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GENERALIZATION OF GOLDEN–THOMPSON TYPE INEQUALITIES FOR NORMAL MATRICES

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Abstract. We survey some well-known matrix exponential formulae, with emphasis on log-majorization results, by using the compound matrix method.

1. Introduction

This is a short survey of some well-known matrix exponential formulae, with emphasis on log-majorization results, by using the compound matrix method. Though proofs are given for the sake of completeness or of showing compound matrix method, the author does not claim originality of these beautiful results.

Let $\mathscr{M}_n$ be the linear algebra of all $n \times n$ complex matrices, let $\mathscr{H}_n$ be the real subspace of all Hermitian matrices, let $\mathscr{P}_n$ be the set of all positive definite matrices in $\mathscr{M}_n$, and let $\mathscr{N}_n$ be the set of all normal matrices in $\mathscr{M}_n$.

The famous Golden-Thompson inequality asserts that

$$\operatorname{tr} e^{A+B} \leqslant \operatorname{tr} e^{A} e^{B} \tag{1.1}$$

for all $A, B \in \mathscr{H}_n$, where $\operatorname{tr} X$ denotes the trace of $X \in \mathscr{M}_n$. Equality in (1.1) holds if and only if $AB = BA$ [19, 25]. This celebrated result was independently discovered by Golden [13], Symanzik [26], and Thompson [27] in the same year of 1965, all motivated by statistical mechanics. Since then, the Golden-Thompson inequality has received intensive attention and been generalized in various ways and applied in many fields (see, for example, [1, 3, 4, 5, 8, 12, 15, 16, 18, 19, 24, 28] and the references therein). For historical aspects, one may see a recent paper by Forrester-Thompson [12].

Motivated by the Golden-Thompson inequality (1.1) and problems in linear-quadratic optimal feedback control, Bernstein [4] proved the following inequality

$$\operatorname{tr} e^{A^*} e^{A} \leqslant \operatorname{tr} e^{A^*+A} \tag{1.2}$$

for all $A \in \mathscr{M}_n$, where A^* denotes the Hermitian adjoint of A. So [25] showed that equality in (1.2) holds if and only if A is normal.

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Another interesting trace inequality is the Lieb-Thirring inequality [20]

$$\operatorname{tr}(A^{1/2}BA^{1/2})^r \leqslant \operatorname{tr}(A^{r/2}B^rA^{r/2}),$$

for all $A,B \in \mathscr{P}_n$ and $r \geqslant 1$. It was further generalized by Araki [2] as

$$\operatorname{tr}(A^{1/2}BA^{1/2})^{rq} \leqslant \operatorname{tr}(A^{r/2}B^rA^{r/2})^q, \qquad \forall q \geqslant 0, \forall r \geqslant 1, \tag{1.3}$$

$$\operatorname{tr}(A^{1/2}BA^{1/2})^{rq} \geqslant \operatorname{tr}(A^{r/2}B^rA^{r/2})^q, \qquad \forall q \geqslant 0, \forall 0 \leqslant r \leqslant 1. \tag{1.4}$$

The trace inequalities (1.1)–(1.4) were generalized in terms of unitarily invariant norms, as well as even sounder form of log-majorization. The main purpose of this article is to give a short survey of these beautiful Golden-Thompson type log-majorization relations and to further generalize them for normal matrices.

