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A COMPARISON TYPE PRINCIPLE FOR A CLASS OF QUASILINEAR ELLIPTIC SYSTEMS: CRITICAL AND SUBCRITICAL CASES

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Abstract. In this paper, we consider the following nonlinear elliptic system:

$$(P)\begin{cases} -\Delta_{p(x)}u = u^{a(x)}v^{b(x)}, & x\in\Omega,\\ -\Delta_{q(x)}v = u^{c(x)}v^{e(x)}, & x\in\Omega,\\ u>0,\ v>0, \end{cases}$$

in a smooth bounded domain $\Omega\subset\mathbb{R}^N$, with different Dirichlet boundary conditions $u=\lambda$, $v=\mu$, $u=v=+\infty$ or $u=\lambda$, $v=+\infty$ on $\partial\Omega$, where $\lambda,\mu>0$. $p,q:\overline{\Omega}\to\mathbb{R}$ are continuous functions with $1<p(x)$, $q(x)<+\infty$, for $x\in\overline{\Omega}$, where $a(x)>p(x)-1$ and $e(x)>q(x)-1$, for $x\in\overline{\Omega}$. The main objective of this paper is to prove existence, nonexistence and uniqueness or multiplicity of positive solutions in both critical and subcritical cases. For this, a comparison type principle is used intensively.

1. Introduction

This paper concerns the study of the following nonlinear elliptic system

$$(P)\begin{cases} -\Delta_{p(x)}u = u^{a(x)}v^{b(x)}, & x\in\Omega,\\ -\Delta_{q(x)}v = u^{c(x)}v^{e(x)}, & x\in\Omega,\\ u>0,\ v>0, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a smooth bounded domain and $-\Delta_{r(x)}u=-\operatorname{div}(|\nabla u|^{r(x)-2}\nabla u)$ is the $r(x)$-Laplacian, $r(.)$ is a function defined on Ω with $1<r(x)<+\infty$, for $x\in\overline{\Omega}$. Moreover, a,b,c and e are continuous functions such that, for $x\in\overline{\Omega}$, $a(x)>p(x)-1$, $e(x)>q(x)-1$ and $b(x),c(x)>0$. This system will be studied under three different types of Dirichlet boundary conditions: both components (u,v) are bounded on $\partial\Omega$, this is the finite case (F), both components (u,v) are blowing up simultaneously, this is the infinite case (I), or one of them is bounded while the other is blowing up, this is the semifinite case (SF). More precisely

$$(F)\begin{cases} u=\lambda, & \text{on } \partial\Omega,\\ v=\mu, & \text{on } \partial\Omega. \end{cases}$$

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$$(I)\begin{cases} u=+\infty, & \text{on } \partial\Omega, \\ v=+\infty, & \text{on } \partial\Omega. \end{cases}$$

Or

$$(SF)\begin{cases} u=\lambda, & \text{on } \partial\Omega, \\ v=+\infty, & \text{on } \partial\Omega. \end{cases}$$

Where $\lambda,\mu>0$. The condition $u=+\infty$ on $\partial\Omega$ means that $u(x)\to+\infty$ as $d(x)\to 0$ with $d(x)=d(x,\partial\Omega)$. Our motivation for this work comes from [27], [18] and [20], where the problem (P) was studied in the quasilinear case (p-Laplacian operator) and in the semilinear case (Laplacian operator, $p=2$). In [18], J. García-Melián obtained necessary and sufficient conditions for the existence and nonexistence of positive solutions. The uniqueness and multiplicity were also obtained, together with the boundary behavior of the solutions in the semifinite case, under the assumptions $(a-p+1)(e-q+1)>bc$ or $(a-p+1)(e-q+1)=bc$. Moreover, in [20], J. García-Meliána and J. D. Rossi showed the existence and nonexistence of positive solutions, uniqueness and non uniqueness, the exact asymptotic behavior of the solutions and their normal derivatives near $\partial\Omega$, assuming that $(a-1)(e-1)>bc$ or $(a-1)(e-1)=bc$, subject to different types of Dirichlet boundary conditions (F), (I) and (SF).

Our objective is to obtain similar results as those in [27] and [18], for the strongly nonlinear system (P). For this, we show that the results for (P) are depending on the sign of $(a(x)-p(x)+1)(e(x)-q(x)+1)-b(x)c(x)$, for $x\in\Omega$. Precisely, we will be interested in the so-called "subcritical" and "critical" cases, given by

$$(a(x)-p(x)+1)(e(x)-q(x)+1)>b(x)c(x),$$

or

$$(a(x)-p(x)+1)(e(x)-q(x)+1)=b(x)c(x),$$

for $x\in\Omega$, respectively. In the subcritical case, we find that the problem (P) has a unique positive solution with each of the boundary conditions. Our proof of existence is based on the method of sub-and-super-solutions. Uniqueness and global estimates are also given, for solution (u,v) satisfying $u=v=+\infty$ on $\partial\Omega$. Moreover, we obtain these global estimates trough an iterative procedure. While in the critical case, we find infinitely many positive solutions. The nonexistence of solutions, both in the subcritical and critical case will be established also through the iterative procedure used for obtaining global bounds.

