Mathematical
Inequalities
& Applications
Volume 19, Number 3 (2016), 1039–1048 doi:10.7153/mia-19-77

CONTINUED FRACTION EXPRESSION OF THE MATHIEU SERIES

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(*Communicated by J. Pečarić*)

Abstract. In this paper, we represent a continued fraction expression of the Mathieu series by a continued fraction formula of Ramanujan. As applications, we obtain some new bounds for the Mathieu series.

1. Introduction

The infinite series

$$S(r) := \sum_{m=1}^{\infty} \frac{2m}{(m^2+r^2)^2}, \quad (r>0) \tag{1}$$

is called a Mathieu series. It was introduced and studied by Émile Leonard Mathieu in his book [19] devoted to the elasticity of solid bodies. Since his introduction, the series $S(r)$ and its various generalizations have attracted many researchers, who established some remarkable properties of these series including the various integral representations, the asymptotic expansions, lower and upper estimates, see e.g. Cerone and Lenard [8], Frontczak [14], Milovanović and Pogány [20], Pogány *et al.* [22], and references quoted therein.

An integral representation for the Mathieu series (1) is given by Emersleben [13] as

$$S(r) = \frac{1}{r} \int_0^{\infty} \frac{x}{e^x-1} \sin(rx) dx. \tag{2}$$

The integral representation was used by Elbert [12] to derive the asymptotic expansion of $S(r)$:

$$S(r) = \sum_{m=0}^{\infty} (-1)^m \frac{\mathscr{B}_{2m}}{r^{2m+2}} = \frac{1}{r^2} - \frac{1}{6r^4} \pm \cdots, \quad (r \to \infty), \tag{3}$$

Mathematics subject classification (2010): 11J70, 40A25, 41A20, 26D15.
Keywords and phrases: Continued fraction, Mathieu series, inequality, asymptotic expansion.
This work is supported by the National Natural Science Foundation of China (Grant No. 11171344) and the Natural Science Foundation of Beijing (Grant No. 1112010).

where $\mathscr{B}_{2n}$ denote the even indexed Bernoulli numbers defined by the generating function

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}\mathscr{B}_n\frac{x^n}{n!},\qquad |x|<2\pi.$$

Throughout the paper, we always use notation $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$. Let $(a_n)_{n\geqslant 1}$ and $(b_n)_{n\geqslant 0}$ be two sequences of real (or complex) numbers with $a_n\neq 0$ for all $n\in\mathbb{N}$. The generalized continued fraction

$$\tau=b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\ddots}}=b_0+\frac{a_1}{b_1}+\frac{a_2}{b_2}+\cdots=b_0+\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right) \tag{4}$$

is defined as the limit of the nth approximant

$$\frac{A_n}{B_n}=b_0+\mathop{\mathbf{K}}_{k=1}^{n}\left(\frac{a_k}{b_k}\right) \tag{5}$$

as n tends to infinity. The numerators A_n and denominators B_n of the approximants satisfy the recurrence relations

$$A_{n+2}=b_{n+2}A_{n+1}+a_{n+2}A_n,\quad B_{n+2}=b_{n+2}B_{n+1}+a_{n+2}B_n \tag{6}$$

with initial values $A_0=b_0, B_0=1, A_1=b_0b_1+a_1$ and $B_1=b_1$, see Berndt [5, p. 105]. For the theory of continued fraction, see Cuyt *et al.* [9] or Lorentzen and Waadeland [17].

Let $r>0$ and $\Re x>\frac{1}{2}$. Let the continued fraction $CF(r;x)$ with a parameter r be defined by

$$CF(r;x)=\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{1}{4}(1+4r^2)+\mathbf{K}_{n=1}^{\infty}\left(\frac{\kappa_n}{\left(x-\frac{1}{2}\right)^2+\lambda_n}\right)}, \tag{7}$$

where for $n\in\mathbb{N}$

$$\kappa_n=-\frac{n^4\left(n^2+4r^2\right)}{4(2n-1)(2n+1)},\qquad \lambda_n=\frac{1}{4}(2n^2+2n+1+4r^2). \tag{8}$$

To the best knowledge of authors, until now no continued fraction expression was established for the Mathieu series. In this paper we will establish the following continued fraction formula of the Mathieu series.

