Mathematical
Inequalities
& Applications
Volume 19, Number 3 (2016), 937–948 doi:10.7153/mia-19-68

HARDY AND RELLICH TYPE INEQUALITIES WITH TWO WEIGHT FUNCTIONS

SEMRA AHMETOLAN AND ISMAIL KOMBE

(*Communicated by I. Perić*)

Abstract. In the present paper we prove several sharp two-weight Hardy, Hardy-Poincaré, and Rellich type inequalities on the sub-Riemannian manifold $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined by the vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial l}, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial l}, \qquad j = 1,2,\dots,n$$

where $(z,y) = (x,y,l) \in \mathbb{R}^{2n+1}$, $|z| = (|x|^2+|y|^2)^{1/2}$ and $k \geqslant 1$.

1. Introduction

The present work is concerned with the two-weight Hardy, Hardy-Poincaré, and Rellich type inequalities on the sub-Riemannian manifold $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined by the vector fields:

$$X_j = \frac{\partial}{\partial x_j} + 2ky_j|z|^{2k-2}\frac{\partial}{\partial l}, \quad Y_j = \frac{\partial}{\partial y_j} - 2kx_j|z|^{2k-2}\frac{\partial}{\partial l}, \qquad j = 1,2,\dots,n \tag{1}$$

where $|z| = (|x|^2+|y|^2)^{1/2}$ and $k \geqslant 1$. The vector fields (1) satisfy Hörmander's condition for any $k \in \mathbb{N}$, i.e, X_j, Y_j and their iterated Lie brackets span the tangent space of $\mathbb{R}^{2n+1}$ at each point [17].

In order to motivate our work, we first present Hardy and Rellich inequalities on the Euclidean space $\mathbb{R}^n$. The classical Hardy inequality for the Laplacian in $\mathbb{R}^n$, $n \geqslant 3$, states that for any smooth, compactly supported function $\phi \in C_0^\infty(\mathbb{R}^n)$, the following inequality holds [15] :

$$\int_{\mathbb{R}^n} |\nabla\phi|^2 dx \geqslant \frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^2}dx. \tag{2}$$

An extension of the Hardy's inequality to second-order derivative is referred as the Rellich inequality [25]:

$$\int_{\mathbb{R}^n} |\Delta\phi|^2 dx \geqslant \frac{n^2(n-4)^2}{16}\int_{\mathbb{R}^n} \frac{|\phi|^2}{|x|^4}dx \tag{3}$$

Mathematics subject classification (2010): 22E30, 26D10, 43A80.
Keywords and phrases: Hardy inequality with two weight functions, Rellich inequality with two weight functions, sharp constant.

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where $\phi \in C_0^\infty(\mathbb{R}^n)$ and $n \geqslant 5$. The coefficients $\frac{(n-2)^2}{4}$ and $\frac{n^2(n-4)^2}{16}$ are sharp, but are never achieved. Hardy and Rellich type inequalities and their improved versions play important role in the study of elliptic and parabolic partial differential equations involving singular potentials, e.g. [1], [2], [3], [5], [6], [9], [10], [12], [13], [18], [20], [21], [24], [26], [27] and references therein.

There has been a growing interest in Hardy and Rellich type inequalities on the sub-Riemannian manifold $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined by the vector fields (1). In the article [28], Niu and Zhang established the following Hardy type inequality for all $\phi \in \mathbb{R}^{2n+1} \setminus \{(0,0)\}$:

$$\int_{\mathbb{R}^{2n+1}} |\nabla_k \phi|^p dw \geqslant \left(\frac{Q-p}{p}\right)^p \int_{\mathbb{R}^{2n+1}} \left(\frac{|z|}{\rho}\right)^{p(2k-1)} \frac{|\phi|^p}{\rho^p} dw \tag{4}$$

where ρ denotes the homogeneous norm: $\rho = (|z|^{4k} + l^2)^{1/4k}$, $Q = 2n + 2k$ the homogeneous dimension, $1 < p < Q$, $\nabla_k \phi = (X_1\phi, \ldots, X_n\phi, Y_1\phi, \ldots, Y_n\phi)$ the sub-elliptic gradient. On the other hand, Niu, Ou and Han [22] obtained (among other inequalities) a weighted version of (4):

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha p} |\nabla_k \phi|^p dw \geqslant \left(\frac{Q-p+\alpha p}{p}\right)^p \int_{\mathbb{R}^{2n+1}} \rho^{\alpha p} \left(\frac{|z|}{\rho}\right)^{p(2k-1)} \frac{|\phi|^p}{\rho^p} dw \tag{5}$$

where $\phi \in \mathbb{R}^{2n+1} \setminus \{(0,0)\}$, $1 < p < Q$ and $Q - p + \alpha p > 0$. Furthermore, the constant $\left(\frac{Q-p+\alpha p}{p}\right)^p$ is sharp.

In our previous work [4], we studied sharp uncertainty principle inequality and Hardy-Poincaré type inequality with radial weight ρ^α. So far only one weight function has been considered in above inequalities and the role of the radial weight function ρ^α is well known.