We underline that all main results obtained in this paper are based on a comparison type principle.

The paper is organized as follows. In Section 2, we give some tools and preliminaries namely definitions of the variable exponent Lebesgue, Sobolev spaces and the comparison principle. In Section 3, we get existence, uniqueness. Moreover, we give nonexistence and global estimates for solutions, in the subcritical case. In Section 4, we are interested in the critical case. Finally, In Section (5) we give an appendix gathering the results of existence of the problem (P), under different types of Dirichlet boundary conditions.

2. Preliminaries

In this paragraph, we recall some definitions and properties of the variable exponent Lebesgue and Sobolev spaces $L^{p(.)}(\Omega)$ and $W^{1,p(.)}(\Omega)$ (for more details, see [11, 15]). We write

$$C_+(\bar{\Omega}) = \{r \in C(\bar{\Omega}) : \min_{x\in\bar{\Omega}} r(x) > 1\}.$$

Let $r \in C_+(\bar{\Omega})$ and denote

$$r^- = \min_{x\in\bar{\Omega}} r(x) \leqslant r(x) \leqslant r^+ = \max_{x\in\bar{\Omega}} r(x).$$

For any $r \in C_+(\bar{\Omega})$, we define the variable exponent Lebesgue space:

$$L^{r(.)}(\Omega) = \left\{ u : u \text{ is a measurable real valued function}, \int_\Omega |u|^{r(x)} dx < +\infty \right\},$$

endowed with the so-called Luxemburg norm

$$\|u\|_{r(.)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{r(x)} dx \leqslant 1 \right\}.$$

The space $(L^{r(.)}(\Omega), \|u\|_{r(.)})$ is a separable and a reflexive Banach space (see [11]). For more details on basic properties of the variable exponent Lebesgue spaces, we refer to [15].

We define the following Sobolev space

$$W^{1,r(.)}(\Omega) = \{u \in L^{r(.)}(\Omega) : |\nabla u| \in L^{r(.)}(\Omega)\}.$$

The space $W^{1,r(.)}(\Omega)$ endowed with the norm

$$\|u\|_{1,r(.)} = \|u\|_{r(.)} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{r(.)}, \quad \forall u \in W^{1,r(.)}(\Omega),$$

is a separable and a reflexive Banach space (see [15]).

We define $W_0^{1,r(.)}(\Omega)$ as the closure of $D(\Omega)$ with respect to the norm $\|u\|_{1,r(.)}$.

We will need the following comparison principle, for weak solutions.

LEMMA 1. (Weak comparison principle). *Let Ω be a bounded domain in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$ and $f : \Omega \times (0,+\infty) \to (0,+\infty)$ is a continuous and decreasing function, in the second variable.*

Assume that, $u, v \in W^{1,r(.)}(\Omega)$ satisfy for all non-negative $\varphi \in W^{1,r(.)}(\Omega)$ the inequalities

$$\int_\Omega |\nabla u|^{r(x)-2} \nabla u \nabla \varphi \, dx \leqslant \int_\Omega f(x,u)\varphi \, dx,$$

and

$$\int_\Omega |\nabla v|^{r(x)-2}\nabla v\nabla\varphi\, dx \geqslant \int_\Omega f(x,v)\varphi\, dx.$$

Then, the inequality $\lim_{x\to\partial\Omega}\sup(u(x)-v(x))\leqslant 0$ *implies that* $u\leqslant v$ *in* Ω.

Proof. Assume that $u>v$. For $\varphi\in W^{1,r(.)}(\Omega)$, $\varphi>0$, we have

$$\int_\Omega |\nabla u|^{r(x)-2}\nabla u\nabla\varphi\, dx - \int_\Omega |\nabla v|^{r(x)-2}\nabla v\nabla\varphi\, dx \leqslant \int_\Omega \big(f(x,u)-f(x,v)\big)\varphi\, dx.$$

For $\varphi=u-v$. The hypotheses on f allow to conclude that the right hand side is negative. Thus, we find that

$$\int_\Omega |\nabla u|^{r(x)-2}\nabla u\nabla\varphi\, dx - \int_\Omega |\nabla v|^{r(x)-2}\nabla v\nabla\varphi\, dx \leqslant 0$$

So, we obtain

$$0\leqslant \langle -\Delta_{r(x)}u+\Delta_{r(x)}v,\varphi(x)\rangle = \int_\Omega \left(|\nabla u|^{r(x)-2}\nabla u - |\nabla v|^{r(x)-2}\nabla v\right)\left(\nabla u-\nabla v\right) dx \leqslant 0$$

Thus, $\nabla\varphi=0$ a.e. in Ω and therefor $\varphi=0$ a.e. in Ω. Since $\lim_{x\to\partial\Omega}\sup(u(x)-v(x))\leqslant 0$. Hence, $u\leqslant v$ a.e in Ω. $\square$

Now, we will establish some properties of positive solutions to an equation related to the system (P). We consider the problem

$$(2.1)\quad \begin{cases} -\Delta_{r(x)}w = d(x)^{-\gamma(x)}w^{m(x)}, & x\in\Omega,\\ w\to+\infty, & x\in\partial\Omega,\\ w>0, \end{cases}$$

such that $\gamma\in C_+(\bar\Omega)$ and for $x\in\bar\Omega$, we have $m(x)>r(x)-1$. This problem has been considered in [1], where they showed all issues concerning the existence, uniqueness and asymptotic behavior of positive solutions near the boundary.

