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CONVEXITY OF $\Gamma(x)\Gamma(1/x)$

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Abstract. It is shown that $\log[\Gamma(x)\Gamma(1/x)]$ is convex. Gautschi's inequality $\Gamma(x)\Gamma(1/x) \geqslant 1$ is an immediate consequence.

1. Introduction

The following inequalities for the gamma function were proved by Gautschi [5] in 1974: for all $x > 0$,

$$\Gamma(x) + \Gamma(1/x) \geqslant 2, \tag{1}$$

$$\Gamma(x)\Gamma(1/x) \geqslant 1. \tag{2}$$

Clearly, (2) implies (1).

A third result of Gautschi, which in turn implies (2), states that the harmonic mean of $\Gamma(x)$ and $\Gamma(1/x)$ is at least 1. Numerous refinements and extensions of these inequalities have appeared in later articles, for example [1], [2], [3], [7], [8].

In this note we present some results on convexity which have (1) and (2) as immediate consequences, so can be regarded as strengthening them. The underlying observation is: if f is a convex, differentiable function on $(0,\infty)$ satisfying $f(1/x) = f(x)$, then $f'(1) = 0$, hence $f(x)$ is decreasing on $(0,1]$ and increasing on $[1,\infty)$, so the least value occurs at $x = 1$.

For (1), this is easy. It was shown in [7, Lemma 2] that the function $\Gamma(1/x)$ is convex on $(0,\infty)$. So $\Gamma(x) + \Gamma(1/x)$ is convex, which implies (1). (The fact that this function is decreasing on $(0,1]$ and increasing on $[1,\infty)$ was shown in [3, Lemma 2] by more elaborate methods.)

The corresponding step for (2) is not quite so simple. We will prove the following theorem:

THEOREM 1. *The function* $\log\Gamma(x) + \log\Gamma(1/x)$, *is convex. Hence* $\Gamma(x)\Gamma(1/x)$ *is log-convex, so convex.*

Two preliminary remarks will help to set the context for this statement. Firstly, in general, a product of two convex functions is convex if both are increasing or both decreasing, but not otherwise. Secondly, while $\log\Gamma(x)$ is convex, $\log\Gamma(1/x)$ is not.

We conclude with a brief outline of some analogous results for $\psi(1/x)$, where $\psi(x)$ is the digamma function $\Gamma'(x)/\Gamma(x)$. For example, $\psi(x) + \psi(1/x)$ is concave.

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2. Proof of Theorem 1

Write $L(x) = \log\Gamma(x) + \log\Gamma(1/x)$. Then

$$L'(x) = \psi(x) - \frac{1}{x^2}\psi(1/x), \tag{3}$$

$$L''(x) = \psi'(x) + \frac{2}{x^3}\psi(1/x) + \frac{1}{x^4}\psi'(1/x). \tag{4}$$

The proof will use a number of estimations for $\psi(x)$, $\psi'(x)$ and $\psi''(x)$, mostly elementary. The function $\psi(x)$ is strictly increasing on $(0,\infty)$, with $\psi(x_0) = 0$, where $x_0 \approx 1.46163$, so $\psi(x) < 0$ on $(0, x_0)$ and $\psi(x) > 0$ on (x_0, ∞), hence $\psi(1/x) > 0$ for $0 < x < 1/x_0 \approx 0.68417$. So it is already clear from (4) that $L''(x) > 0$ on $(0, 1/x_0)$.

Since $\psi(1) = -\gamma$, we have $\psi(x) \geqslant -\gamma$ for $x \geqslant 1$, hence

$$\psi(1/x) \geqslant -\gamma \quad \text{for } 0 < x \leqslant 1. \tag{5}$$

The function $\psi(x)$ is concave and satisfies $\psi(x+1) = \psi(x) + 1/x$. We deduce:

LEMMA 1. *We have*

$$\psi(1/x) \geqslant \frac{1}{x} - x - \gamma \qquad \textit{for} \quad x \geqslant 1. \tag{6}$$

Proof. Since $\psi(1) = -\gamma$, $\psi(2) = 1 - \gamma$ and $\psi(x)$ is concave, we have $\psi(x) + \frac{1}{x} = \psi(1+x) \geqslant x - \gamma$ for $0 < x \leqslant 1$. Substitute x for $1/x$ to obtain (6). □

We now deduce some bounds from the series expressions

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2}, \qquad \psi''(x) = -\sum_{n=0}^{\infty} \frac{2}{(n+x)^3}.$$

Simply taking the first term of the series, we have

$$\psi'(x) \geqslant \frac{1}{x^2}, \qquad \psi''(x) \leqslant -\frac{2}{x^3}, \tag{7}$$

so also $\psi'(1/x) \geqslant x^2$ and $\psi''(1/x) \leqslant -2x^3$ for $x \geqslant 1$.

For $x > 1$, better bounds are found by integral estimation, as follows.

LEMMA 2. *For all $x > 0$,*

$$\frac{1}{x} + \frac{1}{2x^2} \leqslant \psi'(x) \leqslant \frac{1}{x} + \frac{1}{x^2}, \tag{8}$$

$$-\frac{1}{x^2} - \frac{2}{x^3} \leqslant \psi''(x) \leqslant -\frac{1}{x^2} - \frac{1}{x^3}. \tag{9}$$

Proof. For a convex, decreasing, non-negative function f with $\int_0^\infty f(t)\,dt = I$, integral estimation gives

$$I + \tfrac{1}{2} f(0) \leqslant \sum_{n=0}^{\infty} f(n) \leqslant I + f(0),$$

in which the lower bound results from the fact that the trapezium formula overestimates the integral. The stated inequalities follow. □

REMARK 1. Further degrees of accuracy are provided by Euler-Maclaurin summation. For example,

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - r_2(x),$$

where $0 \leqslant r_2(x) \leqslant 1/(30x^5)$.

