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MEAN VALUE BOUNDED VARIATION CONCEPT IN REAL SENSE: AN APPLICATION WITH NEW TECHNIQUES TO WEIGHTED INTEGRABILITY

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Abstract. In this paper, we consider numerical and trigonometric series with a very general monotonicity condition. A necessary and sufficient condition for the weighted integrability of sine and cosine series is proved generalizing a classical theorem of Boas and Heywood. We also remark here the inequality established in Lemma 2.7 does reflex some essential property of MVBV concept in real sense.

1. Introduction

A real sequence $A = \{a_n\}$ is said to satisfy the *mean value bounded variation condition* (in real sense) if there is a $\lambda \geqslant 2$ and a positive constant M depending upon the sequence A and λ only such that for all n we have

$$\sum_{k=n}^{2n} |\Delta a_k| := \sum_{k=n}^{2n} |a_k - a_{k+1}| \leqslant \frac{M}{n} \sum_{k=n/\lambda}^{\lambda n} |a_k|, \tag{1}$$

where $\sum\limits_{k=n/\lambda}^{\lambda n}$ means $\sum\limits_{n/\lambda \leqslant k \leqslant \lambda n}$.

We denote the set of real sequences satisfying (1) as MVBVS (Mean Value Bounded Variation Sequences).

The MVBV concept is generalized from positive sense (see [20]) to real sense in [3].

In Fourier analysis, in many important classical results which play fundamental roles in the field, positivity and monotonicity are two key conditions.

To generalize monotonicity, people spent more than 90 years. Under positive condition, monotonicity was generalized to various quasimonotone conditions and various bounded variation conditions while still keep most important classical results alive. See the papers [7]–[9], [15]–[18] and [20] for various variations, of which (1) in the positive case is the most general one. For positive sequences property (1) was first introduced

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in [20], where it was called the Mean Value Bounded Variation (MVBV) condition, and the papers [2], [12]–[13], [15]–[17], [20] show that (1) in the positive case allows one to derive necessary and sufficient conditions for various properties of trigonometric sums in terms of their coefficient sequences. In the papers [20], [17] etc., it was also shown that from this point of view condition (1) cannot be further weakened. Therefore we can say that MVBV concept in positive sense is the ultimate generalization to monotonicity.

It is ever harder to generalize or remove the positivity to some stage. This kind of work first initiated by Telyakovskii [10] (in 1993) and [11] considering the convergence of series with rarely changing coefficients (piecewisely keeping sign and constant, not necessarily to be nonnegative) and recently by Zhou etc. [19] initially using the piecewise mean value bounded variation concept.

Very recently, our work [3] surprisingly shows that in many situations the positivity assumption can be dropped. In particular, for sine series condition (1) allows us to derive necessary and sufficient conditions for uniform convergence, thereby obtaining a very general extension of the classical result of Jolliffe and Chaundy (see e.g. [21, Theorem V.1.3]).

It possibly displays that condition (1)(MVBV concept in real sense) is not only an ultimate generalization to monotonicity, but also a natural replacement of positivity.

Clearly, classical results which holds for condition (1) are no longer needed to be discussed piecewisely.

After applications in other cases such as L^1-convergence ([5]) and trigonometric inequalities ([4]) were successfully produced, it is of great interest to investigate further whether condition (1) is natural to replace positivity and monotonicity in most classical results or just occasional?

This paper shows that condition (1) can also be applied in weighted integrability case, in which the necessary and sufficient condition has the series form, different from the limit form in uniform and L^1-convergence we already solved successfully.

Let $L_{2\pi}$ be the space of integrable functions of period 2π. We will prove the following theorem:

THEOREM 1.1. *Suppose that a real sequence* $\{a_n\}$ *satisfies condition* (1), *and consider the trigonometric series*

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \sin nx$$

or

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \cos nx,$$

and its sum function is denoted by $f(x)$. *Let* $0<\gamma<1$. *Then* $x^{-\gamma} f(x) \in L_{2\pi}$ *and* $\{a_n\}$ *is the Fourier coefficients of* $f(x)$ *if and only if*

$$\sum_{n=1}^{\infty} n^{\gamma-1} |a_k| < \infty.$$

The claim for nonnegative sequences is in [14] which generalizes a classical result of Boas [1] and Heywood [6].

Note that if $\{a_n\}$ satisfies (1), then $\{|a_n|\}$ satisfies (1) too. We have the following corollary:

COROLLARY 1.2. *Suppose that a real sequence $\{a_n\}$ satisfies condition (1), and consider the trigonometric series*

$$\sum_{n=1}^{\infty} a_n \sin nx$$

or

$$\sum_{n=1}^{\infty} a_n \cos nx,$$

and its sum function is denoted by $f(x)$, also the trigonometric series

$$\sum_{n=1}^{\infty} |a_n| \sin nx$$

or

$$\sum_{n=1}^{\infty} |a_n| \cos nx,$$

and its sum function is denoted by $g(x)$. Let $0<\gamma<1$. Then $x^{-\gamma}f(x)\in L_{2\pi}$ and $\{a_n\}$ is the Fourier coefficients of $f(x)$ if and only if $x^{-\gamma}g(x)\in L_{2\pi}$ and $\{|a_n|\}$ is the Fourier coefficients of $g(x)$.

