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WEIGHTED COMPOSITION OPERATORS FROM DIRICHLET TYPE SPACES TO WEIGHTED TYPE SPACES

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Abstract. The boundedness and compactness of weighted composition operators from Dirichlet type spaces to weighted type spaces on the unit ball of $\mathbb{C}^n$ are characterized in this paper.

1. Introduction

Let $\mathbb{B}_n$ be the open unit ball of $\mathbb{C}^n$ with the boundary $\partial\mathbb{B}_n$, $d\sigma$ the normalized invariant measure on $\partial\mathbb{B}_n$. Let $H(\mathbb{B}_n)$ denote the space of all holomorphic functions on $\mathbb{B}_n$. Let $p \in \mathbb{R}$ and $f \in H(\mathbb{B}_n)$ with the Taylor expansion $f(z) = \sum_{|\beta|\geqslant 0} b_\beta z^\beta$, where $\beta = (\beta_1, \cdots, \beta_n)$, $z^\beta = z_1^{\beta_1}\cdots z_n^{\beta_n}$, $|\beta| = \beta_1 + \cdots + \beta_n$. The Dirichlet type space D_p is the space consisting of all $f \in H(\mathbb{B}_n)$ such that (see [5])

$$\|f\|_{D_p}^2 = \sum_{|\beta|\geqslant 0} (n+|\beta|)^p |b_\beta|^2 \omega_\beta < \infty,$$

where

$$\omega_\beta = \int_{\partial\mathbb{B}_n} |\xi^\beta|^2 d\sigma(\xi) = \frac{(n-1)!\beta!}{(n+|\beta|-1)!}, \quad \beta! = \beta_1!\ldots\beta_n!.$$

When $p = 0$ and $p = -1$, we get the Hardy space H^2 and the Bergman space A^2, respectively. D_n is the Dirichlet space. The Dirichlet type space D_p has been characterized by many authors (see [4, 6, 7]).

Let μ be a positive continuous function on $[0,1)$. We say that μ is normal, if there exist positive numbers s and t, $0 < s < t$, and $\delta \in [0,1)$ such that

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$$\frac{\mu(r)}{(1-r)^s} \text{ is decreasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^s}=0;$$

$$\frac{\mu(r)}{(1-r)^t} \text{ is increasing on } [\delta,1) \text{ and } \lim_{r\to 1}\frac{\mu(r)}{(1-r)^t}=\infty$$

(see [14]).

Let μ be a normal function on $[0,1)$. An $f\in H(\mathbb{B}_n)$ is said to belong to the weighted-type space, denoted by $H^\infty_\mu = H^\infty_\mu(\mathbb{B}_n)$, if (see, e.g., [2, 18])

$$\|f\|_{H^\infty_\mu} = \sup_{z\in\mathbb{B}_n}\mu(|z|)\,|f(z)| < \infty.$$

H^∞_μ is a Banach space under the norm $\|\cdot\|_{H^\infty_\mu}$. We denote by $H^\infty_{\mu,0}$ the space of those $f\in H^\infty_\mu$ such that $\lim_{|z|\to 1}\mu(|z|)|f(z)|=0$. When $\mu(r)=(1-r^2)^\alpha$, the induced spaces H^∞_μ and $H^\infty_{\mu,0}$ become the (classical) weighted space H^∞_α and $H^\infty_{\alpha,0}$, respectively. For some generalizations of weighted-type spaces, see, for example, [30].

Let $\psi\in H(\mathbb{B}_n)$ and φ be a holomorphic self-map of $\mathbb{B}_n$. The weighted composition operator ψC_φ on $H(\mathbb{B}_n)$ is defined as follows:

$$(\psi C_\varphi f)(z) = \psi(z) f(\varphi(z)), \quad f\in H(\mathbb{B}_n),\ z\in\mathbb{B}_n.$$

The weighted composition operator is a generalization of the multiplication operator and the composition operator, where the composition operator is defined by $(C_\varphi f)(z) = f(\varphi(z))$. See [1] for the theory of composition operator.