So, we give the following existence result.

LEMMA 2. *Let* $m(x)>r(x)-1$ *and* $\gamma(x)\in(0,r(x))$, *for all* $x\in\bar\Omega$. *Then, the problem* (2.1) *admits a nonnegative blow-up solution denoted by* $W_{m,\gamma}\in W^{1,r(.)}(\Omega)$.

Proof. See Theorem 2.1, page 149, in [1], with $f(u(x))=u(x)^{m(x)}$ and $g(x)=d(x)^{-\gamma(x)}$, for $x\in\Omega$. $\square$

In this paper, we will use the following comparison result.

LEMMA 3. *Let $w \in W^{1,r(.)}(\Omega)$ satisfy $-\operatorname{div}(|\nabla w|^{r(x)-2}\nabla w) \leqslant Cd(x)^{-\gamma(x)}w^{m(x)}$ in Ω, for some positive constant C and $w = +\infty$ on $\partial\Omega$. Then, for $x \in \Omega$*

$$w(x) \geqslant C^{-\frac{1}{m^- - r^+ + 1}} W_{m,\gamma}(x).$$

Similarly, if $-\operatorname{div}(|\nabla w|^{r(x)-2}\nabla w) \geqslant Cd(x)^{-\gamma(x)}w^{m(x)}$ in Ω with $w = +\infty$ on $\partial\Omega$. Then, for $x \in \Omega$

$$w(x) \leqslant C^{-\frac{1}{m^+ - r^- + 1}} W_{m,\gamma}(x).$$

Proof. For $x \in \Omega$, let $v(x) = C^{\frac{1}{m^- - r^+ + 1}} w(x)$. Then, we have for $\varphi \in W^{1,r(.)}(\Omega)$

$$\begin{aligned}
\int_\Omega |\nabla v|^{r(x)-2} \nabla v \nabla \varphi \, dx &= \int_\Omega C^{\frac{r(x)-1}{m^- - r^+ + 1}} |\nabla w|^{r(x)-2} \nabla w \nabla \varphi \, dx \\
&\leqslant C^{\frac{r^+ - 1}{m^- - r^+ + 1}} \int_\Omega |\nabla w|^{r(x)-2} \nabla w \nabla \varphi \, dx \\
&\leqslant C^{\frac{r^+ - 1}{m^- - r^+ + 1} + 1} \int_\Omega d(x)^{-\gamma(x)} w^{m(x)} \varphi \, dx \\
&\leqslant \int_\Omega C^{\frac{m^-}{m^- - r^+ + 1}} d(x)^{-\gamma(x)} w^{m(x)} \varphi \, dx \\
&\leqslant \int_\Omega d(x)^{-\gamma(x)} v^{m(x)} \varphi \, dx.
\end{aligned}$$

Thus, we obtain that

$$\int_\Omega |\nabla v|^{r(x)-2} \nabla v \nabla \varphi \, dx \leqslant \int_\Omega d(x)^{-\gamma(x)} v^{m(x)} \varphi \, dx.$$

We consider $W_{m,\gamma}$, the solution of the problem (2.1), with $\gamma(x) < r(x)$ and $m(x) > r(x) - 1$, for $x \in \Omega$.

Thus, for $\varphi \in W^{1,r(.)}(\Omega)$, we have

$$\int_\Omega \left|\nabla W_{m,\gamma}\right|^{r(x)-2} \nabla W_{m,\gamma} \nabla \varphi \, dx = \int_\Omega d(x)^{-\gamma(x)} W_{m,\gamma}^{m(x)} \varphi \, dx.$$

Moreover,

$$\begin{aligned}
&\int_\Omega (|\nabla v|^{r(x)-2} \nabla v \nabla \varphi - d(x)^{-\gamma(x)} v^{m(x)} \varphi) \, dx \\
&\leqslant \int_\Omega (\left|\nabla W_{m,\gamma}\right|^{r(x)-2} \nabla W_{m,\gamma} \nabla \varphi - d(x)^{-\gamma(x)} W_{m,\gamma}^{m(x)} \varphi) \, dx.
\end{aligned}$$

We have $\lim\limits_{x\to\partial\Omega}\sup(v(x)-W_{m,\gamma}(x))\geqslant 0$. Thus, by the weak comparison principle, given in Lemma 1, it follows that $C^{\frac{1}{m^{-}-r^{+}+1}}w(x)\geqslant W_{m,\gamma}(x)$.

Hence, $w(x)\geqslant C^{-\frac{1}{m^{-}-r^{+}+1}}W_{m,\gamma}(x)$.

