Mathematical
Inequalities
& Applications
Volume 19, Number 3 (2016), 981–990 doi:10.7153/mia-19-72

WEIGHTED INEQUALITIES FOR THE GENERALIZED FOURIER TRANSFORM ASSOCIATED WITH THE CHEREDNIK OPERATOR ON THE REAL LINE

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(*Communicated by J. Pečarić*)

Abstract. We consider the generalized Fourier transform associated with the Cherednik operator on $\mathbb{R}$ and we give the Hardy-Littelwood inequality and some weighted inequalities associated to this transform as particular case, we obtain the analogue of the Pitt's theorem.

1. Introduction

The classical Hardy and Littelwood inequalities in [8] states that

- If $1 < p < 2$, then exists a positive constant C such that for all f in $L^p(\mathbb{R})$, its classical Fourier transform $\widehat{f}$ exists and satisfies

$$\Big[\int_{\mathbb{R}} |\widehat{f}(\lambda)|^p |\lambda|^{p-2} d\lambda\Big]^{1/p} \leqslant C\Big[\int_{\mathbb{R}} |f(x)|^p dx\Big]^{1/p}. \tag{1}$$

- If $p > 2$, then exists a positive constant C such that for all f in $L^p(\mathbb{R}, |x|^{p-2}dx)$, its classical Fourier transform $\widehat{f}$ exists, belongs to $L^p(\mathbb{R})$ and satisfies

$$\Big[\int_{\mathbb{R}} |\widehat{f}(\lambda)|^p d\lambda\Big]^{1/p} \leqslant C\Big[\int_{\mathbb{R}} |f(x)|^p |x|^{p-2} dx\Big]^{1/p}. \tag{2}$$

Later, in 1983, M. Benjamin has showed that: if we take $1 < p \leqslant q < \infty$, with sufficient conditions on nonnegative pairs of functions U and V are given to imply

$$\Big[\int_{\mathbb{R}^n} |\widehat{f}(\lambda)|^q U(\lambda) d\lambda\Big]^{1/q} \leqslant C\Big[\int_{\mathbb{R}^n} |f(x)|^p V(x) dx\Big]^{1/p}, \tag{3}$$

where C is independent of f. For the case $q = p'$ the sufficient condition is that for all positive r,

$$\Big[\int_{U(\lambda) > Br} U(\lambda) d\lambda\Big]\Big[\int_{V(x) < r^{p-1}} V(x)^{-1/(p-1)} dx\Big] \leqslant A,$$

Mathematics subject classification (2010): 42B10, 42A.
Keywords and phrases: Cherednik operator, generalized Fourier transform, Hardy-Littelwood inequality, weighted inequality, Pitt's theorem.
The authors thank the referee for his valuable remarks and comments.

Paper MIA-19-72

where A and B are positive constants independent of r, see [5] for more details. This result generalized also the Pitt's theorem which is proved first for the Fourier series (see [10]) and generalized for the Fourier transform. It states that:

For $1 < p \leqslant q < \infty$, $a \in]0, n/q[$ and $b \in]0, n(p-1)/p[$ such that $b = a + n(pq - p - q)/pq$, then for all f in $L^1(\mathbb{R}^n)$ we have

$$\Big[\int_{\mathbb{R}^n} |\widehat{f}(\lambda)|^q |\lambda|^{-aq} d\lambda\Big]^{1/q} \leqslant C\Big[\int_{\mathbb{R}^n} |f(x)|^p |x|^{bp} dx\Big]^{1/p}, \tag{4}$$

where C is a positive constant. More later, in 2003, J. J Benedetto and H. P. Heinig [2], gave another proof and a generalization of the inequality (1) on $\mathbb{R}^n$ as follows:

For $1 < p \leqslant q \leqslant p' < \infty$ and for all f in $L^p(\mathbb{R}^n)$ then exists $C > 0$ such that

$$\Big[\int_{\mathbb{R}^n} |\widehat{f}(\lambda)|^q |\lambda|^{n(pq-p-q)/p} d\lambda\Big]^{1/q} \leqslant C\Big[\int_{\mathbb{R}^n} |f(x)|^p dx\Big]^{1/p}. \tag{5}$$

Generally, the above-mentioned results use rearrangement methods. These methods do not yield effective criteria for Fourier transform inequalities in the case of non-monotonic weights, and the constants C become more difficult to compute. Also these results tend to assume one or the other of such constraints as even weights, function weights, monotonic weights, or domain $\mathbb{R}$. Our goal is to construct the theory without rearrangements and with as few constraints as possible. The reasons for such a project are apparent: restriction theorems, uncertainty principle inequalities and effective criteria to establish Fourier transform inequalities for large classes of weights.