The purpose of this paper is to give new sharp weighted Hardy, Hardy-Poincaré and Rellich type inequalities where weight function is given by the product of the radial functions ρ^α and $|z|^t$ on the sub-Riemannian manifold $\mathbb{R}^{2n+1} = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ defined by the vector fields (1). We note that the new radial weight function $|z|^t$ has influence on the sharp constant, brings some difficulties in some computations and also leads to obtain nonnegative remainder term in Rellich-type inequality (see Theorem 4). Furthermore, our results improve and include previously known some results as special cases.

2. Preliminary and notations

In this section, we will introduce some notations, definitions, and preliminary facts which will be used throughout the article. A generic point in $\mathbb{R}^{2n+1}$, $n \geqslant 1$ is defined by $w = (z,l) = (x,y,l) \in \mathbb{R}^{2n+1}$ where $x, y \in \mathbb{R}^n$, $z = x + \sqrt{-1}y$. The sub-elliptic gradient is the $2n$ dimensional vector field given by

$$\nabla_k := (X_1, \ldots, X_n, Y_1, \ldots, Y_n)$$

where X_j and Y_j are the smooth vector fields which are defined as in (1). The generalized Greiner operator on $\mathbb{R}^{2n+1}$ is defined by,

$$\Delta_k = \sum_{j=1}^{n}(X_j^2+Y_j^2) = \Delta_z + 4k^2|z|^{4k-2}\frac{\partial^2}{\partial l^2} + 4k|z|^{2k-2}\frac{\partial}{\partial l}T$$

where $\Delta_z = \sum_{j=1}^{n}(\frac{\partial^2}{\partial x_j^2}+\frac{\partial^2}{\partial y_j^2})$ is the Laplacian in the variable $z=(x,y)\in\mathbb{R}^{2n}$ and T denotes the vector field as $T=\sum_{j=1}^{n}(y_j\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial y_j})$. When $k=1$, Δ_k becomes the sub-Laplacian on the Heisenberg group $\mathbb{H}^n$ [11]. We note that there exists family of dilations

$$\delta_\lambda : (z,l)\mapsto (u,v) := (\lambda z, \lambda^{2k}l), \quad \lambda>0$$

associated with the vector fields in (1). It is easy to see that Δ_k is the homogeneous of degree two with respect to the dilation δ_λ . The change of variable formula for the Lebesgue measure gives that

$$dudv = \lambda^Q dzdl$$

where

$$Q = 2(k+n)$$

is the homogeneous dimension with respect to dilation δ_λ and $dzdl$ denotes the Lebesgue measure on $\mathbb{R}^{2n+1}$. For $w=(z,l)\in\mathbb{R}^{2n}\times\mathbb{R}$, there is a natural norm:

$$\rho = \rho(w) := ((|x|^2+|y|^2)^{2k}+l^2)^{1/4k} = (|z|^{4k}+l^2)^{1/4k}.$$

The norm function ρ is related to the fundamental solution of sub-Laplacian Δ_k at the origin (see, [7], [8], [14], [28]).

A direct computation shows that

$$X_j\rho = \frac{|z|^{2(k-1)}}{\rho^{4k-1}}[x_j|z|^{2k}+y_jl], \quad Y_j\rho = \frac{|z|^{2(k-1)}}{\rho^{4k-1}}[y_j|z|^{2k}-x_jl].$$

Let $\phi=\phi(\rho)$ is a smooth radial function (i.e., ϕ only depends on the function ρ), then using a direct computation, it can be easily shown that [19]

$$|\nabla_k\phi(\rho)| = \frac{|z|^{2k-1}}{\rho^{2k-1}}|\phi'(\rho)|$$

and

$$\Delta_k\phi(\rho) = \frac{|z|^{4k-2}}{\rho^{4k-2}}\Big(\phi''+\frac{Q-1}{\rho}\phi'\Big) = |\nabla_k\rho|^2\Big(\phi''+\frac{Q-1}{\rho}\phi'\Big).$$

In particular

$$\Delta_k\rho = \frac{|\nabla_k\rho|^2}{\rho}(Q-1).$$

The norm function is ∞- harmonic, that is ρ is the solution of

$$\nabla_k(|\nabla_k\rho|^2)\cdot\nabla_k\rho = 0. \tag{6}$$

A direct computations shows that

$$\nabla_k \cdot \left(\frac{\rho}{|\nabla_k \rho|^2} \nabla_k \rho \right) = Q. \tag{7}$$

The open ball with respect to ρ centered at the origin $(0,0) \in \mathbb{R}^{2n} \times \mathbb{R}$ with radius R will be denoted by

$$B_R(0) := \{(z,l) \in \mathbb{R}^{2n} \times \mathbb{R} : \rho < R\}.$$

Let $D = B_{R_2}(0) \setminus \overline{B_{R_1}(0)}$ be an annulus with $0 \leqslant R_1 < R_2$. Introducing the spherical coordinate transformation as in [19], the volume element satisfies the following relation

$$dw = dzdl = \rho^{Q-1} d\rho (\sin\varphi)^{\frac{n-k}{k}} d\varphi \prod_{j=1}^{n-1} \left[\cos\psi_j (\sin\psi_j)^{2(n-j)} d\psi_j \right] \prod_{j=1}^{n} d\theta_j \tag{8}$$

where $0 \leqslant \varphi \leqslant \pi$, $0 \leqslant \psi_j \leqslant \pi/2$, $j = 1,...,n-1$ and $0 \leqslant \theta_j \leqslant 2\pi$, $j = 1,...,n$, and

$$|z|^2 = \rho^2 \sin^{\frac{1}{k}} \varphi. \tag{9}$$

3. Sharp Hardy-Poincaré and Hardy type inequalities

The following theorem is the main result of this section.