2. Log-majorization and unitarily invariant norms

Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be in $\mathbb{R}^n$. Let $x^\downarrow = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})$ denote a rearrangement of the components of x such that $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$. We say that x is *weakly majorized* by y, denoted by $x \prec_w y$, if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n.$$

If, in addition, the equality holds for $k = n$, we say that x is *majorized* by y and denote this by $x \prec y$. When x and y are nonnegative, we say that x is *weakly log-majorized* by y, denoted by $x \prec_{\text{w-log}} y$ if

$$\prod_{i=1}^{k} x_{[i]} \leqslant \prod_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \ldots, n.$$

If, in addition, the equality holds for $k = n$, we say that x is *log-majorized* by y and denote this by $x \prec_{\log} y$. In other words, when x and y are positive, $x \prec_{\log} y$ if and only if $\log x \prec \log y$, and $x \prec_{\text{w-log}} y$ if and only if $\log x \prec_w \log y$. It is known that [15, Proposition 1.3]

$$x \prec_{\log} y \quad \Rightarrow \quad x \prec_{\text{w-log}} y \quad \Rightarrow \quad x \prec_w y. \tag{2.1}$$

Each of the above four types of majorization is a pre-order (and thus induces a partial order) on $\mathbb{R}^n$ or $\mathbb{R}^n_+ =: \{x \in \mathbb{R}^n : x \geqslant 0\}$.

For any $A \in \mathscr{M}_n$, let

$$\lambda(A) = (\lambda_1(A), \ldots, \lambda_n(A))$$

denote the vector of eigenvalues of A whose absolute values are in decreasing order, and let

$$s(A) = (s_1(A), \ldots, s_n(A))$$

denote the vector of singular values of A in decreasing order, and let $|A| = (A^*A)^{1/2}$ so that $\lambda(|A|) = s(A)$.

A matrix norm $|||\cdot|||$ on $\mathscr{M}_n$ is said to be *unitarily invariant* if $|||UAV||| = |||A|||$ for all $A \in \mathscr{M}_n$ and for all unitary $U, V \in \mathscr{M}_n$. As characterized by von Neumann [29], a function $f : \mathscr{M}_n \to \mathbb{R}$ is a unitarily invariant norm if and only if $f(A)$ is a symmetric gauge function on the singular values of A (see also [5, p. 91]). In particular, the operator norm $\|\cdot\|$ defined by

$$\|A\| = \max_{\|x\|=1} \|Ax\| = s_1(A)$$

is unitarily invariant. An equivalent formulation of Fan Dominance Theorem [11] is that for all $A, B \in \mathscr{M}_n$

$$s(A) \prec_{\mathrm{w}} s(B) \quad \Leftrightarrow \quad |||A||| \leqslant |||B||| \text{ for all unitarily invariant norms } |||\cdot|||. \tag{2.2}$$

Combining (2.1) and (2.2), we get that for all $A, B \in \mathscr{M}_n$

$$s(A) \prec_{\log} s(B) \quad \Rightarrow \quad |||A||| \leqslant |||B||| \text{ for all unitarily invariant norms } |||\cdot|||. \tag{2.3}$$

When $A, B \in \mathscr{P}_n$, (2.3) can be rewritten as

$$\lambda(A) \prec_{\log} \lambda(B) \quad \Rightarrow \quad |||A||| \leqslant |||B||| \text{ for all unitarily invariant norms } |||\cdot|||. \tag{2.4}$$

For each $k = 1, \ldots, n$, the k-th compound of $A \in \mathscr{M}_n$ is the $\binom{n}{k} \times \binom{n}{k}$ matrix $C_k(A)$ whose elements are given by

$$C_k(A)_{\alpha,\beta} = \det A[\alpha|\beta],$$

where $\alpha, \beta \in Q_{k,n}$ and $Q_{k,n} = \{\omega = (\omega(1), \ldots, \omega(k)) : 1 \leqslant \omega(1) < \cdots < \omega(k) \leqslant n\}$ is the set of strictly increasing sequences of length k chosen from $1, \ldots, n$. For example, if $n = 3$ and $k = 2$, then

$$C_2(A) = \begin{pmatrix} \det A[1,2|1,2] & \det A[1,2|1,3] & \det A[1,2|2,3] \\ \det A[1,3|1,2] & \det A[1,3|1,3] & \det A[1,3|2,3] \\ \det A[2,3|1,2] & \det A[2,3|1,3] & \det A[2,3|2,3] \end{pmatrix}.$$

In particular, $C_1(A) = A$ and $C_n(A) = \det A$. The following lemma collects some useful properties of compound matrices that will be used later.