In the setting of the unit disk, weighted composition operators on weighted type spaces H^∞_α were studied in [33]. Weighted composition operators from or to weighted-type spaces or its subspaces on various domains in the complex plane $\mathbb{C}$ or in $\mathbb{C}^n$ have been studied considerably, see, e.g., [2, 3, 8, 9, 10, 11, 15, 16, 17, 18, 21, 23, 26, 35]. For some related operators on or to the space H^∞_μ, see, e.g., [22, 27, 32].

In this paper, we study weighted composition operators from Dirichlet type spaces into weighted type spaces H^∞_μ on the unit ball. Some necessary and sufficient conditions for the operator ψC_φ to be bounded and compact are given. We have to say that a closely related problem was studied in papers [28] and [29]. In [28] the author calculated the norm of an integral-type operator from the Dirichlet to the Bloch space on the unit disk, while the case of the unit ball in $\mathbb{C}^n$ was studied in detail in [29]. Namely in [20] Stević introduced an integral-type operator which under the action of radial operator becomes the weighted composition operator (see also [19, 24, 25]). This is related to the fact that the differentiation of the one-dimensional integral-type operators gives weighted composition operators, which now is mapped into a weighted type space. So the problems are closely related. The main novelty in our paper is that our Dirichlet type space is more general than the ones studied in papers [28] and [29].

Constants are denoted by C in this paper, they are positive and may differ from one occurrence to the other. $a\preceq b$ means that there is a positive constant C such that $a\leqslant Cb$. Moreover, if both $a\preceq b$ and $b\preceq a$ hold, then one says that $a\asymp b$.

2. Main results and proofs

In this section, we give our main results and proofs. To this end, we need some auxiliary results which are incorporated in the following lemmas.

LEMMA 1. [34] *Let* $p \in \mathbb{R}$ *and* $f \in D_p$*. Then*

$$|f(z)| \preceq \begin{cases} \|f\|_{D_p} \frac{1}{(1-|z|^2)^{(n-p)/2}} : & p < n; \\ \|f\|_{D_p} \sqrt{\log \frac{1}{1-|z|^2}} : & p = n; \\ \|f\|_{D_p} : & p > n. \end{cases}$$

The following lemma is a natural extension of a lemma in [11]. Its formulation can be found, for example, in Lemma 2.2 in [31]. The proof was not given there for being the same as the one in [11].

LEMMA 2. *Let* μ *be a normal function on* $[0,1)$*. A closed set* K *in* $H^\infty_{\mu,0}$ *is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f \in K} \mu(|z|)|f(z)| = 0.$$

The following lemma follows by the standard Schwartz arguments appearing for the first time in [13].

LEMMA 3. *Let* μ *be a normal function on* $[0,1)$*. Assume that* $p \in \mathbb{R}$*,* $\psi \in H(\mathbb{B}_n)$ *and* φ *is a holomorphic self-map of* $\mathbb{B}_n$*. Then* $\psi C_\varphi : D_p \to H^\infty_\mu$ *is compact if and only if* $\psi C_\varphi : D_p \to H^\infty_\mu$ *is bounded and for any bounded sequence* $(f_k)_{k\in\mathbb{N}}$ *in* H^∞_μ *which converges to zero uniformly on compact subsets of* $\mathbb{B}_n$ *as* $k \to \infty$*, we have* $\|\psi C_\varphi f_k\|_{H^\infty_\mu} \to 0$ *as* $k \to \infty$*.*

When $p > n$, the functions in D_p are indeed Lipschitz continuous. Similarly to the proof of Lemma 3.6 of [12] and using the Arzela-Ascoli Theorem, we have the following result.

LEMMA 4. *Let* $p > n$ *and* $(f_k)_{k\in\mathbb{N}}$ *be a bounded sequence in* D_p *which converges to 0 uniformly on compact subsets of* $\mathbb{B}_n$*, then*

$$\lim_{k\to\infty} \sup_{z\in\mathbb{B}_n} |f_k(z)| = 0.$$

Now, we are in a position to formulate and prove the main results of this paper.