THEOREM 1. *Let $Q = 2n+2k$, $p > 1$, $t + 2n > 0$ and $Q + \alpha + t > 0$. Then the following inequality is valid for all $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$:*

$$\int_{\mathbb{R}^{2n+1}} \frac{|z|^t \rho^{\alpha+p} |\nabla_k \rho \cdot \nabla_k \phi|^p}{|\nabla_k \rho|^{2p}} dzdl \geqslant \left(\frac{Q+\alpha+t}{p} \right)^p \int_{\mathbb{R}^{2n+1}} |z|^t \rho^\alpha |\phi|^p dzdl. \tag{10}$$

Furthermore, the constant $(\frac{Q+\alpha+t}{p})^p$ is sharp.

Proof. Multiply both sides of (7) by the function $\rho^\alpha |z|^t |\phi|^p$ and integrating over $\mathbb{R}^{2n+1}$ yields

$$Q \int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^t |\phi|^p dzdl = \int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^t |\phi|^p dzdl + \int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+1} |z|^t}{|\nabla_k \rho|^2} |\phi|^p \Delta_k \rho dzdl. \tag{11}$$

Applying integration by parts to the second integral in the righthand side, we have,

$$\begin{aligned} \int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+1}|z|^t}{|\nabla_k \rho|^2} |\phi|^p \Delta_k \rho dzdl = & -(\alpha+1) \int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^t |\phi|^p dzdl \\ & - \int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+1}}{|\nabla_k \rho|^2} |\phi|^p \nabla_k \rho \cdot \nabla_k (|z|^t) dzdl \\ & - \int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+1}|z|^t}{|\nabla_k \rho|^2} \nabla_k |\phi|^p \cdot \nabla_k \rho dzdl. \end{aligned} \tag{12}$$

Substituting (12) into (11), we get

$$\frac{Q+\alpha+t}{p}\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^{t}|\phi|^{p}dzdl=-\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+1}|z|^{t}\phi|\phi|^{p-2}}{|\nabla_{k}\rho|^{2}}\nabla_{k}\phi\cdot\nabla_{k}\rho dzdl. \tag{13}$$

Let

$$\mathscr{R}=-\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+1}|z|^{t}\phi|\phi|^{p-2}}{|\nabla_{k}\rho|^{2}}\nabla_{k}\phi\cdot\nabla_{k}\rho dzdl.$$

Applying successively the Cauchy-Schwarz, Hölder and Young inequalities, we get

$$\begin{aligned}\mathscr{R}&\leqslant\Big(\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+p}|z|^{t}}{|\nabla_{k}\rho|^{2p}}|\nabla_{k}\rho\cdot\nabla_{k}\phi|^{p}dzdl\Big)^{1/p}\Big(\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^{t}|\phi|^{p}dzdl\Big)^{(p-1)/p}\\&\leqslant\frac{\varepsilon^{p}}{p}\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+p}|z|^{t}}{|\nabla_{k}\rho|^{2p}}|\nabla_{k}\rho\cdot\nabla_{k}\phi|^{p}dzdl+\frac{(p-1)}{p\varepsilon^{\frac{p}{p-1}}}\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^{t}|\phi|^{p}dzdl\end{aligned} \tag{14}$$

where $\varepsilon>0$. Substituting (14) into (13) and then rearranging terms, we get,

$$\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+p}|z|^{t}}{|\nabla_{k}\rho|^{2p}}|\nabla_{k}\rho\cdot\nabla_{k}\phi|^{p}dzdl\geqslant f(Q,\alpha,p,t;\varepsilon)\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^{t}|\phi|^{p}dzdl$$

where $f(Q,\alpha,p,t;\varepsilon)=\varepsilon^{-p}[(Q+\alpha+t)-(p-1)\varepsilon^{-\frac{p}{(p-1)}}]$. Note that the function f attains the maximum for $\varepsilon=(\frac{p}{Q+\alpha+t})^{(p-1)/p}$ and this maximum value is equal to $(\frac{Q+\alpha+t}{p})^{p}$. Therefore we obtain the desired inequality

$$\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+p}|z|^{t}}{|\nabla_{k}\rho|^{2p}}|\nabla_{k}\rho\cdot\nabla_{k}\phi|^{p}dzdl\geqslant\Big(\frac{Q+\alpha+t}{p}\Big)^{p}\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^{t}|\phi|^{p}dzdl. \tag{15}$$

Now, we need to show that $\big(\frac{Q+\alpha+t}{p}\big)^{p}$ is the best constant. Let C_H be defined as the best constant of (15);

$$C_{H}:=\inf_{0\neq\phi\in C_{0}^{\infty}(\mathbb{R}^{2n+1})}\frac{\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+p}|z|^{t}}{|\nabla_{k}\rho|^{2p}}|\nabla_{k}\rho\cdot\nabla_{k}\phi|^{p}dzdl}{\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^{t}|\phi|^{p}dzdl}.$$

