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INTEGRAL INEQUALITIES OF KANTOROVICH AND FIEDLER TYPES FOR HADAMARD PRODUCTS OF OPERATORS

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Abstract. The scalar Kantorovich inequality is a reverse weighted arithmetic-harmonic mean inequality. In matrix case, this inequality is also a reverse version of Fiedler's inequality. In this paper, we establish several Kantorovich and Fiedler types integral inequalities involving Hadamard products of continuous fields of Hilbert space operators. Kantorovich type inequality in which the product is replaced by an operator mean is also investigated. Such inequalities include discrete inequalities as special cases. Moreover, we obtain the monotonicity of certain maps involving Hadamard products of operators. As special cases, we get some operator versions of Fiedler matrix inequality.

1. Introduction

The scalar Kantorovich inequality [13] is a reverse weighted arithmetic-harmonic mean (AM-HM) inequality. It says that for real numbers a_i and w_i such that $0 < m \leqslant a_i \leqslant M$ and $w_i \geqslant 0$ for all $1 \leqslant i \leqslant n$, we have

$$\left(\sum_{i=1}^{n} w_i a_i\right)\left(\sum_{i=1}^{n} \frac{w_i}{a_i}\right) \leqslant \frac{(m+M)^2}{4mM}\left(\sum_{i=1}^{n} w_i\right)^2. \tag{1.1}$$

Note that the constant $(m+M)^2/(4mM)$ is the square of the ratio between the arithmetic mean and the geometric mean of m and M. Numerical analyses and statisticians use this inequality in the convergence analysis for the method of steepest descent. In the past decades, various extensions, variations and refinements of this inequality in several contexts have been investigated by many authors. This inequality has been proved to be equivalent to many inequalities, e.g. Cauchy-Schwarz-Bunyakovsky inequality and Wielant's inequality; see also [10, 23].

In the literature, there also exists an integral version of Kantorovich inequality as follows. For an integrable function $f : [\alpha,\beta] \to \mathbb{R}$ with $0 < m \leqslant f(x) \leqslant M$ for all $x \in [\alpha,\beta]$, we have

$$\int_\alpha^\beta f(x)^2\,dx \leqslant \frac{(m+M)^2}{4mM}\left(\int_\alpha^\beta f(x)\,dx\right)^2.$$

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This inequality is also called an additive version of Grüss inequality; see more information in [2].

Many matrix versions of Kantorovich inequality were obtained in the literature, e.g. [4, 15, 16]. Denote by $\mathbb{M}_k$ the algebra of k-by-k complex matrices. Recall that the Hadamard product of $A, B \in \mathbb{M}_k$ is defined to be the entrywise product:

$$A \circ B = [a_{ij} b_{ij}] \in \mathbb{M}_k.$$

A matrix analogue of this inequality involving Hadamard product is given in [17] as follows.

THEOREM 1.1. ([17], Theorem 2.2) *For each $i = 1, 2, \ldots, n$, let $A_i \in \mathbb{M}_k$ be a positive definite matrix such that $0 < mI \leqslant A_i \leqslant MI$ and $W_i \in \mathbb{M}_k$ a positive semidefinite matrix. Then*

$$\sum_{i=1}^{n} W_i^{\frac{1}{2}} A_i W_i^{\frac{1}{2}} \circ \sum_{i=1}^{n} W_i^{\frac{1}{2}} A_i^{-1} W_i^{\frac{1}{2}} \leqslant \frac{m^2 + M^2}{2mM} \left(\sum_{i=1}^{n} W_i \circ \sum_{i=1}^{n} W_i \right). \tag{1.2}$$

Kantorovich inequality can be regarded as a reverse of Fiedler's inequality:

THEOREM 1.2. ([1, 7]) *For any positive definite matrix $A \in \mathbb{M}_k$, we have*

$$A \circ A^{-1} \geqslant I. \tag{1.3}$$

Several operator extensions of Kantorovich inequality were also investigated, for instance, in [6, 9, 18, 19, 21] and references therein. Kantorovich type inequality where the product is replaced by an operator mean was considered in [20, 22].

In this paper, we establish various integral inequalities of Kantorovich and Fiedler types for continuous field of Hilbert space operators. Theorems 1.1 and 1.2 are generalized in many ways in terms of Hadamard products of operators. Kantorovich type inequalities involving Kubo-Ando operator means are also investigated. Such integral inequalities include discrete inequalities as special cases. Moreover, we obtain the monotonicity of certain maps involving Hadamard products of operators. As special cases, we get some operator versions of Fiedler's inequality (1.3).

This paper is organised as follows. Section 2 consists of preliminaries on continuous fields of operators and the main assumption used throughout the paper. In Section 3, after providing some useful facts about Hadamard products of operators, we prove Kantorovich type integral inequalities involving Hadamard product of continuous fields of operators. In Section 4, after recalling Kubo-Ando theory of operator means, we derive Kantorovich integral inequalities involving operator means. In the last section, we investigate the monotonicity of certain maps involving Hadamard products of operators. Fiedler type inequalities are also obtained in this section.