Throughout the paper, we always use M to stand for the positive constant appearing in (1), and M_1 denotes a positive constant that may not be necessarily the same at each occurrence. Sometimes, to avoid confusion, we also use $M_1, M_2, \cdots$ to denote different constants.

2. Preliminaries and proofs

If condition (1) is true for a λ then it is true for any larger λ, therefore we may assume that $\lambda > 8$ is an integer and $M>1$ in (1). For a real sequence $\{a_n\}$ set

$$b_n = \frac{1}{n}\sum_{k=n/\lambda}^{\lambda n} |a_k|.$$

LEMMA 2.1. *For all n we have*

$$|a_n| \leqslant 2Mb_n.$$

This is a fundamental inequality. See [3, Lemma 2.2], for example.

THEOREM 2.2. *Let a real sequence* $\{a_n\}$ *satisfy condition* (1), $0<\gamma<1$. *Then, for any* n,

$$\left|\sum_{k=1}^{n} a_k \sin kx\right| = O(x^{-\gamma})$$

holds if and only if

$$n^{1-\gamma} a_n = O(1). \tag{2}$$

REMARK. Theorem 2.2 also holds for cosine series.

Proof. This is an important inequality and is already mentioned and sketched in [4]. As a very useful tool, it will be applied twice in this paper. Here we will give a detailed proof for the sufficient part.

The cases $x=0$ or $x=\pi$ are trivial. Let $x\in(0,\pi)$, set $N=[1/x]$. Write

$$\sum_{k=1}^{n} a_k \sin kx = \sum_{k=1}^{N-1} a_k \sin kx + \sum_{k=N}^{n} a_k \sin kx =: I_1(x)+I_2(x).$$

For the first part, we have, by (2),

$$|I_1(x)| \leqslant \sum_{k=1}^{N-1} k^{\gamma-1}k^{1-\gamma}|a_k| = O\left(N^{\gamma}\right) = O(x^{-\gamma}).$$

On the other hand, by using Abel's transformation we see that

$$|I_2(x)| \leqslant M_1 x^{-1}\left(|a_N|+|a_n|+\sum_{k=N}^{n-1}|\Delta a_k|\right).$$

Take a natural number m such that $2^{m-1}N<n\leqslant 2^m N$, we calculate that, by condition (1) and (2),

$$\sum_{k=N}^{n-1}|\Delta a_k| \leqslant M_1\sum_{j=0}^{m}\frac{1}{2^jN}\sum_{l=2^jN/\lambda}^{\lambda 2^jN}|a_l| \leqslant M_1N^{-1}\sum_{j=0}^{m}2^{-j}\sum_{l=2^jN/\lambda}^{\lambda 2^jN} l^{\gamma-1}.$$

Thus

$$|I_2(x)| \leqslant M_1x^{-1}\left(N^{\gamma-1}+n^{\gamma-1}+N^{-1}\sum_{j=0}^{m}2^{-j}\sum_{l=2^jN/\lambda}^{\lambda 2^jN} l^{\gamma-1}\right)$$
$$= O\left(N^{\gamma}\sum_{j=0}^{\infty}2^{-j(1-\gamma)}\right) = O(x^{-\gamma}).$$

Altogether, we have proved the required result. □

Define, for convenience, that

$$T_j^{(n)} = (\lambda^{j+1}n)^{\gamma-1}\sum_{l=\lambda^jn}^{\lambda^{j+2}n}|a_l|, \quad 0<\gamma<1,\ \ j,n\in\mathbb{N},$$

where $n \in \mathbb{N}$ means that n is a natural number. It is easy to see that $T_j^{(n)} = T_k^{(\lambda^{j-k}n)}$. For any n, by taking $j = \left[\frac{\log n - \log n_0}{\log \lambda}\right] - 1$, we also easily see that

$$n^{\gamma-1} \sum_{k=n/\lambda}^{\lambda n} |a_k| \leqslant M_2(T_j^{(n_0)} + T_{j+2}^{(n_0)}),$$

so that the following lemma is straightforward.

LEMMA 2.3. *If a real sequence* $\{a_n\}$ *satisfies that* $\limsup_{j\to\infty} T_j^{(n_0)} < \infty$ *for a given fixed integer* $n_0 \in \mathbb{N}$, *then* $\limsup_{n\to\infty} n^{\gamma-1} \sum_{k=n/\lambda}^{\lambda n} |a_k| < \infty$.