THEOREM 1. *Let* μ *be a normal function on* $[0,1)$*. Assume that* $\psi \in H(\mathbb{B}_n)$ *and* φ *is a holomorphic self-map of* $\mathbb{B}_n$*. Then* $\psi C_\varphi : D_n \to H^\infty_\mu$ *is bounded if and only if*

$$\sup_{z\in\mathbb{B}_n} \mu(|z|)|\psi(z)|\sqrt{\log \frac{1}{1-|\varphi(z)|^2}} < \infty. \quad (1)$$

Moreover, the following relationship holds.

$$||\psi C_\varphi|| \asymp \sup_{z\in\mathbb{B}_n} \mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}}. \tag{2}$$

Proof. Assume that (1) holds. Let $f\in D_n$. By Lemma 1 we have

$$\begin{aligned}\|\psi C_\varphi f\|_{H_\mu^\infty} &= \sup_{z\in\mathbb{B}_n}\mu(|z|)|(\psi C_\varphi f)(z)| = \sup_{z\in\mathbb{B}_n}\mu(|z|)|\psi(z)||f(\varphi(z))|\\ &\preceq \|f\|_{D_n}\sup_{z\in\mathbb{B}_n}\mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}}.\end{aligned}$$

Therefore $\psi C_\varphi : D_n \to H_\mu^\infty$ is bounded and

$$\|\psi C_\varphi\| \preceq \sup_{z\in\mathbb{B}_n}\mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}}. \tag{3}$$

Conversely, suppose that $\psi C_\varphi : D_n \to H_\mu^\infty$ is bounded. Taking $f\equiv 1$ and using the boundedness of the operator we see that $\psi\in H_\mu^\infty$ and

$$\sup_{w\in\mathbb{B}_n}\mu(|w|)|\psi(w)| \leqslant C\|\psi C_\varphi\| < \infty. \tag{4}$$

For $w\in\mathbb{B}_n$ such that $|\varphi(w)|>1/2$, set

$$f_w(z) = \Big(\log\frac{1}{1-|\varphi(w)|^2}\Big)^{-1/2}\log\frac{1}{1-\langle z,\varphi(w)\rangle}. \tag{5}$$

Then

$$\begin{aligned}\|f_w\|_{D_n}^2 &= \Big(\log\frac{1}{1-|\varphi(w)|^2}\Big)^{-1}\sum_{|\beta|\geqslant 1}(n+|\beta|)^n\Big(\frac{(|\beta|-1)!}{\beta!}\Big)^2|\varphi^\beta(w)|^2\omega_\beta\\ &= \Big(\log\frac{1}{1-|\varphi(w)|^2}\Big)^{-1}\sum_{|\beta|\geqslant 1}(n+|\beta|)^n\Big(\frac{(|\beta|-1)!}{\beta!}\Big)^2|\varphi^\beta(w)|^2\frac{(n-1)!\beta!}{(n+|\beta|-1)!}\\ &\preceq \Big(\log\frac{1}{1-|\varphi(w)|^2}\Big)^{-1}\sum_{k=1}^{\infty}\frac{|\varphi(w)|^2}{k} \preceq 1,\end{aligned} \tag{6}$$

which gives that $f_w\in D_n$ and $\|f_w\|_{D_n}\preceq 1$. Hence,

$$\sup_{z\in\mathbb{B}_n}\mu(|z|)|f_w(\varphi(z))\psi(z)| = \|\psi C_\varphi f_w\|_{H_\mu^\infty} \leqslant \|\psi C_\varphi\|\|f_w\|_{D_n} \preceq \|\psi C_\varphi\| < \infty.$$

Let $z=w$. We have

$$\mu(|w|)|f_w(\varphi(w))||\psi(w)| \preceq \|\psi C_\varphi\| < \infty,$$

which implies that

$$\sup_{|\varphi(w)|>1/2} \mu(|w|)\sqrt{\log\frac{1}{1-|\varphi(w)|^2}}|\psi(w)| \preceq \|\psi C_\varphi\| < \infty. \tag{7}$$

Since $\psi \in H_\mu^\infty$, we get

$$\begin{aligned} &\sup_{|\varphi(w)|\leqslant 1/2} \mu(|w|)\sqrt{\log\frac{1}{1-|\varphi(w)|^2}}|\psi(w)| \\ &\leqslant \sqrt{\log\frac{4}{3}} \sup_{|\varphi(w)|\leqslant 1/2} \mu(|w|)|\psi(w)| \preceq \|\psi C_\varphi\| < \infty. \end{aligned} \tag{8}$$