THEOREM 1. *Let $r>0$ and $CF(r;x)$ be defined as (7). For all positive integer k, we have*

$$S(r)=\sum_{m=1}^{k-1}\frac{2m}{(m^2+r^2)^2}+CF(r;k), \tag{9}$$

where the sum for $k=1$ is stipulated to be zero. In particular,

$$S(r)=CF(r;1). \tag{10}$$

2. The proof of Theorem 1

In order to prove Theorem 1, we will prepare some lemmas. The following continued fraction formula of Ramanujan plays an important role in the proof of Theorem 1.

LEMMA 1. *Let x,l,m, and n denote complex numbers. Define*

$$P = P(x,l,m,n)$$
$$= \frac{\Gamma\left(\frac{1}{2}(x+l+m+n+1)\right)\Gamma\left(\frac{1}{2}(x+l-m-n+1)\right)\Gamma\left(\frac{1}{2}(x-l+m-n+1)\right)\Gamma\left(\frac{1}{2}(x-l-m+n+1)\right)}{\Gamma\left(\frac{1}{2}(x-l-m-n+1)\right)\Gamma\left(\frac{1}{2}(x-l+m+n+1)\right)\Gamma\left(\frac{1}{2}(x+l-m+n+1)\right)\Gamma\left(\frac{1}{2}(x+l+m-n+1)\right)}.$$

Then if either l,m, or n is an integer or if $\Re x > 0$,

$$\frac{1-P}{1+P} = \frac{2lmn}{x^2-l^2-m^2-n^2+1+\mathbf{K}_{j=1}^{\infty}\left(\frac{4(l^2-j^2)(m^2-j^2)(n^2-j^2)}{(2j+1)\left(x^2-l^2-m^2-n^2+2j^2+2j+1\right)}\right)}. \tag{11}$$

Proof. This is Entry 35 of B. C. Berndt [5, p. 157], which was claimed first by Ramanujan [26, 27]. The first published proof was provided by Watson [29]. For the full proof of Entry 35, we refer the reader to L. Jacobsen's paper [15]. □

Let us recall two definitions in the theory of the continued fraction. We say that two continued fractions are *equivalent* if they have the same sequence of classical approximants. We write $b_0+\mathbf{K}(a_n/b_n) \approx d_0+\mathbf{K}(c_n/d_n)$ to express that $b_0+\mathbf{K}(a_n/b_n)$ and $d_0+\mathbf{K}(c_n/d_n)$ are equivalent. Also see [17, p. 73].

We will call $d_0+\mathbf{K}(c_n/d_n)$ a *contraction* of $b_0+\mathbf{K}(a_n/b_n)$ if its classical approximants $\{g_n\}$ form a subsequence of the classical approximants $\{f_n\}$ of $b_0+\mathbf{K}(a_n/b_n)$. In particular, we call $d_0+\mathbf{K}(c_n/d_n)$ a *canonical contraction* of $b_0+\mathbf{K}(a_n/b_n)$ if

$$C_k = A_{n_k}, \quad D_k = B_{n_k} \quad \text{for} \quad k = 0,1,2,\ldots, \tag{12}$$

where C_n, D_n, A_n and B_n are canonical numerators and denominators of $d_0+\mathbf{K}(c_n/d_n)$ and $b_0+\mathbf{K}(a_n/b_n)$ respectively. See [17, p. 83].