LEMMA 2.4. *Let a real sequence* $\{a_n\}$ *satisfy* $\lim_{n\to\infty} a_n = 0$ *and*

$$\limsup_{n\to\infty} n^{\gamma-1} \sum_{k=n/\lambda}^{\lambda n} |a_k| = \infty,$$

then for any given $n_0 \in \mathbb{N}$, *there exists a subsequence of natural numbers* $\{j_k\}$ *such that*

$$T_{j_k}^{(n_0)} + T_{j_k+2}^{(n_0)} = O(T_{j_k+1}^{(n_0)})$$

and

$$\lim_{k\to\infty} T_{j_k+1}^{(n_0)} = \infty$$

hold.

Proof. First assume that $\{T_j^{(n_0)}\}_{j=1}^{\infty}$ has only ∞ as its accumulation point. Then

$$\lim_{j\to\infty} T_j^{(n_0)} = \infty. \tag{3}$$

Suppose to the contrary, the conclusion of Lemma 2.4 does not hold, then it holds that

$$\lim_{j\to\infty} \frac{T_j^{(n_0)} + T_{j+2}^{(n_0)}}{T_{j+1}^{(n_0)}} = \infty.$$

Hence there exists a $J \in \mathbb{N}$ such for all $j > J$ that

$$\frac{T_j^{(n_0)} + T_{j+2}^{(n_0)}}{T_{j+1}^{(n_0)}} > 2\lambda^4.$$

Note that $j > J$, $T_j^{(n_0)}$ cannot be a decreasing sequence for j, otherwise it makes a contradiction to (3). Therefore we assume now for some $j_0 > J$ it holds that $T_{j_0+1}^{(n_0)} >$

$T_{j_0}^{(n_0)}$. With $\lambda \geqslant 2$, and writing two roots of the equation $x^2-2\lambda^4 x+1=0$ as x_1 and x_2, where $0<x_1<\frac{1}{2}<\lambda^4<x_2$, by Viete's formulas, we have

$$T_{m+2}^{(n_0)}-x_1 T_{m+1}^{(n_0)} \geqslant x_2(T_{m+1}^{(n_0)}-x_1 T_m^{(n_0)}), \quad m>J,$$

in particular,

$$T_{j_0+2}^{(n_0)}-x_1 T_{j_0+1}^{(n_0)} > x_2(T_{j_0+1}^{(n_0)}-x_1 T_{j_0}^{(n_0)}) \geqslant 0$$

by the assumption. Therefore,

$$T_{j+2}^{(n_0)} > M_1\left(T_{j_0+1}^{(n_0)}-x_1 T_{j_0}^{(n_0)}\right) x_2^{j-j_0+1} > M_1 \lambda^{4j-4j_0+4}$$

for $j \geqslant j_0$, hence

$$\max_{\lambda^{j+2} n_0 \leqslant k \leqslant \lambda^{j+4} n_0} |a_k| \geqslant \frac{M_1}{\lambda^{j\gamma}} T_{j+2}^{(n_0)} > M_1 \lambda^{4j-\gamma j-4j_0+4}, \quad j \geqslant j_0.$$

However it contradicts to the condition that $\lim\limits_{n\to\infty} a_n = 0$.

Altogether, the conclusion of Lemma 2.4 holds in this case.

Next, in case $\{T_j^{(n_0)}\}_{j=1}^{\infty}$ has at least one finite accumulation point, then, there exists a number L and a subsequence of natural numbers $\{\widetilde{j}_k^{(1)}\}_{k=1}^{\infty}$ satisfying

$$\lim_{k\to\infty} T_{\widetilde{j}_k^{(1)}}^{(n_0)} = L,$$

hence $\{T_{\widetilde{j}_k^{(1)}}^{(n_0)}\}_{k=1}^{\infty}$ has an upper bound S. Applying Lemma 2.3, we see that $\limsup\limits_{j\to\infty} T_j^{(n_0)} = \infty$, and thus select a subsequence $\{\widetilde{j}_k^{(2)}\}$ such for all $k\in\mathbb{N}$ that $T_{\widetilde{j}_k^{(2)}}^{(n_0)} > S$, and

$$\lim_{k\to\infty} T_{\widetilde{j}_k^{(2)}}^{(n_0)} = \infty.$$

Set $j_1^{(1)} = \widetilde{j}_1^{(1)}$, take

$$j_i^{(2)} = \min\{\widetilde{j}_k^{(2)} > j_i^{(1)} : k\in\mathbb{N}\}, \quad i=1,2,\cdots,$$
$$j_{i+1}^{(1)} = \min\{\widetilde{j}_k^{(1)} > j_i^{(2)} : k\in\mathbb{N}\}, \quad i=1,2,\cdots,$$

and define $\widetilde{j}_{2k-1} = j_k^{(1)}$, while define $\widetilde{j}_{2k}$ as the number satisfying

$$T_{\widetilde{j}_{2k}}^{(n_0)} = \max_{\widetilde{j}_{2k-1}<j<\widetilde{j}_{2k+1}} T_j^{(n_0)}.$$