It is clear from (15) that

$$\Big(\frac{Q+\alpha+t}{p}\Big)^{p}\leqslant\frac{\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+p}|z|^{t}}{|\nabla_{k}\rho|^{2p}}|\nabla_{k}\rho\cdot\nabla_{k}\phi|^{p}dzdl}{\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^{t}|\phi|^{p}dzdl} \tag{16}$$

holds for all $\phi\in C_0^{\infty}(\mathbb{R}^{2n+1})$. By taking the infimum in (16), we have

$$\Big(\frac{Q+\alpha+t}{p}\Big)^{p}\leqslant C_{H}. \tag{17}$$

To prove that $C_H=\big(\frac{Q+\alpha+t}{p}\big)^{p}$, we now only need to show that $C_H\leqslant\big(\frac{Q+\alpha+t}{p}\big)^{p}$. Hence, for a given $\varepsilon>0$, we define the radial function $\phi_{\varepsilon}(\rho)\in C_0^{\infty}(\mathbb{R}^{2n+1})$ that can be approximated by smooth functions with compact support in $\mathbb{R}^{2n+1}$:

$$\phi_{\varepsilon}(\rho)=\begin{cases}\rho^{\beta+\varepsilon} & \text{if}\quad 0\leqslant\rho\leqslant 1,\\ \rho^{-(\beta+\varepsilon)} & \text{if}\quad \rho>1,\end{cases} \tag{18}$$

where $\beta = \frac{Q+\alpha+t}{p}$. It is clear that

$$\frac{\rho^{\alpha+p}|z|^t}{|\nabla_k\rho|^{2p}}|\nabla_k\rho\cdot\nabla_k\phi_\varepsilon|^p = \begin{cases}(\beta+\varepsilon)^p|z|^t\rho^{\alpha+p(\varepsilon+\beta)} & \text{if} \quad 0<\rho<1,\\ (\beta+\varepsilon)^p|z|^t\rho^{\alpha-p(\varepsilon+\beta)} & \text{if} \quad \rho>1.\end{cases}$$

In the sequel we indicate $B_1 = \{(z,l)\in\mathbb{R}^{2n+1}, 0\leqslant\rho\leqslant 1\}$ ρ-ball centered at the origin in $\mathbb{R}^{2n+1}$ with radius 1. By direct computation we get

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+p}|z|^t}{|\nabla_k\rho|^{2p}}|\nabla_k\rho\cdot\nabla_k\phi_\varepsilon|^p dzdl =&(\beta+\varepsilon)^p\Big[\int_{B_1}\rho^{\alpha+p(\beta+\varepsilon)}|z|^t dzdl\\ &+\int_{\mathbb{R}^{2n+1}\setminus B_1}\rho^{\alpha-p(\beta+\varepsilon)}|z|^t dzdl\Big]\\ =&(\beta+\varepsilon)^p\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^t|\phi_\varepsilon|^p dzdl\end{aligned}$$

By the spherical coordinates in Section 2, we have

$$\int_{B_1}\rho^{\alpha+p(\beta+\varepsilon)}|z|^t dzdl = \frac{1}{\alpha+p(\beta+\varepsilon)+t+Q}\Big[\int_0^\pi\Big(\sin\varphi\Big)^{\frac{t}{2k}+\frac{n-k}{k}}d\varphi\Big]K_n$$

$$\int_{\mathbb{R}^{2n+1}\setminus B_1}\rho^{\alpha+p(\beta+\varepsilon)}|z|^t dzdl = \frac{1}{\alpha-p(\beta+\varepsilon)+t+Q}\Big[\int_0^\pi\Big(\sin\varphi\Big)^{\frac{t}{2k}+\frac{n-k}{k}}d\varphi\Big]K_n$$

where

$$K_n = \int_0^{\frac{\pi}{2}}\cdots\int_0^{\frac{\pi}{2}}\prod_{j=1}^{n-1}\Big[\cos\psi_j(\sin\psi_j)^{2(n-j)}d\psi_j\Big]\times\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{j=1}^{n}d\theta_j. \tag{19}$$

Note that the improper integral $\int_0^\pi\Big(\sin\varphi\Big)^{\frac{t}{2k}+\frac{n-k}{k}}d\varphi$ converges for $t>-2n$. It is clear that

$$C_H < (\beta+\varepsilon)^p = \frac{\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+p}|z|^t}{|\nabla_k\rho|^{2p}}|\nabla_k\rho\cdot\nabla_k\phi|^p dzdl}{\int_{\mathbb{R}^{2n+1}}\rho^\alpha|z|^t|\phi|^p dzdl}. \tag{20}$$

Letting $\varepsilon\to 0$, it follows that $C_H\leqslant\big(\frac{Q+\alpha+t}{p}\big)^p$. Therefore, from (17) and (20) we have $C_H = \big(\frac{Q+\alpha+t}{p}\big)^p$. □

We now prove the following weighted L^p Hardy-type inequality which plays an important role in the proof of the weighted Rellich type inequalities.

THEOREM 2. *Let $Q=2n+2k$, $p>1$, $\alpha\in\mathbb{R}$, $t\in\mathbb{R}$, $Q+\alpha+t-p>0$ and $t+2kp-p+2n>0$. Then the following inequality is valid for all $\phi\in C_0^\infty(\mathbb{R}^{2n+1})$:*

$$\int_{\mathbb{R}^{2n+1}}\rho^\alpha|z|^t|\nabla_k\phi|^p dzdl\geqslant\Big(\frac{Q+\alpha+t-p}{p}\Big)^p\int_{\mathbb{R}^{2n+1}}\rho^{\alpha-p}|z|^t|\nabla_k\rho|^p|\phi|^p dzdl \tag{21}$$

Furthermore, the constant $(\frac{Q+\alpha+t-p}{p})^p$ is sharp.