LEMMA 2. *The canonical contraction of $b_0+\mathbf{K}(a_n/b_n)$ with*

$$C_k = A_{2k}, \quad D_k = B_{2k} \quad \textit{for} \quad k = 0,1,2,\ldots$$

exists if and only if $b_{2k} \neq 0$ for $k = 0,1,2,\ldots$, and is given by

$$b_0 + \frac{a_1b_2}{a_2+b_1b_2}\,\frac{a_2a_3b_4}{-a_3b_4+b_2(a_4+b_3b_4)}\,\frac{a_4a_5b_2b_6}{-a_5b_6+b_4(a_6+b_5b_6)} - \cdots$$
$$- \frac{a_{2n}a_{2n+1}b_{2n-2}b_{2n+2}}{a_{2n+1}b_{2n+2}+b_{2n}(a_{2n+2}+b_{2n+1}b_{2n+2})} - \cdots.$$

Proof. It follows from Theorem 12 and Eq. (2.4.3) of L. Lorentzen and H. Waadeland [17, pp. 83–84]. For applications, interested readers may refer to Berndt [5, p. 121, Eq. (14.2)] or [5, p. 157]. □

LEMMA 3. *$b_0+\mathbf{K}(a_n/b_n)\approx d_0+\mathbf{K}(c_n/d_n)$ if and only if there exists a sequence $\{r_n\}$ of complex numbers with $r_0=1, r_n\neq 0$ for all $n\in\mathbb{N}$, such that*

$$d_0=b_0,\quad c_n=r_{n-1}r_na_n,\quad d_n=r_nb_n\quad \text{for all } n\in\mathbb{N}.$$

Proof. As to the proof consult [17, p. 73, Theorem 9]. □

Now, we will prove the following lemma, from which Theorem 1 follows readily by the telescoping method.

LEMMA 4. *Let $r>0$, $\Re x>\frac{1}{2}$ and $CF(r;x)$ be defined by (7), then*

$$CF(r;x)-CF(r;x+1)=\frac{2x}{(x^2+r^2)^2}. \tag{13}$$

Proof. Replacing x by $2x-1$ and l by $2ri$ in Lemma 1, respectively, we obtain that for $\Re x>\frac{1}{2}$

$$\frac{1-P}{(1+P)}=\frac{4rmni}{(2x-1)^2+4r^2-m^2-n^2+1+\mathbf{K}_{j=1}^{\infty}\left(\frac{4(-4r^2-j^2)(m^2-j^2)(n^2-j^2)}{(2j+1)\left((2x-1)^2+4r^2-m^2-n^2+2j^2+2j+1\right)}\right)}.$$

By dividing both sides by $4rmni$, we have

$$\frac{1}{4ri}\frac{1-P}{mn(1+P)}=\frac{1}{(2x-1)^2+4r^2-m^2-n^2+1+\mathbf{K}_{j=1}^{\infty}\left(\frac{4(-4r^2-j^2)(m^2-j^2)(n^2-j^2)}{(2j+1)\left((2x-1)^2+4r^2-m^2-n^2+2j^2+2j+1\right)}\right)}. \tag{14}$$

Now let m tend to zero and n tend to zero, successively. On the right hand side, we arrive at

$$\frac{1}{(2x-1)^2+1+4r^2+\mathbf{K}_{j=1}^{\infty}\left(\frac{-4j^4(j^2+4r^2)}{(2j+1)\left((2x-1)^2+2j^2+2j+1+4r^2\right)}\right)}. \tag{15}$$

On the other hand, from the definition of P, we see easily that $\lim_{m\to 0}P=1$. A direct calculation with the use of L'Hôspital's rule gives

$$\lim_{m\to 0}\frac{1-P}{m(1+P)}=\lim_{m\to 0}\frac{1}{1+P}\lim_{m\to 0}\frac{1-P}{m}=\frac{1}{2}\lim_{m\to 0}\frac{1-P}{m}=\frac{1}{2}\lim_{m\to 0}\frac{\partial}{\partial m}(1-P)$$
$$=\frac{1}{2}\left\{-\psi(-\frac{n}{2}+x-ri)+\psi(\frac{n}{2}+x-ri)+\psi(-\frac{n}{2}+x+ri)-\psi(\frac{n}{2}+x+ri)\right\}. \tag{16}$$

