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FRACTIONAL INTEGRALS ON B_σ–WEIGHTED MORREY SPACES

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Abstract. By using B_σ-weighted function spaces, we will investigate the weighted estimates of fractional integrals on B_σ-weighted Morrey spaces, which unify the weighted estimates of them on several function spaces.

1. Introduction

In [8] (cf. [6]), we introduced B_σ-Morrey-Campanato spaces, in order to unify central Morrey spaces, λ-central mean oscillation spaces and usual Morrey-Campanato spaces, and to unify the results of the boundedness of fractional integrals on several classical function spaces because Morrey and Campanato spaces contain L^p, BMO and Lipschitz spaces as particular cases. As a result, for the fractional integrals I_α and the modified fractional integrals $\tilde{I}_\alpha$, $0<\alpha<n$, which are defined in Definition 9 below, the following boundedness results were showed:

(i) For $0<\alpha<n$, $1<p<\infty$, $-n/p \leqslant \lambda < 0$, $-n/q \leqslant \mu = \lambda+\alpha < 0$, $0 \leqslant \sigma < \infty$, $1/q = 1/p - \alpha/n$, $(-\mu)q = (-\lambda)p$ and $\mu+\sigma<0$,

$$I_\alpha : B_\sigma(L_{p,\lambda})(\mathbb{R}^n) \to B_\sigma(L_{q,\mu})(\mathbb{R}^n);$$

(ii) For $0<\alpha<n$, $1<p<\infty$, $-n/p \leqslant \lambda < 0$, $0 \leqslant \mu = \lambda+\alpha < 1$, $0 \leqslant \sigma < \infty$ and $\mu+\sigma<1$,

$$\tilde{I}_\alpha : B_\sigma(L_{p,\lambda})(\mathbb{R}^n) \to B_\sigma(\mathrm{Lip}_\mu)(\mathbb{R}^n).$$

As for the definition of B_σ-functioh spaces, see Definition 3 below. In these boundedness results, (i) unifies

$$I_\alpha : L^{p,\lambda}(\mathbb{R}^n) \to L^{q,\mu}(\mathbb{R}^n) \qquad \text{(see [11])}$$

and

$$I_\alpha : B^{p,\lambda}(\mathbb{R}^n) \to B^{q,\mu}(\mathbb{R}^n) \qquad \text{(see [1])},$$

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Paper MIA-19-71

and (ii) unifies

$$\tilde{I}_\alpha : L^{p,\lambda}(\mathbb{R}^n) \to \mathrm{Lip}_\mu(\mathbb{R}^n) \qquad \text{(see [10])}$$

and

$$\tilde{I}_\alpha : B^{p,\lambda}(\mathbb{R}^n) \to \mathrm{Lip}_{\alpha-n/p,\sigma}(\mathbb{R}^n) \qquad \text{(see [5], [7] and [8])}.$$

Here, note that $L^{p,\lambda}(\mathbb{R}^n)$, $B^{p,\lambda}(\mathbb{R}^n)$, $\mathrm{Lip}_\beta(\mathbb{R}^n)$ and $\mathrm{Lip}_{\beta,\sigma}(\mathbb{R}^n)$ stand for Morrey space, (non-homogeneous) central Morrey space, Lipschitz space and σ-Lipschitz space, respectively.

In this paper, we will consider the weighted estimates of fractional integrals on B_σ-Morrey spaces and unify the results of the boundedness on several weighted function spaces.

2. Notation and definitions

We start by explaining necessary notation. For $r > 0$, by B_r, we mean that $B_r = \{y \in \mathbb{R}^n : |y| < r\}$. And for $x \in \mathbb{R}^n$, we set $B(x,r) = x + B_r = \{x+y : y \in B_r\}$. For a nonnegative measurable function w and a measurable set $G \subset \mathbb{R}^n$, let

$$w(G) = \int_G w(y)\,dy.$$

For a measurable set $G \subset \mathbb{R}^n$, we denote the Lebesgue measure of G by $|G|$ and the characteristic function of G by χ_G. And also, for a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and a measurable set $G \subset \mathbb{R}^n$ with $|G| > 0$, let

$$m_G(f) = \fint_G f(y)\,dy = \frac{1}{|G|}\int_G f(y)\,dy.$$

Further, for $1 < p < \infty$, let p' be the dual exponent of p, i.e., $p' = p/(p-1)$.

Now, we state the definitions of several weighted function spaces by means of weight, which is a nonnegative locally integrable function on $\mathbb{R}^n$ that takes values in $(0,\infty)$ almost everywhere. First we define the weighted Morrey, Lipschitz and BMO spaces on U, where $U = \mathbb{R}^n$ or $U = B_r$ with $r > 0$ as follows:

DEFINITION 1. Let w be a weight and $U = \mathbb{R}^n$ or $U = B_r$ with $r > 0$. For $1 \leqslant p < \infty$, $-\infty < \lambda < \infty$ and $0 \leqslant \beta < 1$, let $L_{p,\lambda}(w)(U)$ and $\mathrm{Lip}_\beta(w)(U)$ be the sets of all functions f on U such that the following functionals are finite:

$$\|f\|_{L_{p,\lambda}(w)(U)} = \sup_{B(x_0,s)\subset U} \frac{1}{s^\lambda}\left(\fint_{B(x_0,s)} |f(y)|^p w(y)\,dy\right)^{1/p},$$

$$\|f\|_{\mathrm{Lip}_\beta(w)(U)} = \sup_{B(x_0,s)\subset U} \inf_{c\in\mathbb{R}} \frac{1}{s^\beta}\frac{1}{w(B(x_0,s))}\int_{B(x_0,s)} |f(y)-c|\,dy.$$

In particular, let $\mathrm{BMO}(w)(U) = \mathrm{Lip}_0(w)(U)$ and

$$\|f\|_{\mathrm{BMO}(w)(U)} = \sup_{B(x_0,s)\subset U} \inf_{c\in\mathbb{R}} \frac{1}{w(B(x_0,s))}\int_{B(x_0,s)} |f(y)-c|\,dy.$$

REMARK 1. Let $E(U)$ be a function space on $U \subset \mathbb{R}^n$ with semi norm $\|\cdot\|_{E(U)}$. In this paper, when $U = \mathbb{R}^n$, we abbreviate $\|f\|_{E(\mathbb{R}^n)}$ to $\|f\|_E$ for the symbol of the norm, while we don't abbreviate $E(\mathbb{R}^n)$ for the symbol of the function space.

Second we define the weighted (non-homogeneous) central Morrey, σ-Lipschitz and σ-BMO spaces as follows:

DEFINITION 2. Let w be a weight. For $1 \leqslant p < \infty$, $-\infty < \lambda < \infty$, $0 \leqslant \beta < 1$ and $0 \leqslant \sigma < \infty$, let $B^{p,\lambda}(w)(\mathbb{R}^n)$ and $\mathrm{Lip}_{\beta,\sigma}(w)(\mathbb{R}^n)$ be the sets of all functions f on $\mathbb{R}^n$ such that the following functionals are finite:

$$\|f\|_{B^{p,\lambda}(w)} = \sup_{r \geqslant 1} \frac{1}{r^\lambda} \left(\fint_{B_r} |f(y)|^p w(y)\, dy \right)^{1/p},$$

$$\|f\|_{\mathrm{Lip}_{\beta,\sigma}(w)} = \sup_{r \geqslant 1} \frac{1}{r^\sigma} \|f\|_{\mathrm{Lip}_\beta(w)(B_r)}.$$

In particular, let $\mathrm{BMO}_\sigma(w)(\mathbb{R}^n) = \mathrm{Lip}_{0,\sigma}(w)(\mathbb{R}^n)$ and

$$\|f\|_{\mathrm{BMO}_\sigma(w)} = \sup_{r \geqslant 1} \frac{1}{r^\sigma} \|f\|_{\mathrm{BMO}(w)(B_r)}.$$

Further, we recall the definition of (non-homogeneous) B_σ-function space. For details, see [8] (cf. [6]).

DEFINITION 3. For each $r > 0$, let $E(B_r)$ be a function space on B_r with semi norm $\|\cdot\|_{E(B_r)}$. Then, for $0 \leqslant \sigma < \infty$, we define a function space $B_\sigma(E)(\mathbb{R}^n)$ as the set of all functions f on $\mathbb{R}^n$ such that $\|f\|_{B_\sigma(E)} < \infty$, where

$$\|f\|_{B_\sigma(E)} = \sup_{r \geqslant 1} \frac{1}{r^\sigma} \|f\|_{E(B_r)}.$$

REMARK 2. We note that $B_\sigma(L_{p,\lambda})(\mathbb{R}^n)$ unifies $L_{p,\lambda}(\mathbb{R}^n)$ and $B^{p,\lambda}(\mathbb{R}^n)$ as follows:

$$B_0(L_{p,\lambda}(w))(\mathbb{R}^n) = L_{p,\lambda}(w)(\mathbb{R}^n), \quad B_{\lambda+n/p}(L_{p,-n/p}(w))(\mathbb{R}^n) = B^{p,\lambda}(w)(\mathbb{R}^n).$$

And we also note that

$$B_0(\mathrm{Lip}_\beta(w))(\mathbb{R}^n) = \mathrm{Lip}_\beta(w)(\mathbb{R}^n), \quad B_\sigma(\mathrm{Lip}_\beta(w))(\mathbb{R}^n) = \mathrm{Lip}_{\beta,\sigma}(w)(\mathbb{R}^n)$$

and

$$B_0(\mathrm{BMO}(w))(\mathbb{R}^n) = \mathrm{BMO}(w)(\mathbb{R}^n), \quad B_\sigma(\mathrm{BMO}(w))(\mathbb{R}^n) = \mathrm{BMO}_\sigma(w)(\mathbb{R}^n).$$

We also need to define some classes of weights. As to the basic properties of weights, see, e.g., [2] and [3].

DEFINITION 4. For $1 < p < \infty$, let A_p be the set of all weights w such that

$$\sup_{\mathbb{R}^n \supset B:\text{ball}} m_B(w)\, m_B(w^{-1/(p-1)})^{p-1} < \infty.$$

DEFINITION 5. Let A_1 be the set of all weights w such that for any ball $B \subset \mathbb{R}^n$,

$$\frac{w(B)}{|B|} \leqslant Cw(x) \quad \text{a.e. } x \in B$$

for some constant C.