LEMMA 2.1. [21, 22, 23] *Let $A, B \in \mathscr{M}_n$. The following statements are true.*

(1) $C_k(AB) = C_k(A)C_k(B)$.

(2) $C_k(A^*) = [C_k(A)]^*$.

(3) If A is unitary (normal, Hermitian, positive definite), so is $C_k(A)$, respectively.

(4) The eigenvalues of $C_k(A)$ are $\prod_{j=1}^{k} \lambda_{\omega(j)}(A)$ for all $\omega \in Q_{k,n}$.

(5) The singular values of $C_k(A)$ are $\prod_{j=1}^{k} s_{\omega(j)}(A)$ for all $\omega \in Q_{k,n}$.

3. Inequalities for normal matrices

We start with the following equivalent Cordes inequalities for the operator norm.

LEMMA 3.1. [9, Cordes] *Let* $A,B \in \mathscr{P}_n$. *Then for the operator norm* $\|\cdot\|$,

$$\|AB\|^r \leqslant \|A^rB^r\|, \qquad \forall r \geqslant 1, \tag{3.1}$$

$$\|A^rB^r\| \leqslant \|AB\|^r, \qquad \forall 0 \leqslant r \leqslant 1. \tag{3.2}$$

For general normal matrix A, the term A^r makes no sense for positive real number r, except r being a positive integer.

COROLLARY 3.2. *Let* $A,B \in \mathscr{N}_n$ *and* $m \in \mathbb{N}$. *Then for the operator norm* $\|\cdot\|$,

$$\|(AB)^m\| \leqslant \|AB\|^m \leqslant \|A^mB^m\|. \tag{3.3}$$

Proof. Obviously $\|(AB)^m\| \leqslant \|AB\|^m$, since $\|\cdot\|$ is a matrix norm. Let $A = UP_A$ and $B = VP_B$ be polar decompositions with $P_A = |A| \in \mathscr{P}_n$ and $P_B = |B| \in \mathscr{P}_n$ and unitary $U,V \in \mathscr{M}_n$. Note that $UP_A = P_AU$ and $VP_B = P_BV$ by the normality of A and B. Then we have for each $m \in \mathbb{N}$

$$\begin{aligned}
\|AB\|^m &= \|UP_AP_BV\|^m \\
&= \|P_AP_B\|^m \\
&\leqslant \|P_A{}^mP_B{}^m\| \qquad \text{(by (3.1))} \\
&= \|U^mP_A{}^mP_B{}^mV^m\| \\
&= \|(UP_A)^m(VP_B)^m\| \\
&= \|A^mB^m\|.
\end{aligned}$$

This completes the proof. □

The following result of Horn, as a generalization of Corollary 3.2 by (2.3), can serve as a typical example of generalizing inequalities for unitarily invariant norms to those for log-majorization. The technique of compound matrix arguments in the proof will be used often.

PROPOSITION 3.3. [17, Horn] *Let* $A,B \in \mathscr{N}_n$ *and* $m \in \mathbb{N}$. *Then*

$$s((AB)^m) \prec_{\log} [s(AB)]^m \prec_{\log} s(A^mB^m). \tag{3.4}$$

Proof. Because $s(X^m) \prec_{\log} [s(X)]^m$ for all $X \in \mathscr{M}_n$ and all $m \in \mathbb{N}$ (see (3.7) below), we have

$$s((AB)^m) \prec_{\log} [s(AB)]^m.$$

It remains to show the second log-majorization. Because $\lambda_1([(AB)^*(AB)]^m) = [\lambda_1((AB)^*(AB))]^m = [s_1(AB)]^{2m} = \|AB\|^{2m}$ and $\lambda_1((A^mB^m)^*(A^mB^m) = [s_1(A^mB^m)]^2 = \|A^mB^m\|^2$, it follows from (3.3) that