By making use of L'Hôspital's rule again, and noting the following classical representation [1, p. 259, Eq. 6.3.16]

$$\psi(z+1) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+z}\right), \quad (z \neq -1, -2, -3, \ldots),$$

where γ denotes Euler-Mascheroni constant, it follows from (16) that

$$\begin{aligned}\lim_{n\to 0}\lim_{m\to 0}\frac{1-P}{mn(1+P)} &= \lim_{n\to 0}\frac{\partial}{\partial n}\left(\lim_{m\to 0}\frac{1}{m}\frac{1-P}{1+P}\right)\\ &= \frac{1}{2}\left(\psi'(x-ri) - \psi'(x+ri)\right)\\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{1}{(x-ri+k)^2} - \sum_{k=0}^{\infty}\frac{1}{(x+ri+k)^2}\right)\\ &= 2ri\sum_{k=0}^{\infty}\frac{x+k}{\left((x+k)^2+r^2\right)^2}. \end{aligned} \tag{17}$$

Combining (15) and (17), we get that for $\Re x > \frac{1}{2}$

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{2(x+k)}{\left((x+k)^2+r^2\right)^2} &= \frac{4}{(2x-1)^2+1+4r^2+\mathbf{K}_{j=1}^{\infty}\left(\frac{-4j^4(j^2+4r^2)}{(2j+1)\left((2x-1)^2+2j^2+2j+1+4r^2\right)}\right)}\\ &= CF(r;x). \end{aligned} \tag{18}$$

Here we used Lemma 3 in the last equality. It is not difficult to check that for $\Re x > \frac{1}{2}$

$$\begin{aligned}CF(r;x) - CF(r;x+1) &= \sum_{k=0}^{\infty}\frac{2(x+k)}{\left((x+k)^2+r^2\right)^2} - \sum_{k=0}^{\infty}\frac{2(x+k+1)}{\left((x+k+1)^2+r^2\right)^2}\\ &= \frac{2x}{\left(x^2+r^2\right)^2}. \end{aligned} \tag{19}$$

This completes the proof of Lemma 4. □

REMARK 1. In fact, Lemma 4 was guessed first by the *multiple-correction method* developed in [6, 7]. We stress that Lemma 4 gives a continued fraction solution of the following difference equation

$$y(x) - y(x+1) = \frac{2x}{\left(x^2+r^2\right)^2}.$$

Mortici [21] made an important contribution in this direction.

3. Some new inequalities for the Mathieu series

The bounds for the Mathieu series attracted many mathematicians like Schröder [28], Emersleben [13], Makai [18] and Diananda [10]. In the past twenty years, many authors like Alzer, Bagdasaryan, Brenner, Guo, Lampret, Milovanović, Mortici, Pogány, Qi, Ruehr, Srivastava, Tomovski, *etc.* have made important contributions to this research topic, see e.g. [2, 3, 11, 16, 20, 21, 22, 23, 24, 25] and references therein. Let us briefly recall some simple results.

Mathieu [19] himself conjectured only the upper bound $S(r) < r^{-2}$, $r > 0$, proved first by Berg [4]. Makai [18] showed the double sided inequalities

$$\frac{1}{r^2+\frac{1}{2}} < S(r) < \frac{1}{r^2+\frac{1}{6}}. \tag{20}$$

Alzer *et al.* [2] improved the lower bound to

$$\frac{1}{r^2+\frac{1}{2\zeta(3)}} < S(r) < \frac{1}{r^2+\frac{1}{6}}, \tag{21}$$

where the constant $1/(2\zeta(3))$ and $1/6$ are sharp.