DEFINITION 6. For $1 < p, q < \infty$, let $A_{p,q}$ be the set of all weights w such that

$$\sup_{\mathbb{R}^n \supset B:\text{ball}} m_B(w^q)^{1/q}\, m_B(w^{-p'})^{1/p'} < \infty.$$

REMARK 3. Note that $w \in A_{p,q}$ if and only if $w^{-p'} \in A_{1+p'/q}$.

DEFINITION 7. For $1 < p < \infty$, let $A_{p,\infty}$ be the set of all weights w such that $w^{-p'} \in A_1$.

REMARK 4. Note that for $1 < p < \infty$ and $1 < q \leqslant \tilde{q} \leqslant \infty$,

$$A_{p,\infty} \subset A_{p,\tilde{q}} \subset A_{p,q},$$

and for $1 < \tilde{p} \leqslant p < \infty$ and $1 < q \leqslant \infty$,

$$A_{\tilde{p},q} \subset A_{p,q}.$$

DEFINITION 8. (strong doubling) For $0 < a < \infty$, let $SD(a)$ be the set of all weights w such that for all balls B and Q provided $B \subset Q$, there exists a constant $C > 0$ such that

$$w(Q) \leqslant C \left(\frac{|Q|}{|B|} \right)^a w(B).$$

REMARK 5. Note that if $w \in A_p$, $1 \leqslant p < \infty$, then $w \in SD(p)$. Moreover, if $w \in A_p$, $1 < p < \infty$, then by self-improvement property of A_p, $w \in SD(a)$ for some $a < p$.

3. Fractional integrals on $B_\sigma(L_{p,\lambda}(w^p))(\mathbb{R}^n)$

In this section we consider the B_σ-weighted function spaces estimates of fractional integral I_α and modified fractional integral $\tilde{I}_\alpha$, $0 < \alpha < n$, which are defined by the following.

DEFINITION 9. For $0<\alpha<n$ and $f\in L^1_{\rm loc}(\mathbb{R}^n)$,

$$I_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}\,dy$$

and

$$\tilde{I}_\alpha f(x)=\int_{\mathbb{R}^n}f(y)\left(\frac{1}{|x-y|^{n-\alpha}}-\frac{1-\chi_{B_1}(y)}{|y|^{n-\alpha}}\right)dy.$$

Then for a fractional integral I_α, the following estimates on the weighted L^p and Morrey spaces are known.

THEOREM A. ([9]) *Let $0<\alpha<n$, $1<p<n/\alpha$ and $1/q=1/p-\alpha/n$. If $w\in A_{p,q}$, then I_α is bounded from $L^p(w^p)(\mathbb{R}^n)$ to $L^q(w^q)(\mathbb{R}^n)$, that is, there exists a constant $C>0$ such that*

$$\|I_\alpha f\|_{L^q(w^q)}\leqslant C\|f\|_{L^p(w^p)},\quad f\in L^p(w^p)(\mathbb{R}^n),$$

where $L^p(w^p)(\mathbb{R}^n)$ is the set of all functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p(w^p)}=\left(\int_{\mathbb{R}^n}|f(y)|^p w^p(y)\,dy\right)^{1/p}<\infty.$$

THEOREM B. ([4]) *Let $0<\alpha<n$, $1<p,q<\infty$, $-n/p\leqslant\lambda<0$, $-n/q\leqslant\mu=\lambda+\alpha<0$, $1/q=1/p-\alpha/n$ and $(-\mu)q=(-\lambda)p$. If $w\in A_{p,-n/\mu}$, then I_α is bounded from $L_{p,\lambda}(w^p)(\mathbb{R}^n)$ to $L_{q,\mu}(w^q)(\mathbb{R}^n)$, that is, there exists a constant $C>0$ such that*

$$\|I_\alpha f\|_{L_{q,\mu}(w^q)}\leqslant C\|f\|_{L_{p,\lambda}(w^p)},\quad f\in L_{p,\lambda}(w^p)(\mathbb{R}^n).$$

On the other hand, for a fractional integral I_α, we can prove the following B_σ-weighted Morrey estimate.

THEOREM 1. *Let $0<\alpha<n$, $1<p,q<\infty$, $-n/p\leqslant\lambda<0$, $-n/q\leqslant\mu=\lambda+\alpha<0$, $0\leqslant\sigma<\infty$, $1/q=1/p-\alpha/n$, $(-\mu)q=(-\lambda)p$ and $\mu+\sigma<0$. If $w\in A_{p,-n/(\mu+\sigma)}$, then I_α is bounded from $B_\sigma(L_{p,\lambda}(w^p))(\mathbb{R}^n)$ to $B_\sigma(L_{q,\mu}(w^q))(\mathbb{R}^n)$, that is, there exists a constant $C>0$ such that*

$$\|I_\alpha f\|_{B_\sigma(L_{q,\mu}(w^q))}\leqslant C\|f\|_{B_\sigma(L_{p,\lambda}(w^p))},\quad f\in B_\sigma(L_{p,\lambda}(w^p))(\mathbb{R}^n).$$

REMARK 6. Note that Theorem 1 with $\sigma=0$ means Theorem B.