2. Continuous field of operators

Throughout this paper, let $\mathbb{H}$ be a separable Hilbert space over the complex numbers. Denote by $B(\mathbb{H})$ the C*-algebra of bounded linear operators acting on $\mathbb{H}$. The identity operator on $\mathbb{H}$ is denoted by I. The spectrum of $A \in B(\mathbb{H})$ is written as $\operatorname{Sp}(A)$. Let Ω be a compact Hausdorff space endowed with a Radon measure μ.

A parametrized family $(A_t)_{t\in\Omega}$ of operators in $B(\mathbb{H})$ is said to be a *continuous field of operators* if the operator-valued function $t \mapsto A_t$ is norm continuous and the real-valued function $t \mapsto \|A_t\|$ is Lebesgue integrable on Ω. The continuity of the field $(A_t)_{t\in\Omega}$ allows us to form the Bochner integral of the operators A_t's as follows. Let $\mathscr{P}$ be a partition of Ω into disjoint Borel subsets and let $\varepsilon > 0$ be a real number. For each operator A_t in $B(\mathbb{H})$, we can approximate A_t by a net of operators in the form

$$F_{\mathscr{P},\varepsilon}(A_t) = \sum_{i=1}^{n} \mu(E_i) A_{t_i}$$

where $E_i \in \mathscr{P}$ and $t_i \in E_i \subseteq \{t \in \Omega : \|A_t - A_{t_i}\| < \varepsilon\}$ for each $1 \leqslant i \leqslant n$. Then the net $F_{\mathscr{P},\varepsilon}(A_t)$ converges uniformly to the Bochner integral

$$\int_\Omega A_t \, d\mu(t).$$

The set of continuous functions from Ω to $B(\mathbb{H})$ is a C*-algebra under the pointwise operations and the C*-norm

$$\|(A_t)_{t\in\Omega}\| = \sup_{t\in\Omega} \|A_t\|.$$

Now, we state the main assumption for most of the results in this paper.

Main hypothesis. Let Ω be a compact Hausdorff space equipped with a Radon measure μ. Let $(A_t)_{t\in\Omega}$ be a continuous field of strictly positive operators in $B(\mathbb{H})$ such that $\operatorname{Sp}(A_t) \subseteq [m,M] \subseteq (0,\infty)$ for each $t \in \Omega$. Let $(W_t)_{t\in\Omega}$ be a continuous field of positive operators in $B(\mathbb{H})$.

PROPOSITION 2.1. *Assume Main hypothesis. For any continuous function* $f : [m,M] \to \mathbb{R}$, *we can form the Bochner integral*

$$\int_\Omega W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}} \, d\mu(t).$$

In addition, if $f([m,M]) \subseteq [0,\infty)$, *then this operator is positive.*

Proof. Since (Ω,μ) is a finite measure space, it suffices to prove the Lebesgue integralability of its norm function. Indeed, we have

$$\begin{aligned}\int_\Omega \|W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}\| \, d\mu(t) &\leqslant \int_\Omega \|W_t^{\frac{1}{2}}\| \cdot \|f(A_t)\| \cdot \|W_t^{\frac{1}{2}}\| \, d\mu(t) \\ &\leqslant \int_\Omega \|W_t\| \cdot \|f\|_\infty \, d\mu(t) \\ &\leqslant \int_\Omega \sup_{t\in\Omega} \|W_t\| \cdot \|f\|_\infty \, d\mu(t) \\ &= \mu(\Omega) \sup_{t\in\Omega} \|W_t\| \cdot \|f\|_\infty \\ &< \infty. \end{aligned}$$

Suppose that f is positive on $[m,M]$. Then $f(A_t)$ is a positive operator for each $t \in \Omega$. It follows that the integral is positive since the integrand is positive. □

3. Kantorovich type integral inequalities for Hadamard product of operators

In this section, we establish many integral analogues of Kantorovich type inequalities concerning Hadamard products of operators. These results includes discrete inequalities as special cases. In particular, we get a reverse version of weighted AM-HM operator inequality.

In what follows, fix an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ for $\mathbb{H}$. The Hadamard product of A and B in $B(\mathbb{H})$ is defined to be the unique operator $A \circ B$ in $B(\mathbb{H})$ such that

$$\langle (A\circ B)e_j, e_j \rangle = \langle Ae_j, e_j \rangle \langle Be_j, e_j \rangle \quad \text{for all } j \in \mathbb{N}.$$

Equivalently, it was shown in [8] that

$$A \circ B = U^*(A \otimes B)U \tag{3.1}$$

where $U : \mathbb{H} \to \mathbb{H} \otimes \mathbb{H}$ is the isometry defined by

$$Ue_j = e_j \otimes e_j \quad \text{for all } j \in \mathbb{N}. \tag{3.2}$$

When $\mathbb{H} = \mathbb{C}^n$, the Hadamard product $A \circ B$ for complex matrices is a principal submatrix of the Kronecker (tensor) product of A and B. For each fixed $X \in B(\mathbb{H})$, the map $A \mapsto A \circ X$ is a bounded linear operator from $B(\mathbb{H})$ to itself. It follows that

$$\int_\Omega A_t \, d\mu(t) \circ X = \int_\Omega (A_t \circ X) \, d\mu(t). \tag{3.3}$$

Moreover, this map preserves positivity when the multiplier is a positive operator.

We start with the following estimation about tensor products.