$$\lambda_1([(AB)^*(AB)]^m) \leqslant \lambda_1((A^mB^m)^*(A^mB^m)). \tag{3.5}$$

Now applying (3.5) on $C_k(A)$ and $C_k(B)$, by Lemma 2.1 we have for all $1 \leqslant k \leqslant n$

$$\begin{aligned}
\prod_{j=1}^{k} \lambda_j([(AB)^*(AB)]^m) &= \lambda_1(C_k([(AB)^*(AB)]^m)) \\
&= \lambda_1([(C_k(A)C_k(B))^*(C_k(A)C_k(B))]^m) \\
&\leqslant \lambda_1(([C_k(A)]^m[C_k(B)]^m)^*([C_k(A)]^m[(C_k(B)]^m)) \\
&= \lambda_1(C_k((A^mB^m)^*(A^mB^m))) \\
&= \prod_{j=1}^{k} \lambda_j((A^mB^m)^*(A^mB^m)).
\end{aligned}$$

Furthermore, determinant consideration yields

$$\det([(AB)^*(AB)]^m) = \det([A^*A]^m)\det([B^*B]^m) = \det([A^mB^m]^*(A^mB^m)).$$

Therefore by definition of log-majorization we have

$$\lambda([(AB)^*(AB)]^m) \prec_{\log} \lambda((A^mB^m)^*(A^mB^m)),$$

which is equivalent to

$$[s(AB)]^m \prec_{\log} s(A^mB^m).$$

This completes the proof. $\square$

Corollary 3.2 and Proposition 3.3 are not valid for general $A,B \in \mathscr{M}_n$, due to the following result of Ky Fan. To see this, compare $s((AB)^m) \prec_{\log} s(A^mB^m)$ and (3.6). We note that (3.7) also follows from Horn's log-majorization result for singular values.

PROPOSITION 3.4. [10, Fan] *Let $A \in \mathscr{M}_n$ and $m \in \mathbb{N}$. The following two relations are equivalent and valid:*

$$\lambda((A^m)^*A^m) \prec_{\log} \lambda((A^*A)^m), \tag{3.6}$$

$$\lambda(|A^m|) = s(A^m) \prec_{\log} [s(A)]^m = (\lambda(|A|)^m. \tag{3.7}$$

Moreover, $(A^m)^*A^m = (A^*A)^m$ if and only if $A \in \mathscr{N}_n$.

Proof. The equivalence of (3.6) and (3.7) follows immediately. By compound matrix arguments similar to the proof of Proposition 3.3, to derive (3.6) it suffices to show

$$\lambda_1((A^m)^*A^m) \leqslant \lambda_1((A^*A)^m). \tag{3.8}$$

But (3.8) is just the first of the following inequalities by Fan [10] (see [7, Theorem 1] for more interesting inequalities):

$$\sum_{j=1}^{k} \lambda_j((A^m)^*A^m) \leqslant \sum_{j=1}^{k} \lambda_j((A^*A)^m), \qquad 1 \leqslant k \leqslant n.$$

If $A \in \mathscr{N}_n$, then $(A^m)^*A^m = (A^*)^m A^m = (A^*A)^m$. Conversely, if $(A^m)^*A^m = (A^*A)^m$, then $\operatorname{tr}(A^m)^*A^m = \operatorname{tr}(A^*A)^m$, and hence $A \in \mathscr{N}_n$ by [25, Theorem 4.4], which states that "if $\operatorname{tr} X^{*p}X^p = \operatorname{tr}(X^*X)^p$ for some $p \geqslant 2$, then X is normal". □

As an application of Proposition 3.4, the following result of Cohen is a generalization of the Bernstein inequality (1.2). It is a matrix version of the scalar identity $|e^{x+iy}| = e^x$ for $x, y \in \mathbb{R}$.