It is clear that $T_{\widetilde{j}_{2k}}^{(n_0)} \geqslant T_{j_k^{(2)}}^{(n_0)}$. Furthermore, $\lim\limits_{k\to\infty} T_{\widetilde{j}_{2k}}^{(n_0)} = \infty$ since $\lim\limits_{k\to\infty} T_{j_k^{(2)}}^{(n_0)} = \infty$. Setting $j_k = \widetilde{j}_{2k}-1$, we get

$$\lim_{k\to\infty} T_{j_k+1}^{(n_0)} = \lim_{k\to\infty} T_{\widetilde{j}_{2k}}^{(n_0)} = \infty.$$

At the same time, by noting $\widetilde{j}_{2k-1} \leqslant j_k < j_k+2 \leqslant \widetilde{j}_{2k+1}$ with $T^{(n_0)}_{\widetilde{j}^{(2)}_k} > S \geqslant T^{(n_0)}_{\widetilde{j}^{(1)}_k}$, we deduce that

$$T^{(n_0)}_{j_k} + T^{(n_0)}_{j_k+2} \leqslant 2T^{(n_0)}_{j_k+1},$$

we also prove the conclusion in this case.

Altogether, Lemma 2.4 is completed. □

LEMMA 2.5. *Under the conditions and symbols of Lemma* 2.4*, set* $n_k = \lambda^{j_k+2} n_0$*, and define*

$$A^{\alpha}_{n_k} = \left\{ l : |a_l| \geqslant \frac{1}{\alpha l} \sum_{j=n_k/\lambda}^{\lambda n_k} |a_j|,\ \ n_k/\lambda \leqslant l \leqslant \lambda n_k \right\}.$$

Then, taking a sufficiently large α_0*, there is a constant* $M_0 > 0$ *such that* $|A^{\alpha_0}_{n_k}| \geqslant M_0 n_k$*, where* $|A^{\alpha}_{n_k}|$ *indicates the number of the elements in* $A^{\alpha}_{n_k}$*.*

Proof. By Lemma 2.4, for fixed n_0, we already know that, there is a natural subsequence $\{j_k\}$ such that

$$\lim_{k\to\infty} T^{(n_0)}_{j_k+1} = \infty; \quad T^{(n_0)}_{j_k} + T^{(n_0)}_{j_k+2} = O(T^{(n_0)}_{j_k+1}).$$

By Lemma 2.1,

$$\sum_{l=n_k/\lambda}^{\lambda n_k} |a_l| \leqslant \sum_{l\in[n_k/\lambda,\lambda n_k]\setminus A^{\alpha}_{n_k}} \frac{1}{\alpha l} \sum_{j=n_k/\lambda}^{\lambda n_k} |a_j| + \sum_{l\in A^{\alpha}_{n_k}} \frac{M_1}{n_k} \sum_{j=n_k/\lambda^2}^{\lambda^2 n_k} |a_j|. \tag{4}$$

We note that,

$$\sum_{j=n_k/\lambda^2}^{\lambda^2 n_k} |a_j| = \sum_{j=\lambda^{j_k} n_0}^{\lambda^{j_k+4} n_0} |a_j| \leqslant M_1 n_k^{1-\gamma} (T^{(n_0)}_{j_k} + T^{(n_0)}_{j_k+2}) \leqslant M_1 n_k^{1-\gamma} T^{(n_0)}_{j_k+1} \leqslant M_1 \sum_{j=n_k/\lambda}^{\lambda n_k} |a_j|. \tag{5}$$

Therefore, with the above symbols, from (4) and (5) it is deduced that

$$T^{(n_0)}_{j_k+1} \leqslant \frac{M_1}{\alpha} \frac{\lambda}{n_k} \left(\lambda n_k - \frac{n_k}{\lambda} - |A^{\alpha}_{n_k}| \right) T^{(n_0)}_{j_k+1} + \frac{M_1}{n_k} |A^{\alpha}_{n_k}| T^{(n_0)}_{j_k+1},$$

i.e.,

$$|A^{\alpha}_{n_k}| \geqslant \frac{n_k}{M_1} \frac{1 - \frac{M_1 \lambda^2}{\alpha}}{1 - \frac{\lambda}{\alpha}}. \tag{6}$$

With (6), taking sufficiently large α_0, we can find a constant $M_0 > 0$ such that $|A^{\alpha_0}_{n_k}| \geqslant M_0 n_k$. □