In the above theorem, if $\lambda=-n/p$ and $\sigma=\lambda+n/p$, then the following corollary is obtained by Remark 2.

COROLLARY 1. *Let $0<\alpha<n$, $1<p,q<\infty$, $-n/p\leqslant\lambda<0$, $-n/q\leqslant\mu=\lambda+\alpha<0$, $1/q=1/p-\alpha/n$, $(-\mu)q=(-\lambda)p$, $\sigma=\lambda+n/p$ and $\mu+\sigma<0$. If $w\in A_{p,-n/(\mu+\sigma)}$, then I_α is bounded from $B^{p,\lambda}(w^p)(\mathbb{R}^n)$ to $B^{q,\mu}(w^q)(\mathbb{R}^n)$, that is, there exists a constant $C>0$ such that*

$$\|I_\alpha f\|_{B^{q,\mu}(w^q)}\leqslant C\|f\|_{B^{p,\lambda}(w^p)},\quad f\in B^{p,\lambda}(w^p)(\mathbb{R}^n).$$

Moreover, in Theorem 1, if we put $\mu = 0$ formally, then the condition of w becomes $w \in A_{p,\infty}$. Therefore, for a modified fractional integral $\tilde{I}_\alpha$, we can show the following B_σ-weighted function spaces estimate on the weighted Morrey spaces.

THEOREM 2. *Let $0 < \alpha < n$, $1 < p < \infty$, $-n/p \leqslant \lambda < 0$, $0 \leqslant \mu = \lambda + \alpha < 1$, $0 \leqslant \sigma < \infty$ and $\mu + \sigma < 1$. If*

$$w \in \begin{cases} A_{p,\infty} & (1 < p < n/\alpha) \\ A_{n/\alpha,\infty} & (n/\alpha \leqslant p < \infty), \end{cases}$$

then $\tilde{I}_\alpha$ is bounded from $B_\sigma(L_{p,\lambda}(w^p))(\mathbb{R}^n)$ to $B_\sigma(\mathrm{Lip}_\mu(w^{-1}))(\mathbb{R}^n)$, that is, there exists a constant $C > 0$ such that

$$\|\tilde{I}_\alpha f\|_{B_\sigma(\mathrm{Lip}_\mu(w^{-1}))} \leqslant C\|f\|_{B_\sigma(L_{p,\lambda}(w^p))}, \quad f \in B_\sigma(L_{p,\lambda}(w^p))(\mathbb{R}^n).$$

In particular, when $\mu = 0$, i.e., $\lambda = -\alpha$,

$$\|\tilde{I}_\alpha f\|_{B_\sigma(\mathrm{BMO}(w^{-1}))} \leqslant C\|f\|_{B_\sigma(L_{p,-\alpha}(w^p))}, \quad f \in B_\sigma(L_{p,-\alpha}(w^p))(\mathbb{R}^n),$$

where $1 < p \leqslant n/\alpha$.

In the above theorem, if $\sigma = 0$, then we obtaine the following corollary by Remark 2.

COROLLARY 2. *Let $0 < \alpha < n$, $1 < p < \infty$, $-n/p \leqslant \lambda < 0$ and $0 \leqslant \mu = \lambda + \alpha < 1$. If*

$$w \in \begin{cases} A_{p,\infty} & (1 < p < n/\alpha) \\ A_{n/\alpha,\infty} & (n/\alpha \leqslant p < \infty), \end{cases}$$

then $\tilde{I}_\alpha$ is bounded from $L_{p,\lambda}(w^p)(\mathbb{R}^n)$ to $\mathrm{Lip}_\mu(w^{-1})(\mathbb{R}^n)$, that is, there exists a constant $C > 0$ such that

$$\|\tilde{I}_\alpha f\|_{\mathrm{Lip}_\mu(w^{-1})} \leqslant C\|f\|_{L_{p,\lambda}(w^p)}, \quad f \in L_{p,\lambda}(w^p)(\mathbb{R}^n).$$

In particular, when $\mu = 0$, i.e., $\lambda = -\alpha$,

$$\|\tilde{I}_\alpha f\|_{\mathrm{BMO}(w^{-1})} \leqslant C\|f\|_{L_{p,-\alpha}(w^p)}, \quad f \in L_{p,-\alpha}(w^p)(\mathbb{R}^n),$$

where $1 < p \leqslant n/\alpha$.

Also if $\lambda = -n/p$ and $\sigma = \lambda + n/p$ in Theorem 2, then by Remark 2, the following corollary is obtained.