From (7) and (8) we get

$$\sup_{w\in\mathbb{B}_n} \mu(|w|)|\psi(w)|\sqrt{\log\frac{1}{1-|\varphi(w)|^2}} \preceq \|\psi C_\varphi\|. \tag{9}$$

From (3) and (9), (2) follows. □

THEOREM 2. *Let μ be a normal function on $[0,1)$. Assume that $\psi \in H(\mathbb{B}_n)$ and φ is a holomorphic self-map of $\mathbb{B}_n$. Then $\psi C_\varphi : D_n \to H_\mu^\infty$ is compact if and only if $\psi \in H_\mu^\infty$ and*

$$\lim_{|\varphi(z)|\to 1} \mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}} = 0. \tag{10}$$

Proof. First we assume that $\psi \in H_\mu^\infty$ and (10) holds. It is easy to see that

$$\sup_{z\in\mathbb{B}_n} \mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}} < \infty.$$

Hence $\psi C_\varphi : D_n \to H_\mu^\infty$ is bounded. Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in D_n with $\sup_{k\in\mathbb{N}} \|f_k\|_{D_n} \leqslant C$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}_n$ as $k \to \infty$. By the assumption, for any $\varepsilon > 0$, there is a constant $\delta, 0 < \delta < 1$, such that

$$\mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}} < \varepsilon$$

whenever $\delta < |\varphi(z)| < 1$. Let $E = \{z \in \mathbb{B}_n : |\varphi(z)| \leqslant \delta\}$. Using Lemma 1, we have

$$\begin{aligned} &\| \psi C_\varphi f_k \|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}_n} \mu(|z|)|(\psi C_\varphi f_k)(z)| \\ &= \sup_{z\in E} \mu(|z|)|\psi(z) f_k(\varphi(z))| + \sup_{z\in\mathbb{B}_n\setminus E} \mu(|z|)|\psi(z) f_k(\varphi(z))| \\ &\preceq \sup_{z\in E} \mu(|z|)|\psi(z) f_k(\varphi(z))| + \sup_{z\in\mathbb{B}_n\setminus E} \mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}}\|f_k\|_{D_n} \\ &\leqslant \sup_{z\in\mathbb{B}_n} \mu(|z|)|\psi(z)| \cdot \sup_{|w|\leqslant\delta} |f_k(w)| + C\varepsilon. \end{aligned}$$

By the assumption we obtain $\limsup_{k\to\infty}\|\psi C_\varphi f_k\|_{H^\infty_\mu} \leqslant C\varepsilon$. Since ε is an arbitrary positive number we have that $\lim_{k\to\infty}\|\psi C_\varphi f_k\|_{H^\infty_\mu}=0$. By Lemma 3 we see that $\psi C_\varphi : D_n \to H^\infty_\mu$ is compact.

Conversely, suppose $\psi C_\varphi : D_n \to H^\infty_\mu$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}_n$ such that $|\varphi(z_k)|\to 1$ as $k\to\infty$. If such a sequence does not exist conditions (10) is automatically satisfied. Take

$$f_k(z)=\Big(\log\frac{1}{1-\langle z,\varphi(z_k)\rangle}\Big)\Big(\log\frac{1}{1-|\varphi(z_k)|^2}\Big)^{-1/2}, k\in\mathbb{N}. \tag{11}$$

Then $f_k\in D_n$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{B}_n$ as $k\to\infty$. Since ψC_φ is compact, by Lemma 3 we have $\|\psi C_\varphi f_k\|_{H^\infty_\mu}\to 0$ as $k\to\infty$. Thus

$$\begin{aligned}\mu(|z_k|)|\psi(z_k)|\sqrt{\log\frac{1}{1-|\varphi(z_k)|^2}} &\leqslant \sup_{z\in\mathbb{B}_n}\mu(|z|)|(\psi C_\varphi f_k)(z)|\\ &=\|\psi C_\varphi f_k\|_{H^\infty_\mu}\to 0,\end{aligned}$$

as $k\to\infty$. From which (10) follows. □

THEOREM 3. *Let μ be a normal function on $[0,1)$. Assume that $\psi\in H(\mathbb{B}_n)$ and φ is a holomorphic self-map of $\mathbb{B}_n$ such that $\psi C_\varphi : D_n\to H^\infty_{\mu,0}$ is bounded. Then $\psi C_\varphi : D_n\to H^\infty_{\mu,0}$ is compact if and only if*

$$\lim_{|z|\to 1}\mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}}=0. \tag{12}$$