LEMMA 3.1. *The minimum constant* α *for which the following inequality*

$$A_1 \otimes A_2^{-1} + A_1^{-1} \otimes A_2 \leqslant \alpha (I \otimes I). \tag{3.4}$$

holds for all selfadjoint operators A_1, A_2 *such that* $mI \leqslant A_1, A_2 \leqslant MI$ *for some* $m, M > 0$ *is given by* $\alpha = (m^2 + M^2)/(mM)$.

Proof. First, note that the minimum constant α for which the scalar inequality

$$\frac{x}{y}+\frac{y}{x}\leqslant\alpha$$

holds for all real numbers x,y such that $x,y\in[m,M]$ is given by $\alpha=(m/M)+(M/m)$.

Now, since $\mathrm{Sp}(A_i)\subseteq[m,M]$ for $i=1,2$, we have $\mathrm{Sp}(A_1\otimes A_2{}^{-1})\subseteq[m/M,M/m]$. Note that $(A_1\otimes A_2{}^{-1})^{-1}=A_1^{-1}\otimes A_2$. It follows from spectral mapping theorem that

$$\begin{aligned}\|A_1\otimes A_2^{-1}+A_1^{-1}\otimes A_2\| &= \sup\{\lambda+\lambda^{-1} : \lambda\in\mathrm{Sp}(A_1\otimes A_2{}^{-1})\}\\ &\leqslant \sup\{\lambda+\lambda^{-1} : \lambda\in[m/M,M/m]\}\\ &= \frac{m^2+M^2}{mM}.\end{aligned}$$

Thus, we arrive at the inequality (3.4). The constant $(m^2+M^2)/(mM)$ cannot be improved since the case $A_1=xI$ and $A_2=yI$ is reduced to the scalar case. □

The following theorem is a Kantorovich type integral inequality.

THEOREM 3.2. *Under Main hypothesis, the following integral inequality holds*

$$\begin{aligned}&\int_\Omega W_t^{\frac12}A_tW_t^{\frac12}\,d\mu(t)\circ\int_\Omega W_t^{\frac12}A_t^{-1}W_t^{\frac12}\,d\mu(t)\\ &\qquad\leqslant K(m,M)\left(\int_\Omega W_t\,d\mu(t)\circ\int_\Omega W_t\,d\mu(t)\right).\end{aligned}\tag{3.5}$$

Here, $K(m,M):=\dfrac{m^2+M^2}{2mM}$ *is the best possible constant.*

Proof. Proposition 2.1 allows the existence of the operators in the left hand of the inequality (3.5). It follows from the property (3.3) that

$$\begin{aligned}&\int_\Omega W_t{}^{\frac12}A_tW_t{}^{\frac12}\,d\mu(t)\circ\int_\Omega W_t{}^{\frac12}A_t{}^{-1}W_t{}^{\frac12}\,d\mu(t)\\ &=\int_\Omega\left(\int_\Omega W_t{}^{\frac12}A_tW_t{}^{\frac12}\,d\mu(t)\right)\circ W_s^{\frac12}A_s^{-1}W_s^{\frac12}\,d\mu(s)\\ &=\iint_{\Omega^2}\left(W_t^{\frac12}A_tW_t^{\frac12}\circ W_s^{\frac12}A_s^{-1}W_s^{\frac12}\right)d\mu(t)\,d\mu(s)\end{aligned}$$

Passing through the isometry U defined by (3.2), we have

$$\begin{aligned}&\int_\Omega W_t{}^{\frac12}A_tW_t{}^{\frac12}\,d\mu(t)\circ\int_\Omega W_t{}^{\frac12}A_t{}^{-1}W_t{}^{\frac12}\,d\mu(t)\\ &=\iint_{\Omega^2}U^*\left(W_t^{\frac12}A_tW_t^{\frac12}\otimes W_s^{\frac12}A_s^{-1}W_s^{\frac12}\right)U\,d\mu(t)\,d\mu(s)\\ &=\iint_{\Omega^2}U^*\left[(W_t\otimes W_s)^{\frac12}\left(A_t\otimes A_s^{-1}\right)(W_t\otimes W_s)^{\frac12}\right]U\,d\mu(t)\,d\mu(s)\\ &=\frac12\iint_{\Omega^2}U^*\left[(W_t\otimes W_s)^{\frac12}\left(A_t\otimes A_s^{-1}+A_t^{-1}\otimes A_s\right)(W_t\otimes W_s)^{\frac12}\right]U\,d\mu(t)\,d\mu(s).\end{aligned}$$

By making use of Lemma 3.1 and the property (3.3), we obtain

$$\begin{aligned}
\int_\Omega W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}}\,d\mu(t) \circ \int_\Omega W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}}\,d\mu(t)
&\leqslant \frac{1}{2}\int_\Omega\int_\Omega 2K(m,M)U^*(W_t\otimes W_s)U\,d\mu(t)\,d\mu(s)\\
&= K(m,M)\int_\Omega\int_\Omega (W_t\circ W_s)\,d\mu(t)\,d\mu(s)\\
&= K(m,M)\int_\Omega \left(\int_\Omega W_t\,d\mu(t)\right)\circ W_s\,d\mu(s)\\
&= K(m,M)\int_\Omega W_t\,d\mu(t)\circ\int_\Omega W_t\,d\mu(t).
\end{aligned}$$

Therefore, we arrive at the desired inequality (3.5). □

Note that $K(m,M)$ is the ratio between the arithmetic mean and the geometric mean of m^2 and M^2. As a special case of Theorem 3.2, we obtain a discrete version of the integral inequality (3.5) as follows.