Proof. Using the identity (6) we get the following inequality $|\nabla_k\rho\cdot\nabla_k(|\nabla_k\rho|\phi)|\leqslant |\nabla_k\rho|^2|\nabla_k\phi|$. We now replace $|\nabla\rho|\phi$ instead of ϕ in the inequality (10) and we get

$$\int_{\mathbb{R}^{2n+1}}\rho^{\alpha+p}|z|^t|\nabla_k\phi|^p dzdl \geqslant \left(\frac{Q+\alpha+t}{p}\right)^p\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^t|\nabla_k\rho|^p|\phi|^p dzdl. \tag{22}$$

Next, using the well known inequality in L_p- spaces, $||f+g||_p\geqslant ||f||_p-||g||_p$, yields;

$$\begin{aligned}&\Big(\int_{\mathbb{R}^{2n+1}}|\rho^{\alpha}|z|^t\nabla_k(\rho\phi)|^p dzdl\Big)^{1/p}+\Big(\int_{\Omega}\rho^{\alpha}|z|^t|\nabla_k\rho|^p|\phi|^p dzdl\Big)^{1/p}\\ &\geqslant\Big(\int_{\mathbb{R}^{2n+1}}\rho^{\alpha+p}|z|^t|\nabla_k\phi|^p dzdl\Big)^{1/p}\end{aligned} \tag{23}$$

Substituting (23) into (22) yields,

$$\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^t|\nabla_k(\rho\phi)|^p dzdl\geqslant\Big(\frac{Q+\alpha+t-p}{p}\Big)^p\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^t|\nabla_k\rho|^p|\phi|^p dzdl.$$

Now, we replace $\frac{\phi}{\rho}$ instead of ϕ to obtain the usual weighted L^p- Hardy inequality:

$$\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^t|\nabla_k\phi|^p dzdl\geqslant\Big(\frac{Q+\alpha+t-p}{p}\Big)^p\int_{\mathbb{R}^{2n+1}}\rho^{\alpha-p}|z|^t|\nabla_k\rho|^p|\phi|^p dzdl.$$

To show that the constant $\big(\frac{Q+\alpha+t-p}{p}\big)^p$ is sharp, we use the same family of functions $\phi_\varepsilon(\rho)$ in Theorem 1 and pass to the limit as $\varepsilon\to 0$. $\square$

4. Sharp weighted Rellich type inequalities with two weight functions

The purpose of this section is to derive sharp weighted Rellich type inequalities with two radial weight functions on the sub-Riemannian manifold $\mathbb{R}^{2n+1}=\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ defined by the vector fields (1). First, we prove the following theorem:

THEOREM 3. *Let $\alpha\in\mathbb{R}$, $t\in\mathbb{R}$, $Q\geqslant 3$, $Q+t+\alpha-2>0$ and $Q+t+2k-2>0$. Then the following inequality is valid for all $\phi\in C_0^\infty(\mathbb{R}^{2n+1})$:*

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}}\frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2 dzdl\geqslant&\zeta_0^2\int_{\mathbb{R}^{2n+1}}\rho^{\alpha-2}|z|^t\phi^2|\nabla_k\rho|^2 dzdl\\ &-2\zeta_0 t\left(\frac{t}{2}-1+n\right)\int_{\mathbb{R}^{2n+1}}\rho^{\alpha}|z|^{t-2}\phi^2\,dzdl\end{aligned} \tag{24}$$

where $\zeta_0=\frac{(Q+t-2)^2-\alpha(\alpha+2t)}{4}$. Furthermore, the constant ζ_0^2 is sharp.

Proof. A straightforward computation shows that

$$\Delta_k\rho^{\alpha}=\xi\rho^{\alpha-2}|\nabla_k\rho|^2, \tag{25}$$

where $\xi = \alpha(Q+\alpha-2)$. Multiplying both sides of (25) by the function $|z|^t\phi^2$ and integrating over $\mathbb{R}^{2n+1}$, we obtain

$$\int_{\mathbb{R}^{2n+1}} |z|^t\phi^2\Delta_k\rho^\alpha dzdl = \xi\int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2}|z|^t\phi^2|\nabla_k\rho|^2dzdl.$$

By integration by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} |z|^t\phi^2\Delta_k\rho^\alpha dzdl =& -2\int_{\mathbb{R}^{2n+1}} |z|^t\phi\nabla_k\rho^\alpha\cdot\nabla_k\phi dzdl\\ &-\int_{\mathbb{R}^{2n+1}} \phi^2\nabla_k(\rho^\alpha)\cdot\nabla_k(|z|^t)dzdl.\end{aligned} \tag{26}$$

Applying integration by parts to the first and second integrals on the right hand side, (26) becomes