Proof. Suppose that (12) holds. From Lemma 2, we see that $\psi C_\varphi : D_n\to H^\infty_{\mu,0}$ is compact if and only if

$$\lim_{|z|\to 1}\sup_{\|f\|_{D_n}\leqslant 1}\mu(|z|)|(\psi C_\varphi f)(z)|=0. \tag{13}$$

On the other hand, by Lemma 1, we have that

$$\mu(|z|)|(\psi C_\varphi f)(z)|\preceq \|f\|_{D_n}\mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}}. \tag{14}$$

Taking the supremum in (14) over the unit ball in the space D_n, then letting $|z|\to 1$ and applying (12) the result follows.

Conversely, suppose that $\psi C_\varphi : D_n\to H^\infty_{\mu,0}$ is compact. Then $\psi C_\varphi : D_n\to H^\infty_\mu$ is compact and hence by Theorem 2,

$$\lim_{|\varphi(z)|\to 1}\mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}}=0. \tag{15}$$

In addition, by the boundedness of $\psi C_\varphi : D_n \to H^\infty_{\mu,0}$ we see that $\psi \in H^\infty_{\mu,0}$. By (15), for every $\varepsilon > 0$, there exists a $\delta \in (0,1)$ such that

$$\mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}} < \varepsilon$$

when $\delta < |\varphi(z)| < 1$. Since $\psi \in H^\infty_{\mu,0}$, for above chosen ε, there exists an $r \in (0,1)$,

$$\mu(|z|)|\psi(z)| < \varepsilon\Big(\log\frac{1}{1-\delta^2}\Big)^{-1/2}$$

when $r < |z| < 1$.

Therefore, when $r < |z| < 1$ and $\delta < |\varphi(z)| < 1$, we have that

$$\mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}} < \varepsilon. \tag{16}$$

If $|\varphi(z)| \leqslant \delta$ and $r < |z| < 1$, we obtain

$$\mu(|z|)|\psi(z)|\sqrt{\log\frac{1}{1-|\varphi(z)|^2}} \leqslant \sqrt{\log\frac{1}{1-\delta^2}}\mu(|z|)|\psi(z)| < \varepsilon. \tag{17}$$

Combing (16) with (17) we get (12), as desired. □

THEOREM 4. *Let μ be a normal function on $[0,1)$. Assume that $p < n$, $\psi \in H(\mathbb{B}_n)$ and φ is a holomorphic self-map of $\mathbb{B}_n$. Then $\psi C_\varphi : D_p \to H^\infty_\mu$ is bounded if and only if*

$$\sup_{z\in\mathbb{B}_n}\frac{\mu(|z|)}{(1-|\varphi(z)|^2)^{\frac{n-p}{2}}}|\psi(z)| < \infty. \tag{18}$$

Moreover, the following relationship holds:

$$||\psi C_\varphi|| \asymp \sup_{z\in\mathbb{B}_n}\frac{\mu(|z|)}{(1-|\varphi(z)|^2)^{\frac{n-p}{2}}}|\psi(z)|. \tag{19}$$

Proof. Assume that (18) holds. Let $f \in D_p$. By Lemma 1,

$$\|\psi C_\varphi f\|_{H^\infty_\mu} = \sup_{z\in\mathbb{B}_n}|f(\varphi(z))|\mu(|z|)|\psi(z)| \preceq \|f\|_{D_p}\sup_{z\in\mathbb{B}_n}\frac{\mu(|z|)}{(1-|\varphi(z)|^2)^{\frac{n-p}{2}}}|\psi(z)|.$$

Therefore $\psi C_\varphi : D_p \to H^\infty_\mu$ is bounded and

$$\|\psi C_\varphi\| \preceq \sup_{z\in\mathbb{B}_n}\frac{\mu(|z|)}{(1-|\varphi(z)|^2)^{\frac{n-p}{2}}}|\psi(z)|. \tag{20}$$