COROLLARY 3.3. *For each $i = 1,2,\ldots,n$, let $A_i \in B(\mathbb{H})$ be a selfadjoint operator such that $\mathrm{Sp}(A_i) \subseteq [m,M] \subseteq (0,\infty)$ and let W_i be a positive operator in $B(\mathbb{H})$. Then we have*

$$\sum_{i=1}^n W_i^{\frac{1}{2}} A_i W_i^{\frac{1}{2}} \circ \sum_{i=1}^n W_i^{\frac{1}{2}} A_i^{-1} W_i^{\frac{1}{2}} \leqslant K(m,M)\left(\sum_{i=1}^n W_i \circ \sum_{i=1}^n W_i\right). \tag{3.6}$$

Proof. Set μ to be the counting measure on $\Omega = \{1,2,\ldots,n\}$ in Theorem 3.2. □

This corollary is an operator extension of [17, Theorem 2.2]. The next result is an integral inequality of Kantorovich type in which the weights are scalars.

COROLLARY 3.4. *Assume Main hypothesis. Let $w : \Omega \to [0,\infty)$ be a continuous function. Then the following integral inequality holds*

$$\int_\Omega w(t)A_t\,d\mu(t)\circ\int_\Omega w(t)A_t^{-1}\,d\mu(t) \leqslant K(m,M)\left(\int_\Omega w(t)\,d\mu(t)\right)^2 I. \tag{3.7}$$

Proof. Set $W_t = w(t)I$ for each $t \in \Omega$ in Theorem 3.2. □

The following result is a discrete version of the inequality (3.7).

COROLLARY 3.5. *For each $i = 1,2,\ldots,n$, let $A_i \in B(\mathbb{H})$ be a selfadjoint operator such that $\mathrm{Sp}(A_i) \subseteq [m,M] \subseteq (0,\infty)$ and let $w_i \geqslant 0$ be a constant. Then*

$$\left(\sum_{i=1}^n w_i A_i\right)\circ\left(\sum_{i=1}^n w_i A_i^{-1}\right) \leqslant K(m,M)\left(\sum_{i=1}^n w_i\right)^2 I. \tag{3.8}$$

Proof. Set μ to be the counting measure on $\Omega = \{1,2,\ldots,n\}$ in Corollary 3.4. □

In this corollary, when the weight w_i is $1/n$ for each i, it holds that

$$\frac{1}{n}\left(A_1 + A_2 + \ldots + A_n\right) \circ \frac{1}{n}\left(A_1^{-1} + A_2^{-1} + \ldots + A_n^{-1}\right) \leqslant K(m,M)I. \tag{3.9}$$

The inequality (3.9) can be viewed as a reverse arithmetic-harmonic mean inequality for operators concerning Hadamard product.

Theorem 3.2 can be extended in the following way:

THEOREM 3.6. *Assume Main hypothesis. Let f be a continuous real-valued function defined on $[m,M] \cup [1/M,1/m]$ such that $f(x)f(1/x) \leqslant 1$ for all $x \in [m,M]$. If $f([m,M]) \subseteq [m,M]$ or $f([m,M]) \subseteq [1/M,1/m]$, then*

$$\begin{aligned} &\int_\Omega W_t^{\frac{1}{2}} f(A_t) W_t^{\frac{1}{2}}\, d\mu(t) \circ \int_\Omega W_t^{\frac{1}{2}} f(A_t^{-1}) W_t^{\frac{1}{2}}\, d\mu(t) \\ &\qquad \leqslant K(m,M)\left(\int_\Omega W_t\, d\mu(t) \circ \int_\Omega W_t\, d\mu(t)\right). \end{aligned} \tag{3.10}$$

Proof. The existence of the operators in the left hand of (3.10) follows from Proposition 2.1. Applying functional calculus yields $f(A_t^{-1}) \leqslant f(A_t)^{-1}$ for each $t \in \Omega$. The desired result is a direct consequence of Theorem 3.2 together with the monotonicity of the Bochner integration and the Hadamard multiplication with a fixed positive operator. Note that $K(m,M) = K(1/M,1/m)$. □

Theorem 3.6 is reduced to Theorem 3.2 by setting $f(x) = x$ or $f(x) = 1/x$.

4. Kantorovich integral inequalities involving Hadamard products and operator means

In this section, we establish integral analogues of Kantorovich inequality involving operator means. To begin with, recall some fundamental facts in Kubo-Ando theory of operator means [14]; see also [11, Section 3] and [12, Chapter 5].

An *operator connection* is a binary operation σ assigned to each pair of positive operators such that for all $A,B,C,D \geqslant 0$,

(M1) monotonicity: $A \leqslant C, B \leqslant D \implies A\,\sigma\,B \leqslant C\,\sigma\,D$

(M2) transformer inequality: $C(A\,\sigma\,B)C \leqslant (CAC)\,\sigma\,(CBC)$

(M3) upper semi-continuity: for $A_n, B_n \in B(\mathbb{H})^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n\,\sigma\,B_n \downarrow A\,\sigma\,B$. Here, $X_n \downarrow X$ indicates that (X_n) is a decreasing sequence converging strongly to X.