This paper gives some results in this direction. It aims to establish similar inequalities to (1), (2), (3) and (5) for a generalized Fourier transform. It is organized as follows. In section two we recall some basic results on the Cherednik operator which is a differential-difference operator on the real line and we give the main properties of the generalized Fourier transform. We establish in the third section an analogue of the Hardy-Littlwood inequality associated to the generalized Fourier transform. In the last section, we prove some weighted inequalities for the generalized transform associated to the Cherednik operator, as particular case, we find the analogue of the Pitt's type theorem.

2. Preliminaries

In this section, we collect some basic facts of the differential-difference operator.

$$\Lambda f(x) = f'(x) + \frac{A'(x)}{A(x)} \frac{f(x) - f(-x)}{2} - \rho f(-x), \tag{6}$$

where

$$A(x) = |x|^{2\alpha+1} B(x), \quad \alpha > -1/2. \tag{7}$$

B being a positive C^∞ even function, with $B(0) = 1$, and $\rho \geqslant 0$. In addition to that the function A satisfies the following conditions.

- For all $x \geqslant 0, A(x)$ is increasing and $\lim_{x \to +\infty} A(x) = +\infty$.

- For all $x>0, \frac{A'(x)}{A(x)}$ is decreasing and $\lim_{x\to+\infty}\frac{A'(x)}{A(x)}=2\rho$.
- There exists a positive constant λ such that for all x in $[x_0,+\infty[$, $x_0>0$, we have
$$\frac{A'(x)}{A(x)}=\begin{cases} 2\rho+e^{-\lambda x}D(x), & \text{if } \rho>0 \\ \frac{2\alpha+1}{x}+e^{-\lambda x}D(x), & \text{if } \rho=0, \end{cases}$$
where D be a C^∞ function on $\mathbb{R}$, bounded together with its derivatives.

For the particular case $\alpha\geqslant\beta\geqslant-1/2$, $\alpha\neq-1/2$,

$$A(x)=(\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1},\quad \rho=\alpha+\beta+1>0,$$

Λ will be denoted by $T^{(\alpha,\beta)}$, which is given by

$$T^{(\alpha,\beta)}f(x)=f'(x)+\Big[(2\alpha+1)\coth x+(2\beta+1)\tanh x\Big]\frac{f(x)-f(-x)}{2}-\rho f(-x).$$

It is referred to as the Jacobi-Cherednik operator on $\mathbb{R}$ and can be written as

$$T^{(k_1,k_2)}f(x)=f'(x)+\left[\frac{2k_1}{1-e^{-2x}}+\frac{4k_2}{1-e^{-4x}}\right]\{f(x)-f(-x)\}-(k_1+2k_2)f(x),$$

where $\alpha=k_1+k_2-\frac{1}{2}$ and $\beta=k_2-\frac{1}{2}$. We use [7] and [12] as main references.

For $A(x)=|x|^{2\alpha+1}$, $\alpha>-1/2$ and $\rho=0$, we regain the differential-difference operator

$$D_\alpha f=f'(x)+\left(\alpha+\frac{1}{2}\right)\frac{f(x)-f(-x)}{x},$$

which is referred to as the Dunkl operator with parameter $\left(\alpha+\frac{1}{2}\right)$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$.