COROLLARY 3.5. [7, Cohen] *Let $A \in \mathscr{M}_n$. The following two relations are equivalent and valid:*

$$\lambda(e^{A^*}e^A) \prec_{\log} \lambda(e^{A^*+A}), \tag{3.9}$$

$$\lambda(|e^A|) = s(e^A) \prec_{\log} s(e^{\operatorname{Re} A}) = \lambda(e^{\operatorname{Re} A}), \tag{3.10}$$

where $\operatorname{Re} A = (A^* + A)/2$ *is the Hermitian part of A. Moreover, $e^{A^*}e^A = e^{A^*+A}$ if and only if $A \in \mathscr{N}_n$.*

Proof. Obviously, (3.9) and (3.10) are equivalent. Applying (3.6) on $e^{A/m}$ and noting that $(e^A)^* = e^{A^*}$, we get

$$\lambda(e^{A^*}e^A) \prec_{\log} \lambda([e^{A^*/m}e^{A/m}]^m), \qquad \forall m \in \mathbb{N}. \tag{3.11}$$

Combining (3.11) with the Lie product formula (see [8, p. 60] for interesting discussions about the name)

$$\lim_{m \to \infty} (e^{X/m}e^{Y/m})^m = e^{X+Y}, \qquad \forall X, Y \in \mathscr{M}_n, \tag{3.12}$$

we conclude that $\lambda(e^{A^*}e^A) \prec_{\log} \lambda(e^{A^*+A})$ by the continuity of eigenvalues.

If $A \in \mathscr{N}_n$, then A^* and A commute so that $e^{A^*}e^A = e^{A^*+A}$ by a property of the matrix exponential function. Conversely, if $e^{A^*}e^A = e^{A^*+A}$, then $\operatorname{tr} e^{A^*}e^A = \operatorname{tr} e^{A^*+A}$, and hence $A \in \mathscr{N}_n$ by [25, Theorem 4.7]. □

The following result of Araki generalizes (1.3) and (1.4).

PROPOSITION 3.6. [2, Araki] *Let $A, B \in \mathscr{H}_n$. Then*

$$\lambda((e^{A/2}e^Be^{A/2})^r) \prec_{\log} \lambda(e^{rA/2}e^{rB}e^{rA/2}), \qquad \forall r \geqslant 1, \tag{3.13}$$

$$\lambda(e^{rA/2}e^{rB}e^{rA/2}) \prec_{\log} \lambda((e^{A/2}e^Be^{A/2})^r), \qquad \forall 0 \leqslant r \leqslant 1. \tag{3.14}$$

Proof. We only show (3.13), since (3.14) is similar. By compound matrix arguments similar to the proof of Proposition 3.3, it suffices to show

$$\lambda_1((e^{A/2}e^Be^{A/2})^r) \leqslant \lambda_1(e^{rA/2}e^{rB}e^{rA/2}),$$

or, equivalently (since eigenvalues respect power and $\lambda(XY) = \lambda(YX)$ for all $X, Y \in \mathscr{M}_n$),

$$[\lambda_1(e^Ae^B)]^r \leqslant \lambda_1(e^{rA}e^{rB}). \tag{3.15}$$

But (3.15) is equivalent to (3.1) for the operator norm, because

$$\|(e^Ae^B)\|^r = \|e^Ae^{2B}e^A\|^{r/2} = [\lambda_1(e^Ae^{2B}e^A)]^{r/2} = [\lambda_1(e^{2A}e^{2B})]^{r/2}$$

and

$$\|e^{rA}e^{rB}\| = \|e^{rA}e^{2rB}e^{rA}\|^{1/2} = [\lambda_1(e^{rA}e^{2rB}e^{rA})]^{1/2} = [\lambda_1(e^{2rA}e^{2rB})]^{1/2}.$$

This completes the proof. $\square$

Proposition 3.6 yields the following result about normal matrices.