From these axioms, it holds that

$$X(A\,\sigma\,B)X = (XAX)\,\sigma\,(XBX), \tag{4.1}$$

$$(A+B)\,\sigma\,(C+D)\ \geqslant\ (A\,\sigma\,C)+(B\,\sigma\,D) \tag{4.2}$$

for any $A,B,C,D \geqslant 0$ and $X>0$.

An *operator mean* is an operator connection σ with idempotent property

$$A\,\sigma\,A\ =\ A, \qquad \text{for all } A \geqslant 0. \tag{4.3}$$

Classical examples of operator means are the arithmetic mean, the harmonic mean, the geometric mean and their weighted versions. For each $\alpha \in [0,1]$, we define the α-weighted geometric mean for strictly positive operators A and B as follows:

$$A\,\#_{\alpha}\,B\ =\ A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha}A^{\frac{1}{2}}. \tag{4.4}$$

A major core of Kubo-Ando theory is the one-to-one correspondence between operator connections and operator monotone functions. Recall (e.g. [5, Chapter V]) that a continuous function $f:[0,\infty)\to\mathbb{R}$ is said to be *operator monotone* if

$$A \leqslant B \implies f(A) \leqslant f(B)$$

holds for any positive operators A and B.

THEOREM 4.1. ([14, Theorem 3.4]) *Given an operator connection σ, there is a unique operator monotone function $f:[0,\infty)\to[0,\infty)$ such that*

$$f(A)\ =\ I\,\sigma\,A, \quad A \geqslant 0. \tag{4.5}$$

In fact, the map $\sigma \mapsto f$ is a bijection.

Such a function f is called the *representing function* of σ. For example, the representing function of $\#_{\alpha}$ is the operator monotone function $f(x)=x^{\alpha}$.

In order to prove the main result in this section, recall the following fact:

LEMMA 4.2. ([3]) *Let σ be an operator connection on $B(\mathbb{H})^{+}$. Then for all positive operators A and B, we have*

$$\|A\,\sigma\,B\|\ \leqslant\ \|A\|\,\sigma\,\|B\|.$$

Here, σ on the right-hand side of this inequality is the operator connection on $[0,\infty)$ defined by $(a\,\sigma\,b)I=(aI)\,\sigma\,(bI)$ for each $a,b \geqslant 0$.

We say that a function $f:E\to\mathbb{R}$ defined on a subset $E\subseteq\mathbb{R}$ is *super-multiplicative* if $f(xy)\geqslant f(x)f(y)$ for all $x,y\in E$.

LEMMA 4.3. *Let σ be an operator connection associated with an operator monotone function $f:[0,\infty)\to[0,\infty)$. If f is super-multiplicative, then*

$$(A\,\sigma\,C)\circ(B\,\sigma\,D) \leqslant (A\circ B)\,\sigma\,(C\circ D)$$

for all positive operators A,B,C,D in $B(\mathbb{H})$.

Proof. By continuity, we may assume that A and B are invertible. For convenience, write $X = A^{-\frac{1}{2}}CA^{-\frac{1}{2}}$ and $Y = B^{-\frac{1}{2}}DB^{-\frac{1}{2}}$. It follows from the the properties (4.1) and (4.5) that

$$\begin{aligned}(A\,\sigma\,C)\otimes(B\,\sigma\,D) &= (A\otimes B)^{\frac{1}{2}}[(I\,\sigma\,X)\otimes(I\,\sigma\,Y)](A\otimes B)^{\frac{1}{2}}\\ &= (A\otimes B)^{\frac{1}{2}}[f(X)\otimes f(Y)](A\otimes B)^{\frac{1}{2}}\\ &\leqslant (A\otimes B)^{\frac{1}{2}}[f(X\otimes Y)](A\otimes B)^{\frac{1}{2}}\\ &= (A\otimes B)^{\frac{1}{2}}[I\,\sigma\,(X\otimes Y)](A\otimes B)^{\frac{1}{2}}\\ &= (A\otimes B)\,\sigma\,(C\otimes D).\end{aligned}$$

Now, passing through the isometry U defined by (3.2), we have

$$\begin{aligned}(A\,\sigma\,C)\circ(B\,\sigma\,D) &= U^*[(A\,\sigma\,C)\otimes(B\,\sigma\,D)]U\\ &\leqslant U^*[(A\otimes B)\,\sigma\,(C\otimes D)]U\\ &\leqslant U^*(A\otimes B)U\,\sigma\,U^*(C\otimes D)U\\ &= (A\circ B)\,\sigma\,(C\circ D).\end{aligned}$$

Here, we use the transformer inequality (M2) in the second inequality. □

The following result can be regarded as a Kantorovich type integral inequality involving operator means.