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} |z|^t\rho^\alpha|\nabla_k\phi|^2dzdl =& \left(\frac{\xi}{2}+\alpha t\right)\int_{\mathbb{R}^{2n+1}} |z|^t\rho^{\alpha-2}\phi^2|\nabla_k\rho|^2dzdl\\ &+t\left(\frac{t}{2}-1+n\right)\int_{\mathbb{R}^{2n+1}} \rho^\alpha|z|^{t-2}\phi^2dzdl\\ &-\int_{\mathbb{R}^{2n+1}} \rho^\alpha|z|^t\phi\Delta_k\phi dzdl.\end{aligned} \tag{27}$$

By using the sharp weighted L^p-Hardy inequality (21) into (27), we get

$$\begin{aligned}-\int_{\mathbb{R}^{2n+1}} \rho^\alpha|z|^t\phi\Delta_k\phi dzdl \geqslant& \zeta_0\int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2}|z|^t\phi^2|\nabla_k\rho|^2dzdl\\ &-t\left(\frac{t}{2}-1+n\right)\int_{\mathbb{R}^{2n+1}} \rho^\alpha|z|^{t-2}\phi^2dzdl\end{aligned} \tag{28}$$

where $\zeta_0 = \frac{(Q+t-2)^2-\alpha(\alpha+2t)}{4}$. Applying Cauchy-Schwarz and Young's inequalities gives

$$\begin{aligned}-\int_{\mathbb{R}^{2n+1}} \rho^\alpha|z|^t\phi\Delta_k\phi dzdl \leqslant& \varepsilon\int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2}|z|^t\phi^2|\nabla_k\rho|^2dzdl\\ &+\frac{1}{4\varepsilon}\int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta\phi|^2dzdl.\end{aligned} \tag{29}$$

Substituting (29) into (28) yields

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2dzdl \geqslant& 4\varepsilon(\zeta_0-\varepsilon)\int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2}|z|^t|\nabla_k\rho|^2\phi^2dzdl\\ &-4\varepsilon t\left(\frac{t}{2}-1+n\right)\int_{\mathbb{R}^{2n+1}} \rho^\alpha\phi^2|z|^{t-2}dzdl.\end{aligned} \tag{30}$$

Inequality (24) follows from (30) by taking $\varepsilon = \frac{\zeta_0}{2}$. We now show that ζ_0^2 is the best constant in (24):

$$C_R := \inf_{0\neq\phi\in C_0^\infty(\mathbb{R}^{2n+1})} \frac{\int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2dzdl}{\int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2}|z|^t\phi^2|\nabla_k\rho|^2dzdl}.$$

It is clear that

$$\zeta_0^2 \leqslant \frac{\int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi|^2 dzdl}{\int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2}|z|^t\phi^2|\nabla_k\rho|^2 dzdl} \tag{31}$$

holds for all $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$. If we take the infimum in (31) we get,

$$\zeta_0^2 \leqslant C_R. \tag{32}$$

We only need to show that $\zeta_0^2 \geqslant C_R$ and for this we use the following family of radial functions

$$\phi_\varepsilon(\rho) = \begin{cases} -(\beta+\varepsilon)(\rho-1)+1 & \text{if } \quad 0 \leqslant \rho \leqslant 1, \\ \rho^{-(\beta+\varepsilon)} & \text{if } \quad \rho > 1, \end{cases} \tag{33}$$

where $\varepsilon > 0$ and $\beta = \dfrac{Q-2}{2} + \dfrac{\sqrt{(2-Q)^2 - 4\zeta_0}}{2}$. First we construct the Rayleigh quotient:

$$\mathbf{R} := \frac{\int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi_\varepsilon|^2 dzdl}{\int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2}|z|^t\phi_\varepsilon^2|\nabla_k\rho|^2 dzdl}.$$

By direct computation, we have,

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi_\varepsilon(\rho)|^2 dzdl =& C(Q,\beta,\varepsilon)\Big(\int_0^1 \rho^{\alpha+t+Q-1}d\rho\Big)\tilde{K} \\ &+ D(Q,\beta,\varepsilon)\Big(\int_1^\infty \rho^{(-2\beta-2\varepsilon+\alpha+t+Q-3)}d\rho\Big)\tilde{K}.\end{aligned}$$

Here $C(Q,\beta,\varepsilon) = (\beta+\varepsilon)^2(Q-1)^2$, $D(Q,\beta,\varepsilon) = (-\beta-\varepsilon)^2(Q-2-\beta-\varepsilon)^2$ and

$$\tilde{K} = K_n \int_0^\pi (\sin\varphi)^{(t-2+Q)/2k} d\varphi$$

where K_n is given in (19). The integral $\int_0^1 \rho^{2\beta+2\varepsilon+\alpha+t+Q-3}d\rho$ is finite and $C(Q,\beta,\varepsilon)$ is positive. Therefore,

$$\int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi_\varepsilon(\rho)|^2 dzdl = D(Q,\beta,\varepsilon)\Big(\int_1^\infty \rho^{(\alpha+t-2\beta-2\varepsilon+Q-3)}d\rho\Big)\tilde{K} + O(1).$$

Similarly we get

$$\int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2}|z|^t\phi_\varepsilon^2(\rho)|\nabla_k\rho|^2 dzdl = \Big(\int_1^\infty \rho^{(\alpha+t-2\beta-2\varepsilon+Q-3)}d\rho\Big)\tilde{K} + O(1).$$