COROLLARY 3. *Let $0 < \alpha < n$, $1 < p < \infty$, $-n/p \leqslant \lambda < 0$, $0 \leqslant \mu = \lambda + \alpha < 1$, $\sigma = \lambda + n/p$ and $\mu + \sigma < 1$. If*

$$w \in \begin{cases} A_{p,\infty} & (1 < p < n/\alpha) \\ A_{n/\alpha,\infty} & (n/\alpha \leqslant p < \infty), \end{cases}$$

then $\tilde{I}_\alpha$ *is bounded from* $B^{p,\lambda}(w^p)(\mathbb{R}^n)$ *to* $\mathrm{Lip}_{\mu,\sigma}(w^{-1})(\mathbb{R}^n)$*, that is, there exists a constant* $C>0$ *such that*

$$\|\tilde{I}_\alpha f\|_{\mathrm{Lip}_{\mu,\sigma}(w^{-1})} \leqslant C\|f\|_{B^{p,\lambda}(w^p)}, \quad f\in B^{p,\lambda}(w^p)(\mathbb{R}^n).$$

In particular, when $\mu=0$*, i.e.,* $\lambda=-\alpha$*,*

$$\|\tilde{I}_\alpha f\|_{\mathrm{BMO}_\sigma(w^{-1})} \leqslant C\|f\|_{B^{p,-\alpha}(w^p)}, \quad f\in B^{p,-\alpha}(w^p)(\mathbb{R}^n),$$

where $1<p\leqslant n/\alpha$*.*

4. Proofs of theorems

In what follows, we use the symbol $A\lesssim B$ to denote that there exists a constant $C>0$ such that $A\leqslant CB$. If $A\lesssim B$ and $B\lesssim A$, we then write $A\sim B$.

First of all, we show the following lemma, which is necessary to prove Theorems 1 and 2.

LEMMA 1. *Let* $\beta\in\mathbb{R}$*,* $1<p<\infty$*,* $-n/p\leqslant\lambda<0$*,* $0\leqslant\sigma<\infty$ *and* $\beta+\lambda+\sigma<0$*. If* $w\in A_{p,-n/(\beta+\lambda+\sigma)}$*, then there exists a constant* $C>0$ *such that*

$$\int_{\mathbb{R}^n\setminus B(x_0,s)} \frac{|f(y)|}{|y-x|^{n-\beta}}\,dy \leqslant Cr^\sigma s^{-n/p'+\beta+\lambda} w^{-p'}(B(x_0,s))^{1/p'}\|f\|_{B_\sigma(L_{p,\lambda}(w^p))}$$

$$\text{for all } f\in B_\sigma(L_{p,\lambda}(w^p))(\mathbb{R}^n),\ x\in B(x_0,s)\subset B_r \text{ and } r\geqslant 1.$$

In particular, when $x=x_0=0$ *and* $s=r$*,*

$$\int_{\mathbb{R}^n\setminus B_r} \frac{|f(y)|}{|y|^{n-\beta}}\,dy \leqslant Cr^{-n/p'+\beta+\lambda+\sigma} w^{-p'}(B_r)^{1/p'}\|f\|_{B_\sigma(L_{p,\lambda}(w^p))}$$

$$\text{for all } f\in B_\sigma(L_{p,\lambda}(w^p))(\mathbb{R}^n) \text{ and } r\geqslant 1.$$

Proof.

Let $f\in B^\sigma(L_{p,\lambda}(w^p))(\mathbb{R}^n)$, $B(x_0,s)\subset B_r$ and $r\geqslant 1$. Then, by Hölder's inequality, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n\setminus B(x_0,s)} &\frac{|f(y)|}{|y-x|^{n-\beta}}\,dy \\
&= \sum_{j=0}^{\infty}\int_{B(x_0,2^{j+1}s)\setminus B(x_0,2^js)} \frac{|f(y)|}{|y-x|^{n-\beta}}\,dy \\
&\lesssim \sum_{j=0}^{\infty}\frac{1}{(2^js)^{n-\beta}}\int_{B(x_0,2^{j+1}s)\setminus B(x_0,2^js)} |f(y)|\,dy \\
&\lesssim s^\beta\sum_{j=0}^{\infty}(2^\beta)^j \fint_{B(x_0,2^{j+1}s)} |f(y)|\,dy
\end{aligned}$$

$$\lesssim s^{\beta}\sum_{j=0}^{\infty}(2^{\beta})^{j}\left(\fint_{B(x_0,2^{j+1}s)}|f(y)|^{p}w^{p}(y)\,dy\right)^{1/p}\left(\fint_{B(x_0,2^{j+1}s)}w^{-p'}(y)\,dy\right)^{1/p'}$$

$$\sim s^{\beta}\sum_{j=0}^{\infty}(2^{\beta})^{j}(2^{j+1}s)^{-n/p'+\lambda}w^{-p'}(B(x_0,2^{j+1}s))^{1/p'}\|f\|_{L_{p,\lambda}(w^{p})(B_{2^{j+1}r})}$$

$$\lesssim r^{\sigma}s^{-n/p'+\beta+\lambda}\sum_{j=0}^{\infty}(2^{-n/p'+\beta+\lambda+\sigma})^{j}w^{-p'}(B(x_0,2^{j+1}s))^{1/p'}\|f\|_{B_{\sigma}(L_{p,\lambda}(w^{p}))}.$$