Milovanović and Pogány [20] stated a composite upper bound of simple structure,

$$S(r) \leqslant \begin{cases} \frac{1}{r^2+\frac{1}{4}}, & 0 \leqslant r \leqslant \frac{\sqrt{3}}{2}, \\ \frac{1}{\sqrt{1+4r^2}-1}, & r > \frac{\sqrt{3}}{2}, \end{cases} \tag{22}$$

which is superior to (21) in the interval $[0, \sqrt{(5+2\sqrt{3})/6}) = [0, 1.18772\ldots)$.

Let $a_1 = 1$, $b_1 = \left(x-\frac{1}{2}\right)^2 - \frac{1}{4} + r^2$, for $n \in \mathbb{N}$

$$a_{2n+1} = \frac{n\left(n^2+4r^2\right)}{2(2n+1)}, \quad a_{2n} = \frac{n^3}{2(2n-1)}, \tag{23}$$

and

$$b_{2n+1} = \left(x-\frac{1}{2}\right)^2 - \frac{1}{4} + \frac{r^2}{2n+1}, \quad b_{2n} = 1. \tag{24}$$

By Lemma 2, it is not difficult to prove that

$$CF(r;x) = \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right). \tag{25}$$

We let $z = (x-\frac{1}{2})^2 - \frac{1}{4}$, $c_1 = 2$, $d_1 = 2z + 2r^2$, for $n \in \mathbb{N}$

$$\begin{cases} c_{2n+1} = n(n^2+4r^2), & c_{2n} = n^3, \\ d_{2n+1} = 2(2n+1)z + 2r^2, & d_{2n} = 1. \end{cases} \tag{26}$$

It follows easily from Lemma 3 that

LEMMA 5. *Let* $r > 0$ *and* $\Re x > \frac{1}{2}$. *With the above notation, we have*

$$CF(r;x) = \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n}{b_n}\right) \approx \mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{c_n}{d_n}\right). \tag{27}$$

LEMMA 6. *Assume that* $r > 0$ *and* $x > \frac{1}{2}$. *Let two sequences* $(a_n)_{n\geqslant 1}, (b_n)_{n\geqslant 1}$ *be defined by* (23) *and* (24), *respectively. For all positive integer* l, *we have*

$$\mathop{\mathbf{K}}_{n=1}^{2l}\left(\frac{a_n}{b_n}\right) < CF(r;x) < \mathop{\mathbf{K}}_{n=1}^{2l-1}\left(\frac{a_n}{b_n}\right). \tag{28}$$

Proof. Let A_m and B_m be the numerators and denominators of the kth approximant of $\mathbf{K}_{n=1}^{\infty}(a_n/b_n)$. As the partial coefficients of the continued fraction $\mathbf{K}_{n=1}^{\infty}(a_n/b_n)$ are positive, it follows from the theory of the continued fraction that the sequence $(A_{2l-1}/B_{2l-1})_{l\geqslant 1}$ is strictly decreasing, $(A_{2l}/B_{2l})_{l\geqslant 1}$ is strictly increasing. Now the bounds (28) are deduced readily from the first equality in Lemma 5.

Let the error term $E_m(x) := CF(r;x) - \mathbf{K}_{n=1}^{m}(a_n/b_n)$. By the recurrence relations (6) and induction, we deduce that

$$\frac{A_{m+1}}{B_{m+1}} - \frac{A_m}{B_m} = \frac{(-1)^m \prod_{j=1}^{m+1} a_j}{B_{m+1}B_m}. \tag{29}$$

Combining (27) and (29), we find that

$$\begin{aligned} |E_m(x)| &< \left| \mathop{\mathbf{K}}_{n=1}^{m+1}\left(\frac{a_n}{b_n}\right) - \mathop{\mathbf{K}}_{n=1}^{m}\left(\frac{a_n}{b_n}\right) \right| \\ &= \left| \frac{A_{m+1}}{B_{m+1}} - \frac{A_m}{B_m} \right| = \frac{\prod_{j=1}^{m+1} a_j}{B_{m+1}B_m}. \end{aligned} \tag{30}$$