For the symmetric inequality, we suppose that

$$-\operatorname{div}(|\nabla w|^{r(x)-2}\nabla w)\geqslant Cd(x)^{-\gamma(x)}w^{m(x)}$$

in Ω. let $v(x)=C^{\frac{1}{m^{+}-r^{-}+1}}w(x)$. Then, we have for $\varphi\in W^{1,r(.)}(\Omega)$

$$\begin{aligned}\int_{\Omega}|\nabla v|^{r(x)-2}\nabla v\nabla\varphi\,dx&=\int_{\Omega}C^{\frac{r(x)-1}{m^{+}-r^{-}+1}}|\nabla w|^{r(x)-2}\nabla w\nabla\varphi\,dx\\&\geqslant C^{\frac{r^{-}-1}{m^{+}-r^{-}+1}}\int_{\Omega}|\nabla w|^{r(x)-2}\nabla w\nabla\varphi\,dx.\end{aligned}$$

By hypothesis, we have

$$\begin{aligned}\int_{\Omega}|\nabla v|^{r(x)-2}\nabla v\nabla\varphi\,dx&\geqslant C^{\frac{r^{-}-1}{m^{+}-r^{-}+1}+1}\int_{\Omega}d(x)^{-\gamma(x)}w^{m(x)}\varphi\,dx\\&\geqslant\int_{\Omega}C^{\frac{m^{+}}{m^{+}-r^{-}+1}}d(x)^{-\gamma(x)}w^{m(x)}\varphi\,dx\\&\geqslant\int_{\Omega}d(x)^{-\gamma(x)}v^{m(x)}\varphi\,dx.\end{aligned}$$

Thus, we obtain that

$$\int_{\Omega}|\nabla v|^{r(x)-2}\nabla v\nabla\varphi\,dx\geqslant\int_{\Omega}d(x)^{-\gamma(x)}v^{m(x)}\varphi\,dx.$$

Moreover, by definition of $W_{m,\gamma}$, we obtain

$$\begin{aligned}&\int_{\Omega}(|\nabla v|^{r(x)-2}\nabla v\nabla\varphi-d(x)^{-\gamma(x)}v^{m(x)}\varphi)\,dx\\\geqslant&\int_{\Omega}(\left|\nabla W_{m,\gamma}\right|^{r(x)-2}\nabla W_{m,\gamma}\nabla\varphi-d(x)^{-\gamma(x)}W_{m,\gamma}^{m(x)}\varphi)\,dx.\end{aligned}$$

Using that $\lim\limits_{x\to\partial\Omega}\sup(v(x)-W_{m,\gamma}(x))\leqslant 0$. Thus, by the weak comparison principle, given in Lemma 1, it follows that $C^{\frac{1}{m^{+}-r^{-}+1}}w(x)\leqslant W_{m,\gamma}(x)$, for $x\in\Omega$, hence, $w(x)\leqslant C^{-\frac{1}{m^{+}-r^{-}+1}}W_{m,\gamma}(x)$, for $x\in\Omega$.

In particular, the quantities

$$A_{m,\gamma}(x)=(\sup_{x\in\Omega}d(x))^{\rho_1}W_{m,\gamma}(x),\tag{2.2}$$

and

$$B_{m,\gamma}(x) = (\inf_{x\in\Omega} d(x))^{\rho_2} W_{m,\gamma}(x), \tag{2.3}$$

are finite and positive, for $x \in \Omega$. Such that,

$$\rho_1(x) = \rho_2(x) = \frac{r(x)-\gamma(x)}{m(x)-r(x)+1}. \quad \square$$

The following lemma certifies that the functions defined by (2.2) and (2.3), are also bounded.

LEMMA 4. *The functions $A_{m,\gamma}$ and $B_{m,\gamma}$ are bounded.*

Proof. We aim to prove that the functions $A_{m,\gamma}$ and $B_{m,\gamma}$ defined by (2.2) and (2.3) respectively are bounded. For this, let $W_{m,\gamma}$ be solution for the problem (2.1) with $\gamma(x) < r(x)$ and $m(x) > r(x)-1$, for $x \in \Omega$.

Thus, for $\varphi \in W^{1,r(.)}(\Omega)$, we have

$$\int_\Omega \left|\nabla W_{m,\gamma}\right|^{r(x)-2} \nabla W_{m,\gamma} \nabla\varphi \, dx = \int_\Omega d(x)^{-\gamma(x)} W_{m,\gamma}^{m(x)} \varphi \, dx. \tag{2.4}$$

First, let $\sigma \in C_+(\bar{\Omega})$ such that $\gamma(x) < \sigma(x)$, for $x \in \bar{\Omega}$. Hence

$$\begin{aligned}\int_\Omega \left|\nabla W_{m,\gamma}\right|^{r(x)-2} \nabla W_{m,\gamma} \nabla\varphi \, dx &= \int_\Omega d(x)^{-\gamma(x)+\sigma(x)} d(x)^{-\sigma(x)} W_{m,\gamma}^{m(x)} \varphi \, dx \\ &\leqslant \int_\Omega (\sup_{x\in\Omega} d(x))^{-\gamma(x)+\sigma(x)} d(x)^{-\sigma(x)} W_{m,\gamma}^{m(x)} \varphi \, dx.\end{aligned}$$