COROLLARY 3.7. *Let $A,B \in \mathscr{N}_n$. Then*

$$\lambda((|e^{A/2}|\cdot|e^B|\cdot|e^{A/2}|)^r) \prec_{\log} \lambda(|e^{rA/2}|\cdot|e^{rB}|\cdot|e^{rA/2}|), \qquad \forall r \geqslant 1, \tag{3.16}$$

$$\lambda(|e^{rA/2}|\cdot|e^{rB}|\cdot|e^{rA/2}|) \prec_{\log} \lambda((|e^{A/2}|\cdot|e^B|\cdot|e^{A/2}|)^r), \qquad \forall 0 \leqslant r \leqslant 1. \tag{3.17}$$

Proof. Note that $|e^A| = e^{(A^*+A)/2}$ by the normality of A, with $(A^*+A)/2 \in \mathscr{H}_n$. Therefore, (3.16) and (3.17) follows from (3.13) and (3.14), respectively. $\square$

The following well-known result is a generalization of the Golden-Thompson inequality (1.1). It first appeared in [28, Lemma 6]. See also [8, Theorem 1] and [5, Theorem IX.3.5]. Instead of applying compound matrix arguments, we offer another proof as an application of Proposition 3.6 and the Lie product formula (3.12).

PROPOSITION 3.8. [28, Thompson] *Let $A,B \in \mathscr{H}_n$. Then*

$$\lambda(e^{A+B}) \prec_{\log} \lambda(e^Ae^B). \tag{3.18}$$

Moreover, $e^{A+B} = e^Ae^B$ if and only if $AB = BA$.

Proof. We first note that (3.13) and (3.14) are both equivalent to

$$\lambda((e^{qA/2}e^{qB}e^{qA/2})^{1/q}) \prec_{\log} \lambda((e^{pA/2}e^{pB}e^{pA/2})^{1/p}), \qquad \forall 0 < q \leqslant p. \tag{3.19}$$

To see this for (3.13), setting $r = p/q$ and replacing A and B by qA and qB, respectively, we get

$$\lambda((e^{qA/2}e^{qB}e^{qA/2})^{p/q}) \prec_{\log} \lambda((e^{pA/2}e^{pB}e^{pA/2})), \qquad \forall 0 < q \leqslant p.$$

Since the matrix products on both sides are positive definite, (3.13) and (3.19) are equivalent. Similarly, (3.14) and (3.19) are equivalent.

Now (3.19) is equivalent to

$$\lambda((e^{qA}e^{qB})^{1/q}) \prec_{\log} \lambda((e^{pA}e^{pB})^{1/p}), \qquad \forall 0 < q \leqslant p. \tag{3.20}$$

Applying the Lie product formula on the left side of (3.20), we have that

$$\lambda(e^{A+B}) \prec_{\log} \lambda((e^{pA}e^{pB})^{1/p}), \qquad \forall p > 0 \tag{3.21}$$

and that $\lambda((e^{pA}e^{pB})^{1/p})$ decreases to $\lambda(e^{A+B})$ as p decrease to 0. In particular, when $p=1$, (3.21) reduces to (3.18).

It follows from [25, Theorem 3.4] that $e^{A+B}=e^Ae^B$ if and only if $AB=BA$. □

Proposition 3.8 yields the following result about normal matrices.

COROLLARY 3.9. *Let* $A,B\in\mathscr{N}_n$. *Then*

$$\lambda(|e^{A+B}|)\prec_{\log}\lambda(|e^A|\cdot|e^B|). \tag{3.22}$$

Proof. Note that Corollary 3.5 yields

$$\lambda(e^{(A+B)^*}e^{A+B})\prec_{\log}\lambda(e^{(A^*+A)+(B^*+B)}). \tag{3.23}$$

By Proposition 3.8 and the normality of A and B, we have

$$\lambda(e^{(A^*+A)+(B^*+B)})\prec_{\log}\lambda(e^{A^*+A}e^{B^*+B})=\lambda(e^{A^*}e^Ae^{B^*}e^B). \tag{3.24}$$

Combining (3.23) and (3.24), we derive the desired (3.22), because eigenvalues respect product and power. □

The following interesting result in [6, Corollary 2.12] is credited to Cohen and Thompson.