As it follows from Remarks 3 and 5 that when $w\in A_{p,-n/(\beta+\lambda+\sigma)}$, there exist $C>0$ and $\varepsilon>0$ such that

$$w^{-p'}(B(x_0,2^{j+1}s))\leqslant C(2^{j+1})^{n-p'(\beta+\lambda+\sigma)-\varepsilon}w^{-p'}(B(x_0,s)),$$

we have

$$\int_{\mathbb{R}^n\setminus B(x_0,s)}\frac{|f(y)|}{|y-x|^{n-\beta}}\,dy$$

$$\lesssim r^{\sigma}s^{-n/p'+\beta+\lambda}\left\{\sum_{j=0}^{\infty}(2^{-\varepsilon/p'})^{j}\right\}w^{-p'}(B(x_0,s))^{1/p'}\|f\|_{B_{\sigma}(L_{p,\lambda}(w^{p}))}$$

$$\sim r^{\sigma}s^{-n/p'+\beta+\lambda}w^{-p'}(B(x_0,s))^{1/p'}\|f\|_{B_{\sigma}(L_{p,\lambda}(w^{p}))}. \qquad \square$$

Next we recall the following two lemmas.

LEMMA 2. (Lemma 3.5 of [8]) *Let* $1\leqslant p<\infty$ *and* $r>0$*. If* $-n/p\leqslant\lambda<0$*, then*

$$\|f\chi_{B_r}\|_{L_{p,\lambda}}\leqslant\|f\|_{L_{p,\lambda}(B_{3r})}\qquad \text{for all } f\in L^{p}_{loc}(\mathbb{R}^n) \text{ with } \|f\|_{L_{p,\lambda}(B_{3r})}<\infty.$$

LEMMA 3. (Theorem 7.2.5 of [2]) *If* $w\in A_p$ *with* $1\leqslant p<\infty$*, then there exists an* $a>1$ *such that* $w^{a}\in A_p$.

Finally, we prepare the following lemma in order to prove the case of $n/\alpha\leqslant p<\infty$ of Theorem 2.

LEMMA 4. *Let* $0<\alpha<n$, $n/\alpha\leqslant p<\infty$ *and* $w\in A_{n/\alpha,\infty}$*. Then there exists an exponent* p_0*, which is sufficiently close to* n/α*, such that* $1<p_0<n/\alpha$, $1/p_0-\alpha/n<1/p$ *and* $w\in A_{p_0,\infty}$.

Proof. Let $n/\alpha\leqslant p<\infty$ and $w\in A_{n/\alpha,\infty}$, i.e., $w^{-(n/\alpha)'}\in A_1$. Then, by Lemma 3, there exists an $a>1$ such that $w^{-(n/\alpha)'a}\in A_1$. Now, if we put $\tilde{p}=((n/\alpha)'a)'$, then $1<\tilde{p}<n/\alpha$ and $w\in A_{\tilde{p},\infty}$. Therefore, we can choose p_0, which is sufficiently close to n/α, such that $\tilde{p}<p_0<n/\alpha$ and $1/p_0-\alpha/n<1/p$. Moreover, it follows from Remark 4 that $w\in A_{p_0,\infty}$. $\square$

Proof of Theorem 1. Let $f\in B^{\sigma}(L_{p,\lambda}(w^{p}))(\mathbb{R}^n)$. We prove that

$$\|I_{\alpha}f\|_{L_{q,\mu}(w^{q})(B_r)}\lesssim r^{\sigma}\|f\|_{B_{\sigma}(L_{p,\lambda}(w^{p}))}$$

for any ball B_r with $r \geqslant 1$.

For $x \in B_r$, let

$$I_\alpha f(x) = I_\alpha(f\chi_{B_{2r}})(x) + I_\alpha(f(1-\chi_{B_{2r}}))(x).$$

First, applying Theorem B, we get

$$\|I_\alpha(f\chi_{B_{2r}})\|_{L_{q,\mu}(w^q)(B_r)} \leqslant \|I_\alpha(f\chi_{B_{2r}})\|_{L_{q,\mu}(w^q)} \lesssim \|f\chi_{B_{2r}}\|_{L_{p,\lambda}(w^p)} \leqslant \|f\|_{L_{p,\lambda}(w^p)(B_{6r})} \lesssim r^\sigma \|f\|_{B_\sigma(L_{p,\lambda}(w^p))}$$

by Lemma 2.

Next, since for $x \in B_r$,

$$|I_\alpha(f(1-\chi_{B_{2r}}))(x)| \lesssim \int_{\mathbb{R}^n \setminus B_{2r}} \frac{|f(y)|}{|y|^{n-\alpha}}\, dy$$

and $w \in A_{p,-n/(\mu+\sigma)}$, it follows from Lemma 1 that for $B(x_0,s) \subset B_r$,

$$\begin{aligned} &\frac{1}{s^\mu}\left(\int_{B(x_0,s)} |I_\alpha(f(1-\chi_{B_{2r}}))(y)|^q w^q(y)\, dy\right)^{1/q} \\ &\lesssim \frac{1}{s^\mu}\left(\frac{1}{|B(x_0,s)|}\right)^{1/q} r^{-n/p'+\alpha+\lambda+\sigma} w^{-p'}(B_r)^{1/p'} \|f\|_{B_\sigma(L_{p,\lambda}(w^p))} \cdot w^q(B(x_0,s))^{1/q}. \end{aligned}$$