So, we obtain

$$\int_\Omega \left|\nabla W_{m,\gamma}\right|^{r(x)-2} \nabla W_{m,\gamma} \nabla\varphi \, dx \leqslant (\sup_x d(x))^{\sigma^+-\gamma^-} \int_\Omega d(x)^{-\sigma(x)} W_{m,\gamma}^{m(x)} \varphi \, dx.$$

Using Lemma 3, we get

$$W_{m,\gamma}(x) \geqslant (\sup_x d(x))^{-\frac{\sigma^+-\gamma^-}{m^- - r^+ + 1}} W_{m,\sigma}(x).$$

Moreover, by the definition of $A_{m,\gamma}$ (2.2), we have for $x \in \Omega$

$$A_{m,\gamma}(x) \geqslant (\sup_x d(x))^{\frac{\gamma^- - \sigma^+}{m^- - r^+ + 1}} A_{m,\sigma}(x). \tag{2.5}$$

Second, let $\rho \in C_+(\bar{\Omega})$ such that $\gamma(x) > \rho(x)$, for $x \in \Omega$. $W_{m,\gamma}$ being solution for (2.1), then

$$\begin{aligned}\int_\Omega \left|\nabla W_{m,\gamma}\right|^{r(x)-2} \nabla W_{m,\gamma} \nabla \varphi \, dx &= \int_\Omega d(x)^{-\gamma(x)+\rho(x)} d(x)^{-\rho(x)} W_{m,\gamma}^{m(x)} \varphi \, dx \\ &\geqslant \int_\Omega (\sup_{x\in\Omega} d(x))^{-\gamma(x)+\rho(x)} d(x)^{-\rho(x)} W_{m,\gamma}^{m(x)} \varphi \, dx \\ &\geqslant (\sup_{x\in\Omega} d(x))^{\rho^- - \gamma^+} \int_\Omega d(x)^{-\rho(x)} W_{m,\gamma}^{m(x)} \varphi \, dx.\end{aligned}$$

So, by using Lemma 3, we obtain

$$A_{m,\gamma}(x) \leqslant (\sup_x d(x))^{\frac{\gamma^+ - \rho^-}{m^+ - r^- + 1}} A_{m,\rho}(x). \tag{2.6}$$

Then, by combining (2.5) and (2.6), we get for $x \in \Omega$

$$(\sup_x d(x))^{\frac{\gamma^- - \sigma^+}{m^- - r^+ + 1}} A_{m,\sigma}(x) \leqslant A_{m,\gamma}(x) \leqslant (\sup_x d(x))^{\frac{\gamma^+ - \rho^-}{m^+ - r^- + 1}} A_{m,\rho}(x).$$

A similar calculation proves that $B_{m,\gamma}$ is bounded in Ω. On the one hand, we define a function δ in $C_+(\bar{\Omega})$ such that $\gamma(x) < \delta(x)$, for $x \in \bar{\Omega}$.

Hence, using Lemma 3 and (2.3), we get

$$B_{m,\gamma}(x) \leqslant (\inf_x d(x))^{\frac{\gamma^+ - \delta^-}{m^+ - r^- + 1}} B_{m,\delta}(x). \tag{2.7}$$

On the other hand, we consider $\lambda \in C_+(\bar{\Omega})$ such that $\gamma(x) > \lambda(x)$, for $x \in \Omega$. So, by using Lemma 3 and the definition of $B_{m,\delta}$, we obtain

$$B_{m,\gamma}(x) \geqslant (\inf_x d(x))^{\frac{\gamma^- - \lambda^+}{m^- - r^+ + 1}} B_{m,\lambda}(x). \tag{2.8}$$

Then, by combining (2.7) and (2.8), we get for $x \in \Omega$

$$(\inf_x d(x))^{\frac{\gamma^- - \lambda^+}{m^- - r^+ + 1}} B_{m,\lambda}(x) \leqslant B_{m,\gamma}(x) \leqslant (\inf_x d(x))^{\frac{\gamma^+ - \delta^-}{m^+ - r^- + 1}} B_{m,\delta}(x). \quad \square$$

3. The subcritical case

In this section, we will state our results. We begin with the subcritical case $(a(x) - p(x) + 1)(e(x) - q(x) + 1) > b(x)c(x)$, for $x \in \Omega$. In this case, the system (P) behaves like a single equation, because the coupling between the two equations is not too strong.

3.1. Existence of solutions

We will prove that the problem (P) admits a solution for each of the boundary conditions $(F),(SF)$ and (I), provided some conditions.

THEOREM 1. *Let us assume that* $(a(x)-p(x)+1)(e(x)-q(x)+1)>b(x)c(x)$ *and* $p(x)=q(x)$, *for* $x\in\Omega$.

(i) Problem (P) *admits a unique positive solution* $(u,v)\in W^{1,p(.)}(\Omega)\times W^{1,q(.)}(\Omega)$ *with the boundary condition* (F).