THEOREM 2.6. *Suppose that a real sequence* $\{a_n\}$ *satisfies condition* (1)*, and consider the trigonometric series*

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \sin nx$$

or

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \cos nx,$$

and its sum function is denoted by $f(x)$*. Let* $0<\gamma<1$*. If* $x^{-\gamma}f(x)\in L_{2\pi}$ *and* $\{a_n\}$ *is the Fourier coefficients of* $f(x)$*, then*

$$\limsup_{n\to\infty} n^{\gamma-1}\sum_{k=n/\lambda}^{\lambda n}|a_k|<\infty.$$

Proof. Now the Fourier coefficients $\{a_n\}$ satisfy $\lim_{n\to\infty} a_n=0$. Suppose that

$$\limsup_{n\to\infty} n^{\gamma-1}\sum_{k=n/\lambda}^{\lambda n}|a_k|=\infty.$$

With the same symbols (e. g., $n_k=\lambda^{j_k+2}n_0$), applying Lemma 2.3 and Lemma 2.4, we have

$$\lim_{k\to\infty} T_{j_k+1}^{(n_0)}=\infty, \tag{7}$$

and there is a natural subsequence $\{n_k\}$ and a sufficiently large α_0 as well as a constant $M_0>0$ such that $|A_{n_k}^{\alpha_0}|\geqslant M_0 n_k$.

We select disjoint subsets $U_1,\ldots,U_{\kappa_k}$ of $[n_k/\lambda,\lambda n_k]$ as follows. Set $m_1^{(k)}=\min A_{n_k}^{\alpha_0}$, and select ν_1 according to the following procedure:

(i) If for $j=0,1,\cdots,j_0$, $n_k/\lambda\leqslant m_1^{(k)}+j\leqslant\lambda n_k$ the numbers $a_{m_1^{(k)}+j}$ have the same sign, and for $j=0,1,\cdots,j_0-1$, $|a_{m_1^{(k)}+j}|\geqslant\frac{T_{j_k+1}^{(n_0)}}{2\alpha_0(m_1^{(k)}+j)^\gamma}$ while $|a_{m_1^{(k)}+j_0}|<\frac{T_{j_k+1}^{(n_0)}}{2\alpha_0(m_1^{(k)}+j)^\gamma}$, then let $\nu_1=j_0$.

(ii) If case (i) is not satisfied for any j_0, then let $\nu_1=k_0$ for which $a_{m_1^{(k)}+k_0}$ is the first element with $m_1^{(k)}+k_0\in[n_k/\lambda,\lambda n_k]$ to become zero or of opposite sign than $a_{m_1^{(k)}}$.

(iii) If neither (i) and (ii) happen, then simply let $\nu_1=l_0$ for which $m_1^{(k)}+l_0$ is the first number greater than λn_k. Define now

$$U_1=\left\{m_1^{(k)},\ m_1^{(k)}+1,\cdots,m_1^{(k)}+\nu_1-1\right\}.$$

Next, set $m_2^{(k)} = \min(A_{n_k}^{\alpha_0} \setminus U_1)$ if this latter set is not empty, and using the same procedure we select ν_2 and define

$$U_2 = \left\{ m_2^{(k)},\ m_2^{(k)}+1, \cdots, m_2^{(k)}+\nu_2-1 \right\}.$$

We continue this procedure until we reach an U_{κ_k} for which $A_{n_k}^{\alpha_0} \setminus (U_1 \cup \cdots \cup U_{\kappa_k}) = \emptyset$. Next we show that the number κ_k of $\{U_i\}$ is bounded as $k \to \infty$.

Indeed, since $\sum\limits_{j=n/\lambda}^{\lambda n} |\Delta a_j| \leqslant \frac{M_1}{n} \sum\limits_{j=n/\lambda^2}^{\lambda^2 n} |a_j|$, applying Lemma 2.1 and Lemma 2.4 we get

$$\frac{M_1}{n_k^\gamma} T_{j_k+1}^{(n_0)} \geqslant \sum_{j=n_k/\lambda}^{\lambda n_k} |\Delta a_j| \geqslant \sum_{i=1}^{\kappa_k} \sum_{j\in U_i} |\Delta a_j| \geqslant \sum_{i=1}^{\kappa_k} \frac{T_{j_k+1}^{(n_0)}}{2\alpha_0 (m_i^{(k)}+\nu_i)^\gamma} \geqslant \frac{M_2 T_{j_k+1}^{(n_0)} \kappa_k}{\alpha_0 n_k^\gamma}.$$

Therefore $\kappa_k \leqslant M_3 \alpha_0$, i.e., the number κ_k is bounded independent of n_k.