We denote by $\phi_\lambda, \lambda\in\mathbb{C}$, the eigenfunction of the operator Λ. It is the unique C^∞ function on $\mathbb{R}$ which is equals to 1 at 0 and satisfies the differential-difference equation

$$\Lambda f(x)=i\lambda f(x).$$

The function ϕ_λ is expressed as follows

$$\phi_\lambda(x)=\begin{cases} \varphi_\lambda(x)+\frac{1}{i\lambda-\rho}\frac{d}{dx}\varphi_\lambda(x) & \text{if} \quad \lambda\neq-i\rho \\ 1+\frac{2\rho}{A(x)}\int_0^x A(t)dt & \text{if} \quad \lambda=-i\rho, \end{cases}$$

where φ_λ is the eigenfunction of the differential equation

$$\begin{cases} \Delta u(x)=-(\lambda^2+\rho^2)u(x), \quad x\in]0,+\infty[, \\ u(0)=1, u'(0)=0, \end{cases}$$

with

$$\Delta = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx}. \tag{8}$$

In addition to that ϕ_λ has the following Laplace type integral representation

$$\phi_\lambda(x) = \int_{-|x|}^{|x|} K(x,y)e^{i\lambda y}dy, \ x \neq 0,$$

where $K(x,.)$ is nonnegative function on $\mathbb{R}$, continuous on $]-|x|,|x|[$ and supported in $[-|x|,|x|]$. See [9, 13] for more details.

REMARK 1. For all $x \in \mathbb{R}$, in [13], the author proved that

$$\int_{-|x|}^{|x|} K(x,y)dy \leqslant 2. \tag{9}$$

The generalized Fourier transform of a function f in $D(\mathbb{R})$, the space of C^∞ function on $\mathbb{R}$ with compact support, is defined by

$$\mathscr{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)\phi_{-\lambda}(x)A(x)dx, \ \lambda \in \mathbb{C}.$$

In addition, it can be written as:

$$\mathscr{F}(f)(\lambda) = \mathscr{F}_\Delta(f_e)(\lambda) + (i\lambda - \rho)\mathscr{F}_\Delta(Jf_o)(\lambda), \tag{10}$$

where f_e (resp f_o) denotes the even (resp odd) part of f, $\mathscr{F}_\Delta$ stands the Fourier transform related to the differential operator Δ given by the relation (8), defined on the subspace of $D(\mathbb{R})$ consisting of even functions, by

$$\mathscr{F}_\Delta h(\lambda) = \int_{\mathbb{R}} h(x)\varphi_\lambda(x)A(x)dx, \quad \forall \lambda \in \mathbb{C} \tag{11}$$

and

$$Jf_o(x) := \int_{-\infty}^{x} f_o(t)dt.$$

The inversion formula for the Fourier transform $\mathscr{F}$ is defined for all $f \in D(\mathbb{R})$ by,

$$f(x) = \int_{\mathbb{R}} \mathscr{F}f(\lambda)\phi_{-\lambda}(-x)d\sigma_1(\lambda),$$

with

$$d\sigma_1(\lambda) = \frac{\lambda - i\rho}{\lambda}\frac{d\lambda}{|c(\lambda)|^2},$$

where $|c(\lambda)|^{-2}$ is an even continuous function on $\mathbb{R}$, and satisfies the estimates: There exist positive constants k_1, k_2, k such that

- If $\rho \geqslant 0$ and $\alpha > -1/2$, then

$$k_1|\lambda|^{2\alpha+1} \leqslant |c(\lambda)|^{-2} \leqslant k_2|\lambda|^{2\alpha+1}, \quad \lambda \in \mathbb{R}, |\lambda| > k. \tag{12}$$

- If $\rho > 0$ and $\alpha > -1/2$, then

$$k_1|\lambda|^2 \leqslant |c(\lambda)|^{-2} \leqslant k_2|\lambda|^2, \quad \lambda \in \mathbb{R},\ |\lambda| \leqslant k. \tag{13}$$

- If $\rho = 0$ and $\alpha > 0$, then

$$k_1|\lambda|^{2\alpha+1} \leqslant |c(\lambda)|^{-2} \leqslant k_2|\lambda|^{2\alpha+1}, \quad \lambda \in \mathbb{R},\ |\lambda| \leqslant k. \tag{14}$$

We have the following Plancherel formula: for all $f \in D(\mathbb{R})$, we have

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathscr{F} f(\lambda)|^2 d\sigma_1(\lambda) + 2\rho \int_{\mathbb{R}} \mathscr{F}_\Delta(f_e)(\lambda)\overline{\mathscr{F}_\Delta\ (J f_o)(\lambda)}\frac{d\lambda}{|c(\lambda)|^2}, \tag{15}$$

where $\mathscr{F}_\Delta f$ given by the relation (11). We refer to [9] for more details.