Letting $\varepsilon \to 0$, it follows that

$$\mathbf{R} := \frac{\int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k\rho|^2}|\Delta_k\phi_\varepsilon|^2 dzdl}{\int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2}|z|^t\phi_\varepsilon^2|\nabla_k\rho|^2 dzdl} \to \zeta_0^2 = \left[\frac{(Q+t-2)^2 - \alpha(\alpha+2t)}{4}\right]^2, \tag{34}$$

and then the constant ζ_0^2 is sharp. □

THEOREM 4. *Let $\alpha \in \mathbb{R}$, $t \in \mathbb{R}$, $Q \geqslant 3$, $Q+t+\alpha-2>0$ and $Q+t+2k-2>0$. Then the following inequality is valid for all $\phi \in C_0^\infty(\mathbb{R}^{2n+1})$:*

$$\int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k \rho|^2} |\Delta_k \phi|^2 dzdl \geqslant \frac{4\zeta_0^2}{(Q+\alpha+t-2)^2} \int_{\mathbb{R}^{2n+1}} |z|^t \rho^\alpha |\nabla_k \phi|^2 dzdl - \zeta_0 t(t+2n-2) \int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^{t-2} \phi^2 dzdl \tag{35}$$

where $\zeta_0 = \frac{(Q+t-2)^2-\alpha(\alpha+2t)}{4}$. Furthermore, the constant $\frac{4\zeta_0^2}{(Q+\alpha+t-2)^2}$ is sharp.

Proof. To prove the theorem, we apply Cauchy-Schwarz inequality to the integral on the left hand side of (27) and we get

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^t |\nabla_k \phi|^2 dzdl \leqslant &\left(\frac{\xi+2\alpha t}{2}+\varepsilon\right) \int_{\mathbb{R}^{2n+1}} \rho^{\alpha-2} |z|^t \phi^2 |\nabla_k \rho|^2 dzdl \\ &+ t\left(\frac{t}{2}+n-1\right) \int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^{t-2} \phi^2 dzdl \\ &+ \frac{1}{4\varepsilon} \int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t |\Delta_k \phi|^2}{|\nabla_k \rho|^2} dzdl.\end{aligned} \tag{36}$$

Now, by using the Rellich type inequality in (24) and rearranging the resulting inequality, we have,

$$\begin{aligned}\int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^t |\nabla_k \phi|^2 dzdl \leqslant &f(Q,\alpha,\varepsilon) \int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t |\Delta_k \phi|^2}{|\nabla_k \rho|^2} dzdl \\ &+ g(Q,\alpha,\varepsilon) \int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^{t-2} \phi^2 dzdl\end{aligned}$$

where

$$f(Q,\xi,\varepsilon) = \frac{1}{\zeta_0^2}\left(\frac{(\xi+2\alpha t)}{2}+\varepsilon\right) + \frac{1}{4\varepsilon}, \quad \zeta_0 = \frac{(Q+t-2)^2-\alpha(\alpha+2t)}{4},$$

and

$$g(Q,\xi,\varepsilon) = \frac{t(t+2n-2)}{\zeta_0}\left(\frac{\xi+2\alpha t}{2}+\varepsilon\right) + t\left(\frac{t}{2}+n-1\right).$$

Note that the function f attains its minimum value $\frac{(Q+\alpha+t-2)^2}{4\zeta_0^2}$ at $\varepsilon = \frac{\zeta_0}{2}$. Therefore we obtain the desired inequality:

$$\int_{\mathbb{R}^{2n+1}} \frac{\rho^{\alpha+2}|z|^t}{|\nabla_k \rho|^2} |\Delta_k \phi|^2 dzdl \geqslant \frac{4\zeta_0^2}{(Q+\alpha+t-2)^2} \int_{\mathbb{R}^{2n+1}} |z|^t \rho^\alpha |\nabla_k \phi|^2 dzdl - \zeta_0 t(t+2n-2) \int_{\mathbb{R}^{2n+1}} \rho^\alpha |z|^{t-2} \phi^2 dzdl.$$

For the sharpness of the constant $\frac{4\zeta_0^2}{(Q+\alpha+t-2)^2}$, we use the family of radial functions

$$\phi_\varepsilon(\rho) = \begin{cases} \rho^{(\beta+\varepsilon)} & \text{if} \quad 0 \leqslant \rho \leqslant 1, \\ \rho^{-(\beta+\varepsilon)} & \text{if} \quad \rho > 1, \end{cases} \tag{37}$$

where $\varepsilon > 0$, $\beta = \dfrac{(Q+\alpha-2)^2+t(2\alpha-t)}{2(Q+\alpha+t-2)}$ and pass to the limit as $\varepsilon \to 0$. □

REFERENCES

[1] B. ABDELLAOUI AND I. PERAL, *Existence and nonexistence results for quasilinear eliptic equations involving the p-Laplacian with a critical potential*, Annali di Matematica **182** (2003), 247–270.

[2] B. ABDELLAOUI, I. PERAL AND A. PRIMA, *Strong regularizing effect of a gradient term in the heat equation with the Hardy potential*, J. Func. Anal **258** (2010), 1247–1272

[3] S. AHMETOLAN AND S. CAVDAR, *On Nonlinear Degenerate Parabolic Equations with Time Dependent Singular Potential Functions*, Mathematische Nachrichten **284**, 10 (2011), 1219–1233.

[4] S. AHMETOLAN AND I. KOMBE, *A sharp uncertainty principle and Hardy-Poincaré inequalities on sub-Riemannian manifolds*, Mathematical Inequalities & Applications **15**, 2 (2012), 457–467.