In addition, as $w \in A_{p,-n/(\mu+\sigma)}$, which means $w^{-p'} \in A_{1-p'(\mu+\sigma)/n}$ by Remark 3, we obtain

$$w^{-p'}(B_r) \leqslant C\left(\frac{r}{s}\right)^{n-p'(\mu+\sigma)} w^{-p'}(B(x_0,s))$$

in terms of Remark 5. Therefore, since $q \leqslant -n/\mu < -n/(\mu+\sigma)$, by using Remark 4,

$$\begin{aligned} &\frac{1}{s^\mu}\left(\int_{B(x_0,s)} |I_\alpha(f(1-\chi_{B_{2r}}))(y)|^q w^q(y)\, dy\right)^{1/q} \\ &\lesssim \frac{1}{s^\mu} r^{-n/p'+\mu+\sigma}\left(\frac{r}{s}\right)^{n/p'-(\mu+\sigma)} w^{-p'}(B(x_0,s))^{1/p'}\left(\frac{w^q(B(x_0,s))}{|B(x_0,s)|}\right)^{1/q} \|f\|_{B_\sigma(L_{p,\lambda}(w^p))} \\ &\lesssim s^\sigma\left(\frac{w^{-p'}(B(x_0,s))}{|B(x_0,s)|}\right)^{1/p'}\left(\frac{w^q(B(x_0,s))}{|B(x_0,s)|}\right)^{1/q} \|f\|_{B_\sigma(L_{p,\lambda}(w^p))} \\ &\lesssim r^\sigma \|f\|_{B_\sigma(L_{p,\lambda}(w^p))}. \end{aligned}$$

Thus, we have for $r \geqslant 1$,

$$\|I_\alpha f\|_{L_{q,\mu}(w^q)(B_r)} \lesssim r^\sigma \|f\|_{B_\sigma(L_{p,\lambda}(w^p))},$$

which shows the conclusion. □

Proof of Theorem 2. Let $f \in B^{\sigma}(L_{p,\lambda}(w^p))(\mathbb{R}^n)$. We prove that

$$\|\tilde{I}_\alpha f\|_{\mathrm{Lip}_\mu(w^{-1})(B_r)} \lesssim r^{\sigma} \|f\|_{B_\sigma(L_{p,\lambda}(w^p))}$$

for any ball B_r with $r \geqslant 1$.

Now, letting for $y \in B(x_0,s) \subset B_r$,

$$\begin{aligned}\tilde{I}_\alpha f(y) &= \tilde{I}_\alpha(f\chi_{B(x_0,2s)})(y) + \tilde{I}_\alpha(f(1-\chi_{B(x_0,2s)}))(y)\\ &= I_\alpha(f\chi_{B(x_0,2s)})(y) + C_{\alpha,x_0,s}f + J_{\alpha,x_0,s}f(y),\end{aligned}$$

where

$$C_{\alpha,x_0,s}f = -\int_{B(x_0,2s)\setminus B_1} \frac{f(z)}{|z|^{n-\alpha}}\,dz + \tilde{I}_\alpha(f(1-\chi_{B(x_0,2s)}))(x_0)$$

and

$$J_{\alpha,x_0,s}f(y) = \int_{\mathbb{R}^n\setminus B(x_0,2s)} f(z)\left(\frac{1}{|y-z|^{n-\alpha}} - \frac{1}{|x_0-z|^{n-\alpha}}\right)dz,$$

we have

$$\begin{aligned}&\int_{B(x_0,s)} \left|\tilde{I}_\alpha f(y) - C_{\alpha,x_0,s}f\right| dy\\ &\qquad\leqslant \int_{B(x_0,s)} \left|I_\alpha(f\chi_{B(x_0,2s)})(y)\right| dy + \int_{B(x_0,s)} \left|J_{\alpha,x_0,s}f(y)\right| dy\\ &\qquad=: I_1 + I_2.\end{aligned}$$

First, by using Lemma 1, we get for $y \in B(x_0,s) \subset B_r$,

$$\begin{aligned}\left|J_{\alpha,x_0,s}f(y)\right| &\lesssim s\int_{\mathbb{R}^n\setminus B(x_0,2s)} \frac{|f(z)|}{|z-x_0|^{n-\alpha+1}}\,dz\\ &\lesssim r^{\sigma} s^{-n/p'+\mu} w^{-p'}(B(x_0,2s))^{1/p'}\|f\|_{B_\sigma(L_{p,\lambda}(w^p))}.\end{aligned}$$

Therefore, since $w \in A_{p,\infty}$, we have for $B(x_0,s) \subset B_r$,

$$\begin{aligned}&\frac{1}{s^{\mu}}\frac{1}{w^{-1}(B(x_0,s))} I_2\\ &\quad\lesssim \frac{1}{w^{-1}(B(x_0,s))}\cdot r^{\sigma} s^{-n/p'} w^{-p'}(B(x_0,2s))^{1/p'}\|f\|_{B_\sigma(L_{p,\lambda}(w^p))}\cdot|B(x_0,s)|\\ &\quad\lesssim r^{\sigma}\cdot\frac{|B(x_0,s)|}{w^{-1}(B(x_0,s))}\left(\frac{w^{-p'}(B(x_0,s))}{|B(x_0,s)|}\right)^{1/p'}\|f\|_{B_\sigma(L_{p,\lambda}(w^p))}\\ &\quad\lesssim r^{\sigma}\|f\|_{B_\sigma(L_{p,\lambda}(w^p))}.\end{aligned}$$