Denote $S_n(x)$ the nth sum of sine series S (the same method also holds for cosine series). Write

$$\phi_{n,m}(x) = \sum_{l=1}^{m} \left(l^{\gamma-1} \sin(n+l)x - l^{\gamma-1} \sin(n-l)x \right) = 2 \sin nx \sum_{l=1}^{m} l^{\gamma-1} \sin lx$$

for $m \leqslant n$. We know from Theorem 2.2 that $|\phi_{n,m}(x)| = O(x^{-\gamma})$. Define

$$V_n(x) = \frac{1}{n} \sum_{k=n}^{2n-1} S_k(x),$$

$$r_j = \min\{m_j^{(k)}, \nu_j\}, \quad j = 1, 2, \cdots, \kappa_k,$$

then it is easy to verify that, there is a constant $r_0 > 0$ only depending upon λ such that

$$r_0 \nu_j \leqslant r_j \leqslant m_j^{(k)}.$$

Therefore, for $j = 1, 2, \cdots, \kappa_k$, in view of that $\{m_j^{(k)}, m_j^{(k)}+1, \cdots, m_j^{(k)}+r_j-1\} \subseteq U_j$,

$$\int_0^\pi x^{-\gamma} \left| f(x) - V_{m_j^{(k)}-2}(x) \right| dx \geqslant M_1 \int_0^\pi \left(f(x) - V_{m_j^{(k)}-2}(x) \right) \phi_{m_j^{(k)}, \nu_j}(x) dx$$

$$\geqslant \frac{M_1}{m_j^{(k)}} \left| \sum_{l=1}^{r_j} l^\gamma a_{m_j^{(k)}+l-1} \right| \geqslant \frac{M_1}{n_k} \sum_{l=1}^{r_j} l^\gamma |a_{m_j^{(k)}+l-1}|,$$

so, since $x^{-\gamma} f(x) \in L_{2\pi}$, for arbitrary j, $1 \leqslant j \leqslant \kappa_k$, with the definition of U_j, we have

$$M_1 \geqslant \frac{M_2}{n_k} \sum_{l=1}^{r_j} \frac{l^\gamma T_{j_k+1}^{(n_0)}}{2\alpha_0 (m_1^{(k)}+l-1)^\gamma} \geqslant \frac{M_3 \nu_j^{\gamma+1}}{n_k^{1+\gamma}} T_{j_k+1}^{(n_0)},$$

that is,

$$\nu_j \leqslant M_4 n_k (T_{j_k+1}^{(n_0)})^{-1/(1+\gamma)}.$$

In other words,

$$\sum_{j=1}^{\kappa_k} |U_j| = \sum_{j=1}^{\kappa_k} \nu_j \leqslant M_4 \kappa_k n_k (T_{j_k+1}^{(n_0)})^{-1/(1+\gamma)},$$

which contradicts the fact that

$$\sum_{j=1}^{\kappa_k} |U_j| \geqslant |A_{n_k}^{\alpha_0}| \geqslant M_0 n_k$$

by Lemma 2.5 and (7). The proof is therefore completed. □

Write $I_k = \{2^k, 2^k+1, \cdots, 2^{k+1}-1\}$, and select disjoint subsets $S_1, \ldots, S_{\kappa_k}$ of I_k according to the property of the sequence $\{a_n\}$ as follows. Set

$$m_1 = \min\{m \in I_k : a_m \neq 0\}.$$

Without loss of generality, we may assume that $m_1 = 2^k$. If $\{a_m\}$ does not change sign in I_k, let $\nu_1 = 2^{k+1}-1$. Otherwise let $\nu_1 = k_0$ for which $a_{m_1+k_0}$ is the first element with $m_1 + k_0 \in I_k$ of opposite sign than a_{m_1}. Define now

$$S_1 = \{m_1,\ m_1+1, \cdots, m_1+\nu_1-1\}.$$

Next, set $m_2 = \min(I_k \setminus S_1)$ if this latter set is not empty, and using the same procedure we select ν_2 and define

$$S_2 = \{m_2,\ m_2+1, \cdots, m_2+\nu_2-1\}.$$

We continue this procedure until we reach an S_{κ_k} for which $I_k \setminus (S_1 \cup \cdots \cup S_{\kappa_k}) = \emptyset$. Set I_k^+ to be the union of all subsets $\{S_j\}$ whose elements a_n keep positive sign, and I_k^- the union of all subsets $\{S_j\}$ whose elements a_n keep negative sign. Also, define

$$J_k^{(1)} = \{\cup S_j : |S_j| \geqslant 2^k/(32\lambda^2 M), 1 \leqslant j \leqslant \kappa_k\},$$

$$J_k^{(2)} = \{\cup S_j : |S_j| < 2^k/(32\lambda^2 M), 1 \leqslant j \leqslant \kappa_k\},$$

and write

$$|a_{\mu_j}| = \max_{m \in S_j} |a_m|.$$

With all the above symbols, we have

LEMMA 2.7. *Suppose that a real sequence $\{a_n\}$ satisfies condition (1), and let $0 < \gamma < 1$. Then, for sufficiently large k we have*

$$\sum_{m \in J_k^{(2)}} m^{\gamma-1}|a_m| \leqslant \frac{1}{4\lambda^2} \sum_{m \in J_k^{(1)}} m^{\gamma-1}|a_m| + \frac{1}{4\lambda^2} \sum_{n=2^k/\lambda}^{2^k-1} n^{\gamma-1}|a_n| + \frac{1}{4\lambda} \sum_{n=2^{k+1}}^{\lambda 2^k} n^{\gamma-1}|a_n|.$$