More of this, by replacing f by $\check{f}$ in the relation (15), where $\check{f}(x) = f(-x)$, we obtain

$$\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \int_{\mathbb{R}} |\mathscr{F} \check{f}(\lambda)|^2 d\sigma_1(\lambda) - 2\rho \int_{\mathbb{R}} \mathscr{F}_\Delta(f_e)(\lambda)\overline{\mathscr{F}_\Delta\ (J f_o)(\lambda)}\frac{d\lambda}{|c(|\lambda|)|^2}. \tag{16}$$

By summing the relations (15) and (16), we obtain

$$2\int_{\mathbb{R}} |f(x)|^2 A(x)dx = \int_{\mathbb{R}} \left(|\mathscr{F} \check{f}(\lambda)|^2 + |\mathscr{F} f(\lambda)|^2\right) \frac{d\lambda}{|c(\lambda)|^2}. \tag{17}$$

- For f belonging to $L^1(\mathbb{R}, A(x)\,dx)$, $\mathscr{F}(f)$ is a bounded continuous function on $\mathbb{R}$, vanishes at infinity and verifies

$$\|\mathscr{F} f\|_\infty \leqslant \|f\|_{1,A}. \tag{18}$$

- If f be in $L^2(\mathbb{R}, A(x)\,dx)$, then

$$\int_{\mathbb{R}} |\mathscr{F} f(\lambda)|^2 \frac{d\lambda}{|c|\lambda||^2} \leqslant 2\int_{\mathbb{R}} |f(x)|^2 A(x)dx. \tag{19}$$

Using the relations (18), (19) and applying the Riesz-Thorin interpolation theorem [[6], p. 200], we can deduce the following result

COROLLARY 1. (The Hausdorff-Young inequality) *Let* $1 < p \leqslant 2$, *then for all* f *in* $L^p(\mathbb{R}, A(x)\,dx)$, *we have*

$$\left[\int_{\mathbb{R}} |\mathscr{F}(f)(\lambda)|^{p'} \frac{d\lambda}{|c(|\lambda|)|^2}\right]^{1/p'} \leqslant C\left[\int_{\mathbb{R}} |f(x)|^p A(x)dx\right]^{1/p} \tag{20}$$

where $\frac{1}{p} + \frac{1}{p'} = 1, C > 0$.

In the squel we will denote by $f \lesssim g$ to mean that there exists a positive constant C such that $f \leqslant C g$.

3. Hardy-Littlewood type inequality for the generalized Fourier transform

This section is devoted to the following result.

THEOREM 1. *Let* $1 < p \leqslant 2, p \leqslant q \leqslant p'$. *Theen for all* f *in* $L^p(\mathbb{R}, A(x)\,dx)$, *the following inequality holds*

$$\Big[\int_{\mathbb{R}} |\mathscr{F}(f)(\lambda)|^q |\lambda|^{\frac{(2\alpha+2)(pq-p-q)}{p}} \frac{d\lambda}{|c(\lambda)|^2}\Big]^{1/q} \lesssim \Big[\int_{\mathbb{R}} |f(x)|^p A(x)dx\Big]^{1/p}.$$

Proof. Let $r = 1 + q'/p'$. Then $r \leqslant 2$ and $r' = 1 + p'/q'$. Denote $\gamma = (2\alpha + 2)/(r' - 1)$. The Hausdroff-Young inequality gives

$$\Big[\int_{\mathbb{R}} \Big[|\lambda|^{\gamma} |\mathscr{F}(f)(\lambda)|\Big]^{r'} |\lambda|^{-r'\gamma} \frac{d\lambda}{|c(\lambda)|^2}\Big]^{1/r'} \lesssim \Big[\int_{\mathbb{R}} |f(x)|^r A(x)dx\Big]^{1/r},$$

for all $f \in L^r(\mathbb{R}, A(x)\,dx)$.