THEOREM 4.4. *Assume Main hypothesis. Let $(B_t)_{t\in\Omega}$ be a continuous field of strictly positive operators in $B(\mathbb{H})$ such that $\mathrm{Sp}(B_t)\subseteq[m,M]\subseteq(0,\infty)$ for each $t\in\Omega$. Let σ be an operator mean with the super-multiplicative representing function. Then*

$$\begin{aligned}\int_\Omega W_t^{\frac{1}{2}}(A_t\,\sigma\,B_t)W_t^{\frac{1}{2}}\,d\mu(t)\circ\int_\Omega W_t^{\frac{1}{2}}(A_t^{-1}\,\sigma\,B_t^{-1})W_t^{\frac{1}{2}}\,d\mu(t)\\ \leqslant K(m,M)\left(\int_\Omega W_t\,d\mu(t)\circ\int_\Omega W_t\,d\mu(t)\right).\end{aligned}\tag{4.6}$$

Proof. The operator-valued function $t\mapsto W_t^{\frac{1}{2}}(A_t\,\sigma\,B_t)W_t^{\frac{1}{2}}$ is Bochner integrable due to the norm estimate in Lemma 4.2. It follows that

$$\begin{aligned}&\int_\Omega W_t^{\frac{1}{2}}(A_t\sigma B_t)W_t^{\frac{1}{2}}\,d\mu(t)\circ\int_\Omega W_t^{\frac{1}{2}}(A_t^{-1}\,\sigma\,B_t^{-1})W_t^{\frac{1}{2}}\,d\mu(t)\\ &\leqslant\int_\Omega\left(W_t^{\frac{1}{2}}A_tW_t^{\frac{1}{2}}\,\sigma\,W_t^{\frac{1}{2}}B_tW_t^{\frac{1}{2}}\right)d\mu(t)\circ\int_\Omega\left(W_t^{\frac{1}{2}}A_t^{-1}W_t^{\frac{1}{2}}\,\sigma\,W_t^{\frac{1}{2}}B_t^{-1}W_t^{\frac{1}{2}}\right)d\mu(t)\\ &\qquad\text{(since }\sigma\text{ satisfies the transformer inequality (M2))}\\ &\leqslant\left[\int_\Omega W_t^{\frac{1}{2}}A_tW_t^{\frac{1}{2}}\,d\mu(t)\,\sigma\int_\Omega W_t^{\frac{1}{2}}B_tW_t^{\frac{1}{2}}\,d\mu(t)\right]\\ &\qquad\circ\left[\int_\Omega W_t^{\frac{1}{2}}A_t^{-1}W_t^{\frac{1}{2}}\,d\mu(t)\,\sigma\int_\Omega W_t^{\frac{1}{2}}B_t^{-1}W_t^{\frac{1}{2}}\,d\mu(t)\right]\end{aligned}$$

(since σ satisfies property (4.2))

$$\leqslant \left[\int_\Omega W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}}\, d\mu(t) \circ \int_\Omega W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}}\, d\mu(t)\right]$$

$$\sigma \left[\int_\Omega W_t^{\frac{1}{2}} B_t W_t^{\frac{1}{2}}\, d\mu(t) \circ \int_\Omega W_t^{\frac{1}{2}} B_t^{-1} W_t^{\frac{1}{2}}\, d\mu(t)\right] \qquad \text{(by Lemma 4.3)}$$

$$\leqslant \left[K(m,M) \int_\Omega W_t\, d\mu(t) \circ \int_\Omega W_t\, d\mu(t)\right] \sigma \left[K(m,M) \int_\Omega W_t\, d\mu(t) \circ \int_\Omega W_t\, d\mu(t)\right]$$

(by Theorem 3.5)

$$= K(m,M) \int_\Omega W_t\, d\mu(t) \circ \int_\Omega W_t\, d\mu(t) \quad \text{(by property (4.3))}. \qquad \square$$

Theorem 4.4 can be reduced to Theorem 3.2 by setting $A_t = B_t$ for all $t \in \Omega$. The next result is a discrete version of inequality (4.6).

COROLLARY 4.5. *For each $i = 1,2,\ldots,n$, let $A_i, B_i, W_i \in B(\mathbb{H})$ be selfadjoint operators such that $\mathrm{Sp}(A_i), \mathrm{Sp}(B_i) \subseteq [m,M] \subseteq (0,\infty)$ and $W_i \geqslant 0$. Then we have*

$$\sum_{i=1}^{n} W_i^{\frac{1}{2}} (A_i \,\sigma\, B_i) W_i^{\frac{1}{2}} \circ \sum_{i=1}^{n} W_i^{\frac{1}{2}} (A_i^{-1} \,\sigma\, B_i^{-1}) W_i^{\frac{1}{2}} \leqslant K(m,M) \left(\sum_{i=1}^{n} W_i \circ \sum_{i=1}^{n} W_i\right). \tag{4.7}$$

The next result is also a special case of Theorem 4.4 in which an operator mean is specified.