[5] G. J. AZORERO AND I. PERAL, *Hardy inequalities and some critical elliptic and parabolic problems*, J. Diff. Equations **144** (1998), 441–476.

[6] P. BARAS AND J. A. GOLDSTEIN, *The Heat Equation with a Singular Potential*, Transactions of the American Mathematical Society **284**, 1 (1984), 121–139.

[7] R. BEALS, B. GAVEAU AND P. GREINER, *On a Geometric Formula for the Fundamental Solution of Subelliptic Laplacians*, Mathematische Nachrichten **181** (1996), 81–163.

[8] R. BEALS, B. GAVEAU AND P. GREINER, *Green's Functions for Some Highly Degenerate Elliptic Operators*, Journal of Functional Analysis **165** (1999), 407–429.

[9] H. BREZIS AND J. L. VÁZQUEZ, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. de la Universidad Complutense de Madrid **10**, 2 (1997), 443–469.

[10] X. CABRÉ AND Y. MARTEL, *Existence versus explosion instantanée pour des équetions de la chaleur linéaires avec, potentiel singulier*, C. R. Acad. Sci. Paris **329**, 1 (1999), 973–978.

[11] G. B. FOLLAND, *Fundamental solution for a subelliptic operator*, Bull. Amer. Math. Soc. **79**, 2 (1973), 373–376.

[12] J. A. GOLDSTEIN AND I. KOMBE, *Nonlinear Degenerate Parabolic Equations with Singular Lower-Order Term*, Advances in Differential Equations **8**, 10 (2003), 1153–1192.

[13] G. R. GOLDSTEIN, J. A. GOLDSTEIN AND A. RHANDI, *Weighted Hardy's inequality and the Kolmogorov equation perturbed by an inverse-square potential*, Appl. Anal. **91**, 11 (2012), 2057–2071.

[14] P. C. GREINER, *A fundamental Solution for non-elliptic partial differential operator*, Canadian Journal of Mathematices **31** (1979), 1107–1120.

[15] G. H. HARDY, *Note on a theorem of Hilbert*, Math. Zeitschr. **6** (1920), 314–317.

[16] D. HAUER AND A. RHANDI, *A weighted Hardy inequality and nonexistence of positive solutions*, Arch. Math. (Basel) **100**, 3 (2013), 273–287.

[17] L. HÖRMANDER, *Hypoelliptic Second Order Differential Equations*, Acta Mathematica **119** (1967), 147–171.

[18] I. KOMBE, *The Linear Heat Equation with Highly Oscillating Potential*, Proceeding of the American Mathematical Society **132**, 9 (2004), 2683–2691.

[19] I. KOMBE, *On the Nonexistence of Positive Solutions to Nonlinear Degenerate Parabolic Equations with Singular Coefficients*, Applicable Analysis **85**, 5 (2006), 467–478.

[20] I. KOMBE, *Sharp Weighted Rellich and Uncertainty Principle inequalities on Carnot Groups*, Communications in Applied Analysis 14 (2010), no. 2, 251 **14**, 2 (2010), 251–272.

[21] A. MORADIFAM AND N. GHOUSSOUB, *Bessel pairs and optimal Hardy and Hardy-Rellich inequalities*, Math. Ann. **349**, 1 (2011), 1–57.

[22] P. NIU, Y. OU AND J. HAN, *Several Hardy type inequalities with Weights related to generalized greiner operator*, Canadian Math. Bulletin **53**, 1 (2010), 153–162.

[23] P. NIU, H. ZHANG AND Y. WANG, *Hardy Thype and Rellich Type Inequalities on the Heisenberg Group*, Proceeding. Amer. Math. Soc. **129** (2001), 3623–3630.

[24] I. PERAL AND J. L. VÁZQUEZ, *On the stability or insitability of the singular solution of the semilinear heat equation with exponential reaction term*, Arch. Rational Mech. Anal. **129** (1995), 201–224.

[25] F. RELLICH, *Halbbeschränkte Differentialoperatoren häherer Ordnung*, Proceedings of the International Congress of Mathematicians, Edit by J. C. H. Gerretsen et all, Amsterdam **3** (1956), 243–250.

[26] J. VANCOSTENOBLE AND E. ZUAZUA, *Hardy inequalities, observability, and control for the wave and Schöndirger equations with singular potentials*, SIAM journal on Math. Anal. **41**, 4 (2009), 1508–1532.

[27] J. L. VÁZQUEZ AND E. ZUAZUA, *The Hardy constant and the asymptotic behaviour of the heat equation with an inverse-square potential*, J. Funct. Anal. **173** (2000), 103–153.

[28] H. ZHANG AND P. NIU, *Hardy-type inequalities and Pohozaev-type identities for a class of p-degenerate subelliptic operators and applications*, Nonlinear Anal. **54**, 1 (2003), 165–186.

(Received August 3, 2015)

Semra Ahmetolan
Istanbul Technical University
Faculty of Arts and Sciences, Mathematical Engineering
Istanbul, Turkey
e-mail: ahmetola@itu.edu.tr

Ismail Kombe
Istanbul Commerce University
Faculty of Arts and Sciences, Mathematics
Istanbul, Turkey
e-mail: ikombe@ticaret.edu.tr