(ii) Problem (P) *admits a positive solution* $(u,v)\in W^{1,p(.)}(\Omega)\times W^{1,q(.)}(\Omega)$ *with the boundary condition* (I), *if and only if* $c(x)<a(x)-p(x)+1$ *and* $b(x)<e(x)-p(x)+1$, *for* $x\in\Omega$.

(iii) Problem (P) *admits a positive solution* $(u,v)\in W^{1,p(.)}(\Omega)\times W^{1,q(.)}(\Omega)$ *with the boundary condition* (SF), *if and only if* $c(x)<a(x)-p(x)+1$, *for* $x\in\Omega$.

EXAMPLE 1. We give an example illustrating the main assumptions of Theorem 1. For this, we define $\Omega\subset\mathbb{R}^2$ by

$$\Omega=\{x=(x_1,x_2)\in\mathbb{R}^2_+ : x_1^2+x_2^2<1\}.$$

There exist $c_1,c_2,\varepsilon,\varepsilon'>0$ satisfying the inequality $c_1c_2<(\varepsilon+1)(\varepsilon'+1)$ such that, for $x\in\Omega$,

$$\begin{cases} p(x)=q(x)=x_1+x_2+1, \\ a(x)=x_1+x_2+c_1,\ e(x)=x_1+x_2+c_2, \\ b(x)=x_1+\varepsilon,\ c(x)=x_2+\varepsilon'. \end{cases}$$

By the hypothesis of the constants c_1,c_2,ε and ε', it is easy to see that the functions p,q,a,b,c and e satisfy the assumptions of Theorem 1. In particular, if $c_1<\varepsilon+1$ and $c_2<\varepsilon'+1$, it is easy to see that the functions p,q,a,b,c and e satisfy the assumptions of Theorem 1$((ii),(iii))$.

Proof. (i) We consider the finite case, that is the boundary condition (F). We will use sub-and-super-solutions method (see Theorem 4 in the Appendix). Indeed, we can take a pair $(\underline{u},\underline{v})=(0,m)$ as a subsolution and $(\bar{u},\bar{v})=(M,0)$ as a supersolution, for a small positive constant m and for a large positive constant M.

Then, Theorem 4 guarantees the existence of a positive solution.

(ii) Now, we consider the boundary condition (I). We employ sub-and-super-solutions method to prove the existence of (P), looking for a subsolution of the form $(\underline{u},\underline{v})=(\varepsilon^{-\delta^-}U_{a,\gamma},\varepsilon U_{e,\sigma})$, where $\delta,\gamma,\sigma\in C_+(\bar{\Omega})$ and $\varepsilon>0$ is small enough with $\gamma(x)<p(x)$ and $\sigma(x)<p(x)$, for $x\in\Omega$.

The definition of $U_{a,\gamma}$ gives

$$\begin{aligned} -\operatorname{div}(|\nabla\underline{u}|^{p(x)-2}\nabla\underline{u}) &= \varepsilon^{-\delta^-(p(x)-1)}d(x)^{-\gamma(x)}U_{a,\gamma}^{a(x)} \\ &= \varepsilon^{\delta^-(a(x)-p(x)+1)-b(x)}d(x)^{-\gamma(x)}U_{e,\sigma}^{-b(x)}\underline{u}^{a(x)}\underline{v}^{b(x)} \\ &\leqslant \varepsilon^{\delta(x)(a(x)-p(x)+1)-b(x)}d(x)^{-\gamma(x)}U_{e,\sigma}^{-b(x)}\underline{u}^{a(x)}\underline{v}^{b(x)}. \end{aligned}$$

With the same procedure, we prove that

$$-\operatorname{div}(|\nabla \underline{v}|^{p(x)-2}\nabla \underline{v}) = \varepsilon^{p(x)-1} d(x)^{-\sigma(x)} U_{e,\sigma}^{e(x)}$$
$$\leqslant \varepsilon^{\sigma(x)c(x)-e(x)-p(x)+1} d(x)^{-\sigma(x)} U_{a,\gamma}^{-c(x)} \underline{u}^{c(x)} \underline{v}^{e(x)}.$$

Hence, $(\underline{u},\underline{v})$ will be a subsolution, if and only if

$$\varepsilon^{b(x)-\delta(x)(a(x)-p(x)+1)} d(x)^{\gamma(x)} U_{e,\sigma}^{b(x)} \geqslant 1,$$

and

$$\varepsilon^{e(x)-p(x)+1-\delta(x)c(x)} d(x)^{\sigma(x)} U_{a,\gamma}^{c(x)} \geqslant 1.$$

Since $(a(x)-p(x)+1)(e(x)-p(x)+1) > b(x)c(x)$, we may choose δ such that $e(x)-p(x)+1 > \delta(x)c(x)$ and $b(x) < \delta(x)(a(x)-p(x)+1)$, that is

$$\frac{b(x)}{a(x)-p(x)+1} < \delta(x) < \frac{e(x)-p(x)+1}{c(x)}, \quad \text{for all } x \in \Omega.$$

It is shown in a similar way that $(\bar{u},\bar{v}) = (MU_{a,\gamma}, M^{-\sigma^+}U_{e,\sigma})$ is a supersolution for the same choice of δ, γ and σ, provided that M is large enough.