This inequality implies that the operator $T : f \longmapsto |\lambda|^{\gamma}\mathscr{F}(f)$ is of type (r, r') between the spaces $L^r(\mathbb{R}, A(x)\,dx)$ and $L^{r'}(\mathbb{R}, |\lambda|^{-r'\gamma}\frac{d\lambda}{|c(\lambda)|^2})$. Next, we need to prove that T is of weak type $(1,1)$.

By an easy calculation, using the relation (12), we obtain

$$\int_{|\lambda|^{\gamma} > s} |\lambda|^{-r'\gamma} \frac{d\lambda}{|c(\lambda)|^2} \lesssim \int_{|\lambda|^{\gamma} > s} |\lambda|^{-r'\gamma} |\lambda|^{2\alpha+1} d\lambda \leqslant C\frac{1}{\gamma s},$$

for all $s > 0$.

Using the last inequality and the relation (18), we deduce that

$$s \int_{|\lambda|^{\gamma}|\mathscr{F}(f)(\lambda)| > s} |\lambda|^{-r'\gamma} \frac{d\lambda}{|c(\lambda)|^2} \lesssim \|f\|_{A,1},$$

which means that T is of weak type $(1,1)$.

Now we apply the Marcinkiewicz interpolation theorem ([11], p. 184). By taking $t = 1/q - 1/p'$, the real numbers r_t and s_t, defined by

$$\frac{1}{r_t} = \frac{1-t}{r} + t, \quad \text{and} \quad \frac{1}{s_t} = \frac{1-t}{r'} + t,$$

satisfy $r_t = p$ and $s_t = q$. Thus T is of type (p,q). So the equality

$$q\gamma - r'\gamma = (2\alpha + 2)(pq - p - q)/p$$

gives the result. □

4. Weighted inequalities for the generalized transform

In this section, we will prove a weighted inequality for the generalized Fourier transform. First, we need the following lemma

LEMMA 1. *Let $V: A \to B$ be a bijective and measurable function such that $A \subset \mathbb{R}$, $B \subset [0,+\infty)$ and V^{-1} is measurable. If f is a nonzero and integrable function on $\mathbb{R}$ with respect to a positive measure ν, then for all $0 < k < \|f\|_{1,\nu}$, there exists $R_k > 0$ such that*

$$\int_{2V(x)<R_k} |f(x)|\,d\nu(x) = k,$$

Proof. Let $f \in L^1(\mathbb{R}, d\nu)$. Define $h : [0,\infty) \longrightarrow [0, \|f\|_{1,\nu}]$ by

$$h(r) = \int_{|y|<r/2} |f \circ V^{-1}(y)|d\nu_V(y) = \int_{|V(x)|<r/2} |f(x)|\,d\nu(x) \leqslant \|f\|_{1,\nu}$$

where $d\nu_V$ be the image measure of ν by the function V. By using the theorem of continuity in sign integral we can easily prove that the function h is continuous on $[0,\infty)$, then for $0 < k < \|f\|_{1,\nu}$, there exist R_k such that

$$h(R_k) = k,$$

that is

$$\int_{2V(x)<R_k} |f(x)|d\nu(x) = k$$

and this completes the proof. □

THEOREM 2. *Let $1 < p \leqslant 2$, $p \leqslant q \leqslant p'$, $\gamma = (2\alpha+2)(pq-p-q)/p$, $r > 0$, U be a nonnegative functions on $\mathbb{R}$, V as in Lemma 1 and there is a positive constant M_1, independent of r, such that*

$$\Big[\int_{\{|\lambda|^{-\gamma}U(\lambda)\}^{\frac{p'}{q}}>r} U(\lambda)\frac{d\lambda}{|c(\lambda)|^2}\Big]\Big[\int_{V(x)<r^{p-1}} V(x)^{\frac{-1}{p-1}}A(x)dx\Big]^{q/p'} \leqslant M_1, \tag{21}$$

then we have

$$\Big[\int_{\mathbb{R}} |\mathscr{F}f(\lambda)|^q U(\lambda)\frac{d\lambda}{|c(\lambda)|^2}\Big]^{1/q} \lesssim \Big[\int_{\mathbb{R}} |f(x)|^p V(x)A(x)dx\Big]^{1/p}. \tag{22}$$

