holder's inequality, we have

$$\begin{aligned}
&|\mathscr{H}(k,n,\alpha,a,b)(f)| \\
&\leqslant \left(\int_0^1 1dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{q(\alpha+k-1)}\left|f^{(k)}\left(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 1dt\right)^{\frac{1}{p}}\left(\int_0^1 (1-t)^{q(\alpha+k-1)}\left|f^{(k)}\left(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leqslant \left(\int_0^1 (1-t)^{q(\alpha+k-1)}\left\{\left(\frac{n+t}{n+1}\right)^s |f^{(k)}(a)|^q+\left(\frac{1-t}{n+1}\right)^s |f^{(k)}(b)|^q\right\}dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 (1-t)^{q(\alpha+k-1)}\left\{\left(\frac{1-t}{n+1}\right)^s |f^{(k)}(a)|^q+\left(\frac{n+t}{n+1}\right)^s |f^{(k)}(b)|^q\right\}dt\right)^{\frac{1}{q}} \\
&= \frac{1}{(n+1)^{\frac{s}{q}}}\left[\left(\mathscr{E}(\alpha,k,s,q,t)|f^{(k)}(a)|+\frac{1}{q(\alpha+k-1)+s+1}|f^{(k)}(b)|\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\frac{1}{q(\alpha+k-1)+s+1}|f^{(k)}(a)|+\mathscr{E}(\alpha,k,s,q,t)|f^{(k)}(b)|\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof of inequality (25). The sharpness is obtained for the first degree polynomials as in the proof of the sharpness of (22). $\square$

THEOREM 3.6. *Let us suppose that $k \in \mathbb{N}^*$ and $f : I \to \mathbb{R}$ is of k-th order differentiable, $a,b \in I$, $a < b$ and $n \in \mathbb{N}^*$. If $|f^{(k)}|^q$ is s-Breckner convex function where $q > 1$, then*

$$\begin{aligned}
(n+1)^{\frac{s}{q}}|\mathscr{H}(k,n,\alpha,a,b)(f)| \leqslant &\left(\mathscr{E}(\alpha,k,s,t)|f^{(k)}(a)|+\frac{1}{k+s+\alpha}|f^{(k)}(b)|\right)^{\frac{1}{q}} \\
&+ \left(\frac{1}{k+s+\alpha}|f^{(k)}(a)|+\mathscr{E}(\alpha,k,s,t)|f^{(k)}(b)|\right)^{\frac{1}{q}},
\end{aligned} \tag{27}$$

where $\mathscr{E}(\alpha,k,s,t)$ is given by (26).

The inequality (27) is sharp.

Proof. Using Lemma 3.1, the fact that $|f^{(k)}|$ is s-Breckner convex function and the power-mean inequality, we have

$$
\begin{aligned}
&|\mathscr{H}(k,n,\alpha,a,b)(f)| \\
&\leqslant \Big(\int_0^1 (1-t)^{\alpha+k-1}dt\Big)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^{\alpha+k-1}\Big|f^{(k)}\Big(\frac{n+t}{n+1}a+\frac{1-t}{n+1}b\Big)\Big|^q dt\right)^{\frac{1}{q}} \\
&\quad + \Big(\int_0^1 (1-t)^{\alpha+k-1}dt\Big)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^{\alpha+k-1}\Big|f^{(k)}\Big(\frac{1-t}{n+1}a+\frac{n+t}{n+1}b\Big)\Big|^q dt\right)^{\frac{1}{q}} \\
&\leqslant \left(\frac{1}{\alpha+k}\right)^{1-\frac{1}{q}} \left[\left(\int_0^1 (1-t)^{\alpha+k-1}\Big\{\Big(\frac{n+t}{n+1}\Big)^s |f^{(k)}(a)|^q + \Big(\frac{1-t}{n+1}\Big)^s |f^{(k)}(b)|^q\Big\}dt\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\int_0^1 (1-t)^{\alpha+k-1}\Big\{\Big(\frac{1-t}{n+1}\Big)^s |f^{(k)}(a)|^q + \Big(\frac{n+t}{n+1}\Big)^s |f^{(k)}(b)|^q\Big\}dt\right)^{\frac{1}{q}}\right] \\
&= \frac{1}{(n+1)^{\frac{s}{q}}}\left[\Big(\mathscr{E}(\alpha,k,s,t)|f^{(k)}(a)| + \frac{1}{k+s+\alpha}|f^{(k)}(b)|\Big)^{\frac{1}{q}}\right. \\
&\quad \left. + \Big(\frac{1}{k+s+\alpha}|f^{(k)}(a)| + \mathscr{E}(\alpha,k,s,t)|f^{(k)}(b)|\Big)^{\frac{1}{q}}\right].
\end{aligned}
$$

This completes the proof of inequality (27). The sharpness is obtained for the polynomials of at most first degree, as in the proof of the sharpness of (22). □

REMARK 3.7. The set of functions, which was identified as the sharpness class of each inequality may not be maximal. The identification of the maximal sharpness class remains open. To derive better inequalities means, in this case, to obtain inequalities having a larger sharpness class.

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