Furthermore, the number of sets S_j in $J_k^{(1)}$ is bounded independent of k.

Proof. It is easy to see that, by condition (1),

$$\frac{1}{2}\sum_{S_j\subseteq J_k^{(2)}}|a_{\mu_j}| \leqslant \frac{1}{2}\sum_{j=1}^{\kappa_k}|a_{\mu_j}| \leqslant \sum_{n=2^k}^{2^{k+1}}|\Delta a_n| \leqslant \frac{M}{2^k}\sum_{n=2^k/\lambda}^{\lambda 2^k}|a_n|,$$

or in other words,

$$\frac{1}{2}\sum_{S_j\subseteq J_k^{(2)}}|a_{\mu_j}| \leqslant \frac{M}{2^k}\left(\sum_{S_j\subseteq J_k^{(2)}}|S_j||a_{\mu_j}| + \sum_{n\in J_k^{(1)}}|a_n|\right) + \frac{M}{2^k}\left(\sum_{n=2^k/\lambda}^{2^k-1}|a_n| + \sum_{n=2^{k+1}}^{\lambda 2^k}|a_n|\right).$$

A direct calculation yields that

$$\begin{aligned}\frac{1}{4}\sum_{S_j\subseteq J_k^{(2)}}|a_{\mu_j}| &\leqslant \frac{1}{2}\sum_{S_j\subseteq J_k^{(2)}}|a_{\mu_j}|(1-M|S_j|/2^k)\\ &\leqslant \frac{M}{2^k}\sum_{n\in J_k^{(1)}}|a_n| + \frac{M}{2^k}\left(\sum_{n=2^k/\lambda}^{2^k-1}|a_n| + \sum_{n=2^{k+1}}^{\lambda 2^k}|a_n|\right)\end{aligned} \tag{8}$$

by noticing that $|S_j| \leqslant 2^k/(32\lambda^2 M)$ for $S_j \subseteq J_k^{(2)}$. At the same time,

$$\sum_{n\in J_k^{(2)}}|a_n| \leqslant \sum_{S_j\subseteq J_k^{(2)}}|S_j||a_{\mu_j}| \leqslant \frac{2^k}{32\lambda^2 M}\sum_{S_j\subseteq J_k^{(2)}}|a_{\mu_j}|,$$

with (8), we have

$$\sum_{n\in J_k^{(2)}}|a_n| \leqslant \frac{1}{8\lambda^2}\sum_{n\in J_k^{(1)}}|a_n| + \frac{1}{8\lambda^2}\left(\sum_{n=2^k/\lambda}^{2^k-1}|a_n| + \sum_{n=2^{k+1}}^{\lambda 2^k}|a_n|\right).$$

From this the required result of Lemma 2.7 immediately follows. Furthermore, since $|S_j| \geqslant 2^k/(32\lambda^2 M)$ for $S_j \subseteq J_k^{(1)} \subseteq \{2^k, 2^k+1, \cdots, 2^{k+1}-1\}$, the number of $S_j \in J_k^{(1)}$ is definitely bounded independent of k. □

We also remark here the inequality established in Lemma 2.7 does reflex some essential property of MVBV concept in real sense, although it needs further investigation in the future.

COROLLARY 2.8. *Let* $0<\gamma<1$, $\{a_n\}$ *satisfy condition* (1). *For sufficiently large* k_0 *and arbitrary* N *we have*

$$\sum_{k=k_0}^{N}\sum_{m\in J_k^{(2)}} m^{\gamma-1}|a_m| \leqslant 2\left(\sum_{k=k_0}^{N}\sum_{m\in J_k^{(1)}} m^{\gamma-1}|a_m| + \sum_{n=2^{k_0}/\lambda}^{2^{k_0}-1} n^{\gamma-1}|a_n| + \sum_{n=2^{N+1}}^{\lambda 2^N} n^{\gamma-1}|a_n|\right).$$