Proof. Let $1 < p \leqslant q \leqslant p'$, $\gamma = (2\alpha+2)(pq-p-q)/p$ and for all $j \in \mathbb{Z}$,

$$E_j = \{\lambda \in \mathbb{R}\backslash\{0\},\, 2^{jq/p'}|\lambda|^{\gamma} < U(\lambda) \leqslant 2^{jq/p'+1}|\lambda|^{\gamma}\}. \tag{23}$$

We denote by

$$I_1 = \sum_{j\in\mathbb{Z}} \int_{E_j} \Big|\int_{V(x)\geqslant 2^{j(p-1)-1}} f(x)\phi_\lambda(x)A(x)dx\Big|^q U(\lambda)\frac{d\lambda}{|c(\lambda)|^2} \tag{24}$$

and

$$I_2 = \sum_{j\in\mathbb{Z}} \int_{E_j} \Big| \int_{V(x)<2^{j(p-1)-1}} f(x)\phi_\lambda(x)A(x)dx \Big|^q U(\lambda)\frac{d\lambda}{|c(\lambda)|^2}. \tag{25}$$

So, by using the convexity of the function $x \mapsto x^q$, we obtain that

$$\int_{\mathbb{R}} |\mathscr{F} f(\lambda)|^q U(\lambda)\frac{d\lambda}{|c(\lambda)|^2} \leqslant 2^{q-1}(I_1+I_2),$$

We need to prove that I_1 and I_2 are bounded by

$$\Big[\int_{\mathbb{R}} |f(x)|^p V(x)A(x)dx\Big]^{q/p}. \tag{26}$$

To estimate the relation (24), we can see that

$$I_1 \leqslant \sum_{j\in\mathbb{Z}} 2^{jq/p'+1}\Big[\int_{\mathbb{R}} \Big|\mathscr{F}(f\chi_{\{x\in\mathbb{R},\,V(x)\geqslant 2^{j(p-1)-1}\}})(\lambda)\Big|^q |\lambda|^\gamma \frac{d\lambda}{|c(\lambda)|^2}\Big].$$

From the theorem 1, we can see that the right side is bounded by

$$2\sum_{j\in\mathbb{Z}} 2^{jq/p'}\Big[\int_{\mathbb{R}} |f\chi_{\{x\in\mathbb{R},\,V(x)\geqslant 2^{j(p-1)-1}\}}(x)|^p A(x)dx\Big]^{q/p}.$$

Since $q/p \geqslant 1$, we obtain

$$I_1 \leqslant 2\Big[\int_{\mathbb{R}} |f(x)|^p h(x)A(x)dx\Big]^{q/p}, \tag{27}$$

where

$$h(x) = \sum_{j\in\mathbb{Z}} 2^{jp/p'}\chi_{\{x\in\mathbb{R},\,V(x)\geqslant 2^{j(p-1)-1}\}}(x). \tag{28}$$

Now for a given x, let J is the largest integer satisfying $2^{J(p-1)} \leqslant 2V(x)$. So

$$h(x) = \sum_{j\leqslant J} 2^{j(p-1)} = \frac{1}{1-2^{1-p}}2^{J(p-1)} \leqslant cV(x).$$

This completes the proof of the first part.