Rewritten for $1/x$, (8) and (9) become:

$$x + \tfrac{1}{2}x^2 \leqslant \psi'(1/x) \leqslant x + x^2, \tag{10}$$

$$-x^2 - 2x^3 \leqslant \psi''(1/x) \leqslant -x^2 - x^3. \tag{11}$$

Proof of Theorem 1. We have already noted that $L''(x) > 0$ for $0 < x < 1/x_0$. We now consider the cases $\frac{1}{4} \leqslant x \leqslant 1$ and $x > 1$ separately. We will estimate $\psi'(y)$ (where y is x or $1/x$) by (7) if $y \leqslant 1$ and by (6) or (8) if $y \geqslant 1$.

Case $\frac{1}{4} \leqslant x \leqslant 1$. In (4), we estimate $\psi'(x)$ by (7), $\psi(1/x)$ by (5) and $\psi'(1/x)$ by (10), to obtain

$$\begin{aligned} x^4 L''(x) &\geqslant x^2 - 2\gamma x + (x + \tfrac{1}{2}x^2) \\ &= x\left(\tfrac{3}{2}x - (2\gamma - 1)\right) \\ &> 0 \qquad \text{for} \quad x \geqslant \tfrac{1}{4}. \end{aligned}$$

Case $x > 1$. We now estimate $\psi'(x)$ by (8), $\psi(1/x)$ by (6) and $\psi'(1/x)$ by (7), to obtain

$$\begin{aligned} x^4 L''(x) &\geqslant x^4 \left(\frac{1}{x} + \frac{1}{2x^2}\right) + 2x\left(\frac{1}{x} - x - \gamma\right) + x^2 \\ &= x^3 - \tfrac{1}{2}x^2 - 2\gamma x + 2. \end{aligned}$$

Denote this by $p(x)$. Then $p(1) = \frac{5}{2} - 2\gamma > 0$ and $p'(x) = 3x^2 - x - 2\gamma > 0$, hence $p(x) > 0$ and $L''(x) > 0$, for $x \geqslant 1$. □

REMARK 2. The method shows that $\log\Gamma(1/x)$ is convex on $(0,1]$. However, for $x > 1/(x_0 - 1)$, it is elementary that $\psi(1/x) < -x$. With (10), this shows that $\log\Gamma(1/x)$ is concave for such x.

3. Some corresponding results for $\psi(1/x)$

The following lemma (and indeed a more general statement) is known, but, with the help of our Lemma 2, a very brief proof is available for the case we want, so we include it for completeness.

LEMMA 3. *For all $x > 0$, $x^2\psi'(x)$ is increasing, while $x\psi'(x)$ is decreasing.*

Proof. We have

$$x^2\psi'(x) = \sum_{n=0}^{\infty} \frac{x^2}{(n+x)^2}.$$

Since $x/(n+x) = 1 - 1/(n+x)$ is increasing, so is $x^2\psi'(x)$. Also, by Lemma 2,

$$\frac{d}{dx}[x\psi'(x)] = \psi'(x) + x\psi''(x) \leqslant \frac{1}{x} + \frac{1}{x^2} - \left(\frac{1}{x} + \frac{1}{x^2}\right) = 0. \quad \square$$

REMARK 3. Lemma 3 is a special case of [2, Lemma 1], which states: for integers $k \geqslant 1$, $x^{k+1}|\psi^{(k)}(x)|$ is increasing, while $x^k|\psi^{(k)}(x)|$ is decreasing. The proof in [2] uses the integral representation of $\psi(x)$. A completely different proof of the second statement was given in [6, Corollary 4.5].

We now state two results for $\psi(1/x)$ that follow easily. The first one may be known, but I am not aware of a previous reference for it.

PROPOSITION 1. *The function $\psi(1/x)$ is convex.*

Proof. We have

$$\frac{d}{dx}\psi\left(\frac{1}{x}\right) = -\frac{1}{x^2}\psi'\left(\frac{1}{x}\right) = -y^2\psi'(y),$$

where $y = 1/x$. By Lemma 3, this is decreasing with y, hence increasing with x. $\square$

PROPOSITION 2. *For all $x > 0$, we have*

$$\psi(x) + \psi(1/x) \leqslant -2\gamma. \tag{12}$$

Proof. Write $\psi(x) + \psi(1/x) = P(x)$. Then

$$P'(x) = \psi'(x) - \frac{1}{x^2}\psi'\left(\frac{1}{x}\right) = \frac{1}{x}\left[x\psi'(x) - \frac{1}{x}\psi'\left(\frac{1}{x}\right)\right].$$

Let $x > 1$ and write $y = 1/x$. Then $x > y$, so by Lemma 3, $x\psi'(x) \leqslant y\psi'(y)$, hence $P'(x) \leqslant 0$. So $P(x)$ is decreasing on $[1, \infty)$, hence increasing on $(0, 1]$, and the maximum value is $P(1) = -2\gamma$. $\square$

Like (1) and (2), this inequality can be derived from a convexity result: $P(x)$ is concave. The proof is similar to the proof of Theorem 1, given that

$$P''(x) = \psi''(x) + \frac{2}{x^3}\psi'(1/x) + \frac{1}{x^4}\psi''(1/x).$$

Lemma 1 is replaced by the inequality $\psi'(1/x) \leqslant x^2 + \zeta(2) - 1/x$. We omit the details here, because this result is presented (with a slightly different proof) in [4], where it is also shown that the harmonic mean of $\psi(x)$ and $\psi(1/x)$ is not less than $-\gamma$.

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