Conversely, assume that $\psi C_\varphi : D_p \to H_\mu^\infty$ is bounded. Similarly to the proof of Theorem 1 we get $\psi \in H_\mu^\infty$. For any $w \in \mathbb{B}_n$ such that $|\varphi(w)| > 1/2$, set

$$f_w(z) = \frac{1-|\varphi(w)|^2}{(1-\langle z, \varphi(w)\rangle)^{1+\frac{n-p}{2}}}. \tag{21}$$

Using the Stirling formula, we have

$$\begin{aligned}
&\|f_w\|_{D_p}^2 \\
&= (1-|\varphi(w)|^2)^2 \sum_{|\beta|\geqslant 0} (n+|\beta|)^p \Big(\frac{\Gamma(|\beta|+1+\frac{n-p}{2})}{\beta!\Gamma(1+\frac{n-p}{2})}\Big)^2 |\varphi^\beta(w)|^2 \omega_\beta \\
&= (1-|\varphi(w)|^2)^2 \sum_{|\beta|\geqslant 0} (n+|\beta|)^p \Big(\frac{\Gamma(|\beta|+1+\frac{n-p}{2})}{\beta!\Gamma(1+\frac{n-p}{2})}\Big)^2 |\varphi^\beta(w)|^2 \frac{(n-1)!\beta!}{(n+|\beta|-1)!} \\
&\preceq (1-|\varphi(w)|^2)^2 \frac{(n-1)!}{\Gamma^2(1+\frac{n-p}{2})} \sum_{k=0}^\infty \frac{(n+k)^p \Gamma^2(k+1+\frac{n-p}{2})}{k!(n+k-1)!} |\varphi(w)|^{2k} \\
&\preceq (1-|\varphi(w)|^2)^2 \sum_{k=1}^\infty k |\varphi(w)|^{2k} \\
&\preceq 1,
\end{aligned} \tag{22}$$

which implies that $f_w \in D_p$ and $\|f_w\|_{D_p} \preceq 1$. Therefore, the boundedness of $\psi C_\varphi : D_p \to H_\mu^\infty$ implies that

$$\sup_{z\in\mathbb{B}_n} \mu(|z|)|f_w(\varphi(z))\psi(z)| = \|\psi C_\varphi f_w\|_{H_\mu^\infty} \leqslant \|\psi C_\varphi\| \|f_w\|_{D_p} \preceq \|\psi C_\varphi\|.$$

Taking $z = w$ in the above inequality, we get

$$\frac{\mu(|w|)}{(1-|\varphi(w)|^2)^{\frac{n-p}{2}}} |\psi(w)| \preceq \|\psi C_\varphi\| < \infty. \tag{23}$$

Hence

$$\sup_{|\varphi(w)|>1/2} \frac{\mu(|w|)}{(1-|\varphi(w)|^2)^{\frac{n-p}{2}}} |\psi(w)| \preceq \|\psi C_\varphi\| < \infty. \tag{24}$$

The rest of the proof is similar to the proof of Theorem 1 and hence we omit it. □

By taking the sequece

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\langle z, \varphi(z_k)\rangle)^{1+\frac{n-p}{2}}}, \quad k = 1, 2, \ldots, \tag{25}$$

and similar to the proof of Theorem 2, we obtain the following result.

THEOREM 5. *Let* μ *be a normal function on* $[0,1)$. *Assume that* $p<n$, $\psi \in H(\mathbb{B}_n)$ *and* φ *is a holomorphic self-map of* $\mathbb{B}_n$ *such that* $\psi C_\varphi : D_p \to H_\mu^\infty$ *is bounded. Then* $\psi C_\varphi : D_p \to H_\mu^\infty$ *is compact if and only if* $\psi \in H_\mu^\infty$ *and*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(|z|)}{(1-|\varphi(z)|^2)^{\frac{n-p}{2}}}|\psi(z)| = 0. \tag{26}$$

Similarly to the proof of Theorem 3, we get the following result. We omit the proof also.