Next, to estimate the relation (25), observe first that from (23), we can write

$$I_2 \lesssim \int_{\mathbb{R}} \Big[\int_{2V(x)<[(U(\lambda)|\lambda|^{-\gamma})^{\frac{p'}{q}}]^{p-1}} |f(x)|A(x)dx\Big]^q U(\lambda)\frac{d\lambda}{|c(\lambda)|^2}. \tag{29}$$

Now let J be the least integer such that $2^J \geqslant \|f\|_{1,A}$, and let $r_J = \infty$. We apply the Lemma 1 with $k = 2^j$, for $j < J$. Let r_j be chosen so that

$$\int_{2V(x)<r_j^{p-1}} |f(x)|A(x)dx = 2^j. \tag{30}$$

Then from the relation (29), we deduce that

$$I_2 \leqslant \sum_{j\leqslant J} \int_{r_{j-1}<\left(|\lambda|^{-\gamma}U(\lambda)\right)^{p'/q}\leqslant r_j} \left| \int_{2V(x)<r_j^{p-1}} |f(x)|A(x)dx \right|^q U(\lambda)\frac{d\lambda}{|c(\lambda)|^2} \tag{31}$$

using the relation (30) and the definition of r_J, we have for $j \leqslant J$,

$$\int_{2V(x)<r_j^{p-1}} |f(x)|A(x)dx = 2^j = 4\int_{D_j} |f(x)|A(x)dx, \tag{32}$$

where $D_j = \{x \in \mathbb{R},\, r_{j-2}^{p-1} \leqslant 2V(x) < r_{j-1}^{p-1}\}$. Furthermore, by using the relations (32) and (31), we have

$$I_2 \lesssim \sum_{j\in\mathbb{Z}} \left[\int_{r_{j-1}<\left(|\lambda|^{-\gamma}U(\lambda)\right)^{p'/q}\leqslant r_j} U(\lambda)\frac{d\lambda}{|c(\lambda)|^2}\right]\left[\int_{D_j}\left[|f(x)|V(x)^{1/p}\right]V(x)^{-1/p}A(x)dx\right]^q. \tag{33}$$

By Hölder's inequality

$$\left[\int_{D_j}\left[|f(x)|V(x)^{\frac{1}{p}}\right]V(x)^{\frac{-1}{p}}A(x)dx\right]^q \leqslant \left[\int_{2V(x)<r_{j-1}^{p-1}} V(x)^{\frac{-p'}{p}}A(x)dx\right]^{\frac{q}{p'}} \times \left[\int_{D_j}|f(x)|^pV(x)A(x)dx\right]^{\frac{q}{p}}. \tag{34}$$

Then since $q/p \geqslant 1$ and by using the hypothesis (21), we conclude that

$$I_2 \lesssim \left[\sum_{j\in\mathbb{Z}}\int_{D_j}|f(x)|^pV(x)A(x)dx\right]^{q/p}.$$

Now, we get the result by using the fact taht the sets D_j are disjoints. □

REMARK 2. For $\alpha > 0$ and $\rho = 0$ or $\alpha > -1/2$ and $\rho > 0$, we have the following result which is the analogue of the Pitt's theorem.

THEOREM 3. *$1 < p \leqslant q \leqslant p'$. Put $U(\lambda) = |\lambda|^{-aq}$, $V(x) = |x|^{bp}$ with a and b are such that $(2\alpha+2)(1/q - /p') < a < \frac{\min(2\alpha+2,3)}{q}$ and $b = a + (2\alpha+2)(1/p' - 1/q)$. Then for all f in $L^p(\mathbb{R}, |x|^{bp}A(x)dx)$, we have*

$$\left[\int_{\mathbb{R}}\left|\mathscr{F}(f)(\lambda)\right|^q|\lambda|^{-aq}\frac{d\lambda}{|c(|\lambda|)|^2}\right]^{1/q} \lesssim \left[\int_{\mathbb{R}}\left|f(x)\right|^p|x|^{bp}A(x)dx\right]^{1/p}.$$

Proof. With the choice of a and b, we can see that the hypothesis of theorem 2 is satisfied. □

REMARK 3. In the case $\rho = 0$ and $\alpha > 0$, if we take $A(x) = |x|^{2\alpha+1}$, we obtain the Dunkl operator with parameter $(\alpha + 1/2)$ associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ and in this case S. Ben Farah and M. Mili [4], proved that the condition on a and b which we make in the Theorem 3 is optimal.

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(Received October 8, 2015)

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