COROLLARY 3.10. [7, 28] *Let* $A,B\in\mathscr{M}_n$. *Then*

$$\lambda(|e^{A+B}|)\prec_{\log}\lambda(e^{\mathrm{Re}\,A+\mathrm{Re}\,B})\prec\lambda(e^{\mathrm{Re}\,A}e^{\mathrm{Re}\,B}). \tag{3.25}$$

Proof. The first log-majorization follows from (3.10) and the second (3.18). □

REMARK 3.11.

(1) Because of (2.3), many of the log-majorization relations can be written in terms of unitarily invariant norms (and hence operator norm and trace). For example, (3.10) implies that

$$|||\,e^A\,|||\leqslant|||\,e^{\mathrm{Re}\,A}\,||| \tag{3.26}$$

for all unitarily invariant norms $|||\cdot|||$. The inequality (3.26) appeared in [5, p. 258] and [24, Theorem 18], as a rephrasing of part of [7, Theorem 2].

(2) Proposition 3.6 is indeed a generalization of (1.3) and (1.4). To see this, we show (3.13) implies (1.3). Note that (3.13) implies

$$\lambda((e^{A/2}e^Be^{A/2})^{rq})\prec_{\log}\lambda((e^{rA/2}e^{rB}e^{rA/2})^q),$$

which yields

$$\lambda((e^{A/2}e^Be^{A/2})^{rq})\prec_{\mathrm{w}}\lambda((e^{rA/2}e^{rB}e^{rA/2})^q),$$

and hence (1.3).

(3) By a similar argument as in the equivalence of (3.13) and (3.19), we note that (3.20) is equivalent to

$$\lambda((e^Ae^B)^r) \prec_{\log} \lambda(e^{rA}e^{rB}), \qquad \forall r \geqslant 1, \tag{3.27}$$
$$\lambda(e^{rA}e^{rB}) \prec_{\log} \lambda((e^Ae^B)^r), \qquad \forall 0 \leqslant r \leqslant 1. \tag{3.28}$$

Because $\lambda((e^Ae^B)^r) = [\lambda(e^Ae^B)]^r$, (3.27) and (3.28) yields respectively (3.15) and

$$\lambda_1(e^{rA}e^{rB}) \leqslant [\lambda_1(e^Ae^B)]^r, \qquad \forall 0 \leqslant r \leqslant 1. \tag{3.29}$$

As shown in the proof of Proposition 3.6, (3.15) and (3.29) are respectively equivalent to following Cordes inequalities

$$\|e^Ae^B\|^r \leqslant \|e^{rA}e^{rB}\|, \qquad \forall r \geqslant 1, \tag{3.30}$$
$$\|e^{rA}e^{rB}\| \leqslant \|e^Ae^B\|^r, \qquad \forall 0 \leqslant r \leqslant 1. \tag{3.31}$$

Thus the relations (3.13), (3.14), (3.19), (3.20), (3.27), (3.28), (3.15), (3.29), (3.30), (3.31) are all pairwise equivalent.

(4) Most results in the article can be obtained by applying [5, Theorem IX.3.5] for an appropriate function in the class $\mathscr{T}$. It is known that every symmetric gauge function of the numbers $|\lambda_1(X)|, \ldots, |\lambda_n(X)|$, where $X \in \mathscr{M}_n$, is in $\mathscr{T}$. Because of the usefulness of [5, Theorem IX.3.5], it would be nice to give a characterization of $\mathscr{T}$ (maybe similar to (but definitely not the same) the one for unitarily invariant norms in terms of symmetric gauge functions given by von Neumann [29]).

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