COROLLARY 4.6. *Assume Main hypothesis. Then the following integral inequality holds for any $\alpha \in [0,1]$:*

$$\begin{aligned} &\int_\Omega W_t^{\frac{1}{2}} (A_t \,\#_\alpha\, B_t) W_t^{\frac{1}{2}}\, d\mu(t) \circ \int_\Omega W_t^{\frac{1}{2}} (A_t^{-1} \,\#_\alpha\, B_t^{-1}) W_t^{\frac{1}{2}}\, d\mu(t) \\ &\qquad \leqslant K(m,M) \left(\int_\Omega W_t\, d\mu(t) \circ \int_\Omega W_t\, d\mu(t)\right). \end{aligned} \tag{4.8}$$

Proof. From Theorem 4.4, set σ to be the α-weighted geometric mean $\#_\alpha$. Note that its representing function $f(x) = x^\alpha$ is super-multiplicative. $\square$

COROLLARY 4.7. *Assume Main hypothesis. Suppose that $1 \in [m,M]$. Then the following integral inequality holds for any $\alpha \in [-1,1]$:*

$$\begin{aligned} &\int_\Omega W_t^{\frac{1}{2}} A_t^{\alpha} W_t^{\frac{1}{2}}\, d\mu(t) \circ \int_\Omega W_t^{\frac{1}{2}} A_t^{-\alpha} W_t^{\frac{1}{2}}\, d\mu(t) \\ &\qquad \leqslant K(m,M) \left(\int_\Omega W_t\, d\mu(t) \circ \int_\Omega W_t\, d\mu(t)\right). \end{aligned} \tag{4.9}$$

Proof. Note that if $\mathrm{Sp}(A_t) \subseteq [m,M]$, then $\mathrm{Sp}(A_t^{-1}) \subseteq [1/M, 1/m]$ for each $t \in \Omega$. Replacing A_t with A_t^{-1} for each $t \in \Omega$ in (4.8) does not effect the constant $K(m,M)$. Hence, it suffices to consider only the case $\alpha \in [0,1]$. Indeed, replacing A_t and B_t with I and A_t in (4.8), respectively yields the inequality (4.9) via the formula (4.4). $\square$

5. Monotonicity of certain maps and Fiedler type inequalities for Hadamard products of operators

In this section, we consider the monotonicity of the Hadamard product map

$$\alpha \mapsto \int_\Omega W_t^{\frac{1}{2}} A_t^\alpha W_t^{\frac{1}{2}}\, d\mu(t) \circ \int_\Omega W_t^{\frac{1}{2}} A_t^{-\alpha} W_t^{\frac{1}{2}}\, d\mu(t)$$

where α is a constant. In particular, we get Fiedler type inequalities involving Hadamard products of operators.

The next lemma is a preparation.

LEMMA 5.1. *For each $A > 0$, the map $\alpha \mapsto A^\alpha + A^{-\alpha}$ is increasing on $[0,\infty)$, decreasing on $(-\infty,0]$ and attaining its minimum at $\alpha = 0$.*

Proof. For each fixed $t > 0$, consider the function $f(\alpha) = t^\alpha + t^{-\alpha}$. Differential analysis tells us that f is increasing on $[0,\infty)$, decreasing on $(-\infty,0]$ and attaining its minimum at $\alpha = 0$. Now, the claim follows by spectral mapping theorem. □

THEOREM 5.2. *Under Main hypothesis, the map*

$$\alpha \mapsto \int_\Omega W_t^{\frac{1}{2}} A_t^\alpha W_t^{\frac{1}{2}}\, d\mu(t) \circ \int_\Omega W_t^{\frac{1}{2}} A_t^{-\alpha} W_t^{\frac{1}{2}}\, d\mu(t) \tag{5.1}$$

is increasing on $[0,\infty)$, decreasing on $(-\infty,0]$ and attaining its minimum at $\alpha = 0$.

Proof. Proposition 2.1 allows the Bochner integralability of the map

$$t \mapsto W_t^{\frac{1}{2}} A_t^\lambda W_t^{\frac{1}{2}}$$

for any $\lambda \in \mathbb{R}$. It is enough to prove only that the map (5.1) is increasing on $[0,\infty)$. Consider $0 \leqslant \alpha < \beta$. It follows from the properties (3.3) and (3.1) that

$$\begin{aligned}
&\int_\Omega W_t^{\frac{1}{2}} A_t^\alpha W_t^{\frac{1}{2}}\, d\mu(t) \circ \int_\Omega W_t^{\frac{1}{2}} A_t^{-\alpha} W_t^{\frac{1}{2}}\, d\mu(t) \\
&= \iint_{\Omega^2} (W_t^{\frac{1}{2}} A_t^\alpha W_t^{\frac{1}{2}} \circ W_s^{\frac{1}{2}} A_s^{-\alpha} W_s^{\frac{1}{2}})\, d\mu(t)\, d\mu(s) \\
&= \frac{1}{2} \iint_{\Omega^2} (W_t^{\frac{1}{2}} A_t^\alpha W_t^{\frac{1}{2}} \circ W_s^{\frac{1}{2}} A_s^{-\alpha} W_s^{\frac{1}{2}}) + (W_t^{\frac{1}{2}} A_t^{-\alpha} W_t^{\frac{1}{2}} \circ W_s^{\frac{1}{2}} A_s^{\alpha} W_s^{\frac{1}{2}})\, d\mu(t)\, d\mu(s) \\
&= \frac{1}{2} \iint_{\Omega^2} U^* \left[(W_t^{\frac{1}{2}} A_t^\alpha W_t^{\frac{1}{2}} \otimes W_s^{\frac{1}{2}} A_s^{-\alpha} W_s^{\frac{1}{2}}) + (W_t^{\frac{1}{2}} A_t^{-\alpha} W_t^{\frac{1}{2}} \otimes W_s^{\frac{1}{2}} A_s^{\alpha} W_s^{\frac{1}{2}}) \right] U\, d\mu(t)\, d\mu(s) \\
&= \frac{1}{2} \iint_{\Omega^2} U^* (W_t \otimes W_s)^{\frac{1}{2}} \left[(A_t \otimes A_s^{-1})^\alpha + (A_t \otimes A_s^{-1})^{-\alpha} \right] (W_t \otimes W_s)^{\frac{1}{2}} U\, d\mu(t)\, d\mu(s).
\end{aligned}$$