Hence, Theorem 5 in the Appendix implies the existence of a positive solution (u,v) to (P), satisfying $u = v = +\infty$ on $\partial\Omega$.

(iii) We now turn to the boundary conditions (SF). It is not hard to show that the condition $c(x) < a(x)-p(x)+1$ is necessary.

Using Theorem 6 in the Appendix, we prove the existence of (P). For this, we consider a large positive constant M and $\gamma \in C_+(\bar{\Omega})$, such that $(\bar{u},\bar{v}) = (M^{-\delta^+}U_{a,\gamma}, MV_{e,0})$ will be a supersolution provided that

$$M^{b(x)-\delta(x)(a(x)-p(x)+1)} d(x)^{\gamma(x)} V_{e,0}^{b(x)} \leqslant 1, \quad x \in \Omega,$$

and

$$M^{e(x)-p(x)+1-\delta(x)c(x)} U_{a,\gamma}^{c(x)} \leqslant 1, \quad x \in \Omega.$$

We may choose δ such as, $e(x)-p(x)+1 > \delta(x)c(x)$ and $b(x) < \delta(x)(a(x)-p(x)+1)$, for $x \in \Omega$. That is

$$\frac{b(x)}{a(x)-p(x)+1} < \delta(x) < \frac{e(x)-p(x)+1}{c(x)}, \quad \text{for all } x \in \Omega.$$

This is possible thanks to the subcriticality condition $(a(x)-p(x)+1)(e(x)-p(x)+1) > b(x)c(x)$. So, we have a supersolution. It is shown in a similar way that for a small positive ε and $\gamma \in C_+(\bar{\Omega})$, $(\underline{u},\underline{v}) = (\varepsilon^{-\delta^-}U_{a,\gamma}, \varepsilon V_{e,0})$ is a subsolution.

Thus, it follows from Theorem 6 in Appendix that there exists a solution to (P). $\square$

3.2. Uniqueness results

In this section, we obtain a new uniqueness result for the system (P), with finite boundary conditions.

In the linear operator (Laplacian operator), we can use the fact that, if u,v are C^2 functions in a domain Ω, with $u \leqslant v$ and $u(x_0) = v(x_0)$, for $x_0 \in \Omega$, then $\Delta u(x_0) = \Delta v(x_0)$.

In our approach, the situation is more delicate, when we use nonlinear operator with variable exponent, mainly due to the fact that solutions are not in general C^2. For that, let us announce the following result.

LEMMA 5. *Let $f,g \in C(\bar{\Omega})$ and $u,v \in W^{1,p(.)}(\Omega) \times W^{1,q(.)}(\Omega)$ be weak solutions to the following problem*

$$\begin{cases} -\Delta_{p(x)} u = f(x), & x \in \Omega, \\ -\Delta_{q(x)} v = g(x), & x \in \Omega, \end{cases}$$

with $u \leqslant v$ and $u = v$ at some point of Ω. Let us assume moreover that $u < v$ on $\partial\Omega$.

Then, there exists $x_0 \in \Omega$ such that $u(x_0) = v(x_0)$ and $f(x_0) \leqslant g(x_0)$.

Proof. Let $E = \{x \in \Omega : u(x) = v(x)\}$. By assumptions, E is nonempty and it is strictly contained in Ω $(E \subset \Omega)$.

Assuming by contradiction that $f > g$ in E. Then, we can choose an open neighborhood U of E such that $f > g$ in U and $u < v$ on ∂U.

Then, for a small $\varepsilon > 0$, we have $u + \varepsilon \leqslant v$ on ∂U together with

$$-\Delta_{p(x)}(u+\varepsilon) - \Delta_{p(x)} v > (f-g)(x), \quad x \in U.$$

By the comparison principle, we obtain that $u + \varepsilon \leqslant v$ in U, which is clearly a contradiction since $E \subset U$. Thus, $f > g$ is not possible in E and there exists $x_0 \in E$ with $f(x_0) \leqslant g(x_0)$. $\square$

Next, we give a uniqueness result for the problem (P), with a finite boundary condition.

THEOREM 2. *Let $(u_1,v_1),(u_2,v_2)$ be positive weak solutions to the system*

$$\begin{cases} -\Delta_{p(x)} u = u^{a(x)} v^{b(x)}, & x \in \Omega, \\ -\Delta_{q(x)} v = u^{c(x)} v^{e(x)}, & x \in \Omega, \\ u = \lambda,\ v = \mu, & x \in \partial\Omega, \end{cases}$$

with $\lambda,\mu > 0$ on $\partial\Omega$ such that, for $x \in \Omega$, $a(x) > p(x) - 1$, $e(x) > q(x) - 1$ and $(a(x) - p(x) + 1)(e(x) - q(x) + 1) > b(x)c(x)$.

Then, $u_1 = u_2$ and $v_1 = v_2$ in Ω.