Proof. The number of repeat points of $\{2^{k_0}, 2^{k_0}+1, \cdots, 2^N-1\}$ with $\{m : 2_k/\lambda \leqslant m < 2^k\}$, $k = k_0+1, \cdots, N$ and $\{m : 2^k \leqslant m < \lambda 2^k\}$, $k = k_0, \cdots, N-1$ is surely bounded by 2λ. Therefore, we can easily have the conclusion of Corollary 2.8 by a simple calculation. $\square$

By using the symbols of Lemma 2.7, for sufficiently large k_0 and $k = k_0, k_0+1, \cdots$, set

$$d_m = \begin{cases} m^{\gamma-1}, & m \in J_k^{(1)} \cap I_k^+, \\ -m^{\gamma-1}, & m \in J_k^{(1)} \cap I_k^-, \\ 0, & m \in J_k^{(2)}. \end{cases}$$

LEMMA 2.9. *Under the above symbols,* $\{d_m\}$ *satisfies condition* (1).

Proof. Since the number of sets S_j in $J_k^{(1)}$ is bounded independent of k, $|S_j| \geqslant 2^k/(32\lambda^2 M)$ for all $S_j \subseteq J_k^{(1)}$, and $d_{2m} \approx d_m$ for every $m \geqslant k_0$ and $d_m \neq 0$, $d_{2m} \neq 0$, the fact that $\{d_m\}$ satisfies condition (1) can be easily verified. $\square$

LEMMA 2.10. *Suppose that a real sequence* $\{a_n\}$ *satisfies condition* (1) *and consider the trigonometric series*

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \sin nx$$

or

$$S(x) \equiv \sum_{n=1}^{\infty} a_n \cos nx,$$

and its sum function is denoted by $f(x)$. *Let* $0 < \gamma < 1$. *If* $x^{-\gamma} f(x) \in L_{2\pi}$ *and* $\{a_n\}$ *is the Fourier coefficients of* $f(x)$, *then for any* $N \geqslant k_0$, *we have*

$$\sum_{k=k_0}^{N} \sum_{m \in J_k^{(1)}} m^{\gamma-1} |a_m| \leqslant M_1 \int_0^{\pi} x^{-\gamma} |f(x)| dx.$$

Proof. We need only prove the conclusion for sine series, the other case can be treated in the same manner. It is clear to see that $f \in L_{2\pi}$. Hence

$$\sum_{k=k_0}^{N} \sum_{m \in J_k^{(1)}} m^{\gamma-1} |a_m| = \sum_{k=k_0}^{N} \left(\sum_{m \in J_k^{(1)} \cap I_k^+} m^{\gamma-1} a_m + \sum_{m \in J_k^{(1)} \cap I_k^-} m^{\gamma-1} (-a_m) \right)$$

$$= \frac{2}{\pi} \sum_{k=k_0}^{N} \left(\sum_{m \in J_k^{(1)} \cap I_k^+} m^{\gamma-1} \int_0^{\pi} f(x) \sin mx dx + \sum_{m \in J_k^{(1)} \cap I_k^-} (-m^{\gamma-1}) \int_0^{\pi} f(x) \sin mx dx \right)$$

$$= \frac{2}{\pi} \int_0^{\pi} f(x) \left(\sum_{m=2^{k_0}}^{2^N-1} d_m \sin mx \right) dx,$$

or

$$\sum_{k=k_0}^{N}\sum_{m\in J_k^{(1)}} m^{\gamma-1}|a_m| \leqslant \frac{2}{\pi}\int_0^\pi |f(x)|\left|\sum_{m=2^{k_0}}^{2^N-1} d_m \sin mx\right| dx.$$

By Theorem 2.2 and Lemma 2.9, we immediately get

$$\left|\sum_{m=2^{k_0}}^{2^N-1} d_m \sin mx\right| = O(x^{-\gamma}),$$

so that

$$\sum_{k=k_0}^{N}\sum_{m\in J_k^{(1)}} m^{\gamma-1}|a_m| \leqslant M_1\int_0^\pi x^{-\gamma}|f(x)|dx.$$

Lemma 2.10 is proved. □

Proof of Theorem 1.1. It is deduced immediately from Corollary 2.8 and Lemma 2.10 that

$$\sum_{m=2^{k_0}}^{2^N} m^{\gamma-1}|a_m| \leqslant M_1\left(\int_0^\pi x^{-\gamma}|f(x)|dx + \sum_{n=2^{k_0}/\lambda}^{2^{k_0}-1} n^{\gamma-1}|a_n| + \sum_{n=2^{N+1}}^{\lambda 2^N} n^{\gamma-1}|a_n|\right),$$

in combining with Theorem 2.6 we have

$$\sum_{m=2^{k_0}}^{2^N} m^{\gamma-1}|a_m| \leqslant M_1\int_0^\pi x^{-\gamma}|f(x)|dx + O(1),$$

that already finish the proof of necessity.

Sufficiency can be derived from Corollary 2.3 in [3]. □

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