Next, when $1 < p < n/\alpha$, let $1/q = 1/p - \alpha/n$. Then, applying Hölder's inequality, Remark 4 and Theorem A, we obtain

$$\begin{aligned}
I_1 &\leqslant \left(\int_{B(x_0,s)} \left|I_\alpha(f\chi_{B(x_0,2s)})(y)\right|^q w^q(y)\,dy\right)^{1/q} \left(\int_{B(x_0,s)} w^{-q'}(y)\,dy\right)^{1/q'} \\
&\lesssim \left(\int_{\mathbb{R}^n} \left|(f\chi_{B(x_0,2s)})(y)\right|^p w^p(y)\,dy\right)^{1/p} \left(\int_{B(x_0,s)} w^{-q'}(y)\,dy\right)^{1/q'} \\
&\leqslant \left(\int_{B(x_0,2s)} |f(y)|^p w(y)^p\,dy\right)^{1/p} \left(\int_{B(x_0,s)} w^{-p'}(y)\,dy\right)^{1/p'} \cdot |B(x_0,s)|^{1/q'-1/p'} \\
&\lesssim r^\sigma (2s)^\lambda |B(x_0,2s)|^{1/p} \|f\|_{B_\sigma(L_{p,\lambda}(w^p))} \cdot \left(⨍_{B(x_0,s)} w^{-p'}(y)\,dy\right)^{1/p'} |B(x_0,s)|^{1/q'} \\
&\lesssim r^\sigma (2s)^{\lambda+\alpha} \|f\|_{B_\sigma(L_{p,\lambda}(w^p))} \left(⨍_{B(x_0,s)} w^{-p'}(y)\,dy\right)^{1/p'} \cdot |B(x_0,s)| \\
&\lesssim r^\sigma s^\mu \|f\|_{B_\sigma(L_{p,\lambda}(w^p))} ⨍_{B(x_0,s)} w^{-1}(y)\,dy \cdot |B(x_0,s)| \\
&= r^\sigma s^\mu \|f\|_{B_\sigma(L_{p,\lambda}(w^p))} \cdot w^{-1}(B(x_0,s)).
\end{aligned}$$

When $n/\alpha \leqslant p < \infty$, in terms of Lemma 4, we choose p_0 such that $1 < p_0 < n/\alpha$, $1/p_0 - \alpha/n < 1/p$ and $w \in A_{p_0,\infty}$, and let $1/q_0 = 1/p_0 - \alpha/n$. Then, applying Hölder's inequality, Remark 4 and Theorem A, we obtain

$$\begin{aligned}
I_1 &\leqslant \left(\int_{B(x_0,s)} \left|I_\alpha(f\chi_{B(x_0,2s)})(y)\right|^{q_0} w^{q_0}(y)\,dy\right)^{1/q_0} \left(\int_{B(x_0,s)} w^{-q_0'}(y)\,dy\right)^{1/q_0'} \\
&\lesssim \left(\int_{\mathbb{R}^n} \left|(f\chi_{B(x_0,2s)})(y)\right|^{p_0} w^{p_0}(y)\,dy\right)^{1/p_0} \left(\int_{B(x_0,s)} w^{-q_0'}(y)\,dy\right)^{1/q_0'} \\
&\leqslant \left(\int_{B(x_0,2s)} |f(y)|^p w(y)^p\,dy\right)^{1/p} |B(x_0,2s)|^{1/p_0-1/p} \\
&\qquad \times \left(\int_{B(x_0,s)} w^{-p'}(y)\,dy\right)^{1/p'} |B(x_0,s)|^{1/q_0'-1/p'} \\
&\lesssim r^\sigma s^{\lambda+\alpha} \|f\|_{B_\sigma(L_{p,\lambda}(w^p))} \cdot w^{-p'}(B(x_0,s))^{1/p'} |B(x_0,s)|^{1/p} \\
&\lesssim r^\sigma s^\mu \|f\|_{B_\sigma(L_{p,\lambda}(w^p))} \cdot w^{-1}(B(x_0,s)).
\end{aligned}$$

Hence, it follows that for $1 < p < \infty$,

$$\frac{1}{s^\mu} \frac{1}{w^{-1}(B(x_0,s))} I_1 \lesssim r^\sigma \|f\|_{B_\sigma(L_{p,\lambda}(w^p))}.$$

Thus, we have for $r \geqslant 1$,

$$\|\tilde{I}_\alpha f\|_{\mathrm{Lip}_\mu(w^{-1})(B_r)} \lesssim r^\sigma \|f\|_{B_\sigma(L_{p,\lambda}(w^p))},$$

which concludes the proof of Theorem 2. $\square$

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