We observe that both B_{2l-1} and B_{2l} ($l \in \mathbb{N}$) are polynomials of degree $2l$ in x. Thus, for odd m

$$\limsup_{x\to\infty} x^{4m} |E_m(x)| \leqslant C_m, \tag{31}$$

and for even m

$$\limsup_{x\to\infty} x^{4m+2} |E_m(x)| \leqslant C_m, \tag{32}$$

where C_m is a computable constant. The above two results show that the error term $E_m(x)$ has a better convergence for large "x". □

Motivated by Mortici [21], the following theorem tells us how to obtain the double sided inequalities of continued fraction structure for the Mathieu series.

THEOREM 2. *Let* $r>0$, $k,l\in\mathbb{N}$, *and and* $CF(r;x)$ *be defined as* (7). *Let two sequences* $(a_n)_{n\geqslant 1}, (b_n)_{n\geqslant 1}$ *be defined by* (23) *and* (24) *with* $x=k$, *respectively. Then*

$$\sum_{m=1}^{k-1}\frac{2m}{(m^2+r^2)^2}+\mathop{\mathbf{K}}_{n=1}^{2l}\left(\frac{a_n}{b_n}\right)<S(r)<\sum_{m=1}^{k-1}\frac{2m}{(m^2+r^2)^2}+\mathop{\mathbf{K}}_{n=1}^{2l-1}\left(\frac{a_n}{b_n}\right), \tag{33}$$

where the sum for $k=1$ *is stipulated to be zero. In particular,*

$$\frac{2}{(1+r^2)^2}+\frac{1}{5/2+r^2}<S(r)<\frac{2}{(1+r^2)^2}+\frac{1}{2+r^2}, \tag{34}$$

$$\frac{2}{(1+r^2)^2}+\frac{4}{(4+r^2)^2}+\frac{1}{13/2+r^2}<S(r)<\frac{2}{(1+r^2)^2}+\frac{4}{(4+r^2)^2}+\frac{1}{6+r^2}. \tag{35}$$

Proof. (33) follows readily from Theorem 1 and Lemma 6. Taking $(k,l)=(2,1)$ and $(k,l)=(3,1)$ in (33), respectively, we can obtain (34) and (35). □

REMARK 2. For comparison, our upper bound in (34) improves (22) when $0\leqslant r<\sqrt{-2+\sqrt{7}}\approx 0.803587$. It is not hard to check that if

$$r\in\left(\sqrt{\frac{-6+5\zeta(3)}{2+\sqrt{-2+11\zeta(3)-5\zeta^2(3)}}},\sqrt{\frac{2+\sqrt{-2+11\zeta(3)-5\zeta^2(3)}}{\zeta(3)-1}}\right)$$
$$=(0.0507\ldots,4.4490\ldots),$$

then our lower bound in (34) is superior to Alzer's in (21). In addition, the bounds in (35) are always superior to the bounds in (34) for all $r>0$.

REMARK 3. Taking $k=2$ in Theorem 1, letting r tend to zero, and then applying the second equality in Lemma 5, we can obtain the following continued fraction representation for Apéry number $\zeta(3)$

$$\zeta(3)=1+\frac{1}{2^2\cdot 1}\,\frac{1^3}{+1}+\frac{1^3}{2^2\cdot 3}+\frac{2^3}{1}+\frac{2^3}{2^2\cdot 5}+\cdots.$$

Also see Berndt [5, p. 155].

OPEN PROBLEM. For $r^2\in\mathbb{Q}$, the Mathieu series $S(r)$ is an irrational number.

Acknowledgements. The authors would like to thank the referee for his/her careful reading of the manuscript and insightful comments which led to improvement of our original manuscript, especially the referee proposed the above open problem.

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(Received November 12, 2015)

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