THEOREM 6. *Let* μ *be a normal function on* $[0,1)$. *Assume that* $p<n$, $\psi \in H(\mathbb{B}_n)$ *and* φ *is a holomorphic self-map of* $\mathbb{B}_n$ *such that* $\psi C_\varphi : D_p \to H_{\mu,0}^\infty$ *is bounded. Then* $\psi C_\varphi : D_p \to H_{\mu,0}^\infty$ *is compact if and only if*

$$\lim_{|z|\to 1} \frac{\mu(|z|)}{(1-|\varphi(z)|^2)^{\frac{n-p}{2}}}|\psi(z)| = 0.$$

THEOREM 7. *Let* μ *be a normal function on* $[0,1)$. *Assume that* $n<p<\infty$, $\psi \in H(\mathbb{B}_n)$ *and* φ *is a holomorphic self-map of* $\mathbb{B}_n$. *Then the following statements are equivalent.*

(i) The operator $\psi C_\varphi : D_p \to H_\mu^\infty$ *is compact;*
(ii) The operator $\psi C_\varphi : D_p \to H_\mu^\infty$ *is bounded;*
(iii) $\psi \in H_\mu^\infty$.

Proof. $(i) \Rightarrow (ii)$. This implication is obvious.

$(ii) \Rightarrow (iii)$. Taking $f(z)=1$, then using the boundedness of $\psi C_\varphi : D_p \to H_\mu^\infty$ the implication follows.

$(iii) \Rightarrow (i)$. Suppose that $\psi \in H_\mu^\infty$. For any $f \in D_p$, by Lemma 1 we have

$$\mu(|z|)(\psi C_\varphi f)(z)| = \mu(|z|)|f(\varphi(z))||\psi(z)| \preceq \|f\|_{D_p}\mu(|z|)|\psi(z)|. \tag{27}$$

From the above inequality we see that $\psi C_\varphi : D_p \to H_\mu^\infty$ is bounded since $\psi \in H_\mu^\infty$. Let $(f_k)_{k\in\mathbb{N}}$ be any bounded sequence in D_p and $f_k \to 0$ uniformly on compact subsets of $\mathbb{B}_n$ as $k\to\infty$. Employing Lemma 4 we have

$$\|\psi C_\varphi f_k\|_{H_\mu^\infty} = \sup_{z\in\mathbb{B}_n} \mu(|z|)|f_k(\varphi(z))\psi(z)| \leqslant \|\psi\|_{H_\mu^\infty} \sup_{z\in\mathbb{B}_n} |f_k(\varphi(z))| \to 0,$$

as $k\to\infty$. Then the result follows from Lemma 3. □

THEOREM 8. *Let* μ *be a normal function on* $[0,1)$. *Assume that* $n<p<\infty$, $\psi \in H(\mathbb{B}_n)$ *and* φ *is a holomorphic self-map of* $\mathbb{B}_n$. *Then the following statements are equivalent.*

(i) The operator $\psi C_\varphi : D_p \to H_{\mu,0}^\infty$ *is compact;*
(ii) The operator $\psi C_\varphi : D_p \to H_{\mu,0}^\infty$ *is bounded;*
(iii) $\psi \in H_{\mu,0}^\infty$.

Proof. $(i) \Rightarrow (ii)$. It is obvious.

$(ii) \Rightarrow (iii)$. Taking $f(z) = 1$, then using the boundedness of $\psi C_\varphi : D_p \to H^\infty_{\mu,0}$ we get the desired result.

$(iii) \Rightarrow (i)$. Suppose that $\psi \in H^\infty_{\mu,0}$. For any $f \in D_p$ with $\|f\|_{D_p} \leqslant 1$, we have

$$\mu(|z|)|(\psi C_\varphi f)(z)| \preceq \|f\|_{D_p}\mu(|z|)|\psi(z)| \preceq \mu(|z|)|\psi(z)|,$$

which implies that

$$\lim_{|z|\to 1} \sup_{\|f\|_{D_p}\leqslant 1} \mu(|z|)|(\psi C_\varphi f)(z)| \preceq \lim_{|z|\to 1} \mu(|z|)|\psi(z)| = 0.$$

From Lemma 2 we see that $\psi C_\varphi : D_p \to H^\infty_{\mu,0}$ is compact, as desired. □

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