Making use of Lemma 5.1 yields

$$\begin{aligned}
&\int_{\Omega} W_t^{\frac{1}{2}} A_t^{\alpha} W_t^{\frac{1}{2}}\, d\mu(t) \circ \int_{\Omega} W_t^{\frac{1}{2}} A_t^{-\alpha} W_t^{\frac{1}{2}}\, d\mu(t)\\
&\leqslant \frac{1}{2}\iint_{\Omega^2} U^*(W_t\otimes W_s)^{\frac{1}{2}}\left[(A_t\otimes A_s^{-1})^{\beta}+(A_t\otimes A_s^{-1})^{-\beta}\right](W_t\otimes W_s)^{\frac{1}{2}}U\, d\mu(t)\, d\mu(s)\\
&= \frac{1}{2}\iint_{\Omega^2} U^*\left[(W_t^{\frac{1}{2}}A_t^{\beta}W_t^{\frac{1}{2}}\otimes W_s^{\frac{1}{2}}A_s^{-\beta}W_s^{\frac{1}{2}})+(W_t^{\frac{1}{2}}A_t^{-\beta}W_t^{\frac{1}{2}}\otimes W_s^{\frac{1}{2}}A_s^{\beta}W_s^{\frac{1}{2}})\right]U\, d\mu(t)\, d\mu(s)\\
&= \iint_{\Omega^2} (W_t^{\frac{1}{2}}A_t^{\beta}W_t^{\frac{1}{2}}\circ W_s^{\frac{1}{2}}A_s^{-\beta}W_s^{\frac{1}{2}})\, d\mu(t)\, d\mu(s)\\
&= \int_{\Omega} W_t^{\frac{1}{2}}A_t^{\beta}W_t^{\frac{1}{2}}\, d\mu(t) \circ \int_{\Omega} W_t^{\frac{1}{2}}A_t^{-\beta}W_t^{\frac{1}{2}}\, d\mu(t). \quad \square
\end{aligned}$$

As a special case, we obtain a discrete version of Theorem 5.2 as follows.

COROLLARY 5.3. *For each $i=1,2,\ldots,n$, let $A_i>0$ and $W_i\geqslant 0$. Then the map*

$$\alpha \mapsto \sum_{i=1}^{n} W_i^{\frac{1}{2}}A_i^{\alpha}W_i^{\frac{1}{2}} \circ \sum_{i=1}^{n} W_i^{\frac{1}{2}}A_i^{-\alpha}W_i^{\frac{1}{2}}$$

is increasing on $[0,\infty)$, decreasing on $(-\infty,0]$ and attaining its minimum at $\alpha=0$.

This corollary is an operator extension of [17, Theorem 2.5]. The case $n=1$ in Corollary 5.3 says that the map

$$\alpha \mapsto A^{\alpha}\circ A^{-\alpha}$$

is increasing on $[0,\infty)$, decreasing on $(-\infty,0]$ and attaining its minimum at $\alpha=0$.

The next result is also a special case of Theorem 5.2 in which the weights are scalars.

COROLLARY 5.4. *Assume Main hypothesis. For any continuous function $w:\Omega\to[0,\infty)$, the map*

$$\alpha \mapsto \int_{\Omega} w(t)A_t^{\alpha}\, d\mu(t)\circ \int_{\Omega} w(t)A_t^{-\alpha}\, d\mu(t) \tag{5.2}$$

is increasing on $[0,\infty)$, decreasing on $(-\infty,0]$ and attaining its minimum at $\alpha=0$.

Proof. Set $W_t=w(t)I$ for all $t\in\Omega$ in Theorem 5.2. We see that $(W_t)_{t\in\Omega}$ is a continuous field of operators. $\square$

The minimality condition of the map (5.1) leads to the following Fiedler type integral inequality:

COROLLARY 5.5. *Assume Main hypothesis. Suppose that $\int_{\Omega} W_t\, d\mu(t)=I$. For each real number α, we have*

$$\left(\int_{\Omega} W_t^{\frac{1}{2}}A_t^{\alpha}W_t^{\frac{1}{2}}\, d\mu(t)\right)\circ\left(\int_{\Omega} W_t^{\frac{1}{2}}A_t^{-\alpha}W_t^{\frac{1}{2}}\, d\mu(t)\right)\geqslant I. \tag{5.3}$$

The final result is an operator version of Fiedler's inequality (1.3).

COROLLARY 5.6. *For each strictly positive operator A, we have*

$$A \circ A^{-1} \geqslant I.$$

Proof. In Corollary 5.5, set $W_t = I$ and $A_t = A$ for each $t \in \Omega$ and put μ to be a probability measure on Ω. □

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