Proof. Since the solutions are positive, we can choose a large $k > 1$. First, we consider that

$$u_1(x) \geqslant k^{-\frac{b(x)}{a(x)-p(x)+1}} u_2(x), \quad x \in \Omega, \tag{3.1}$$

and

$$v_1(x) \leqslant k v_2(x), \quad x \in \Omega. \tag{3.2}$$

Using Lemma 5, we obtain a point $x_0 \in \Omega$ with

$$u_1(x_0) = k^{-\frac{b(x_0)}{a(x_0)-p(x_0)+1}} u_2(x_0).$$

$$u_1(x_0)^{a(x_0)} v_1(x_0)^{b(x_0)} \geqslant k^{-\frac{b(x_0)(p(x_0)-1)}{a(x_0)-p(x_0)+1}} u_1(x_0)^{a(x_0)} v_1(x_0)^{b(x_0)}.$$

Which implies by (3.2), that $v_1(x_0) = k v_2(x_0)$.

Hence, we may apply Lemma 5 to get a point $x_0 \in \Omega$ verifying

$$v_1(x_0) = k v_2(x_0),$$

and

$$u_1(x_0)^{c(x_0)} v_1(x_0)^{e(x_0)} \leqslant k^{q(x_0)-1} u_2(x_0)^{c(x_0)} v_2(x_0)^{e(x_0)}.$$

Which gives by hypothesis

$$u_1(x_0) \leqslant k^{-\frac{e(x_0)-q(x_0)+1}{c(x_0)}} u_2(x_0).$$

But, the first inequality (3.1) gives that

$$k^{\frac{(a(x_0)-p(x_0)+1)(e(x_0)-q(x_0)+1)-b(x_0)c(x_0)}{c(x_0)(a(x_0)-p(x_0)+1)}} \leqslant 1.$$

We get contradiction since $k > 1$ and $(a(x)-p(x)+1)(e(x)-q(x)+1) > b(x)c(x)$, for $x \in \Omega$. That contradiction shows $k \leqslant 1$, that is, $u_1 \geqslant u_2$ and $v_1 \leqslant v_2$ in Ω.

Second, we consider the reversed inequalities

$$u_1(x) \leqslant k u_2(x), \quad x \in \Omega,$$

and

$$v_1(x) \geqslant k^{-\frac{c(x)}{e(x)-q(x)+1}} v_2(x), \quad x \in \Omega.$$

Thus, by a similar argument, we prove that $u_1 \leqslant u_2$ and $v_1 \geqslant v_2$ in Ω. $\square$

REMARK 1. Lemma 5 is also useful for obtaining an alternative proof of uniqueness of positive solutions to the problem

$$\begin{cases} -\Delta_{r(x)} w = g(x) f(w), & x \in \Omega, \\ w = \lambda, & x \in \partial\Omega, \end{cases} \tag{3.3}$$

when f and g are non decreasing continuous functions such that $\frac{f(w)}{w^{r(x)-1}}$ decreasing in w and $\lambda > 0$. Moreover, for fixed x, $\liminf_{w \to 0^+} \frac{f(w)}{w^{r(x)-1}} \geqslant c$. Indeed, if w_1 and w_2 are positive solutions for (3.3), we can choose a large $k>1$, such that $w_1 \geqslant kw_2$ in Ω. Lemma 5 can be applied to give the existence of a point $x_0 \in \Omega$, such that

$$w_1(x_0)=kw_2(x_0) \quad \text{and} \quad f(kw_2(x_0)) \geqslant k^{r(x_0)-1} f(w_2(x_0)).$$

It is incompatible with the monotonicity of $\frac{f(w)}{w^{r(x)-1}}$.

Thus, $k \leqslant 1$ and $w_1 \leqslant w_2$. The reversed argument gives $w_1 = w_2$.

3.3. Global estimates for solutions

In this section, we will obtain the boundary behavior of solutions using a blow-up argument. For this, we need some rough estimates of all positive solutions. The present proof is modeled on that of the semilinear case contained in [20].

LEMMA 6. *Let (u,v) be a positive solution to (P) with $a(x)>p(x)-1$, $e(x)>p(x)-1$ and $(a(x)-p(x)+1)(e(x)-p(x)+1)>b(x)c(x)$, such that*

$$c(x) < a^- - p^+ + 1, \tag{3.4}$$

and

$$b(x) < e^- - p^+ + 1, \tag{3.5}$$

for $x \in \Omega$. Then, there exist positive constants C_1, C_2 such that

$$C_1 d(x)^{-\alpha} \leqslant u(x) \leqslant C_2 d(x)^{-\alpha},$$

and

$$C_1 d(x)^{-\beta} \leqslant v(x) \leqslant C_2 d(x)^{-\beta}.$$

Where

$$\alpha = \frac{p(x)(e(x)-p(x)+1-b(x))}{(a(x)-p(x)+1)(e(x)-p(x)+1)-b(x)c(x)},$$

and

$$\beta = \frac{p(x)(a(x)-p(x)+1-c(x))}{(a(x)-p(x)+1)(e(x)-p(x)+1)-b(x)c(x)}.$$

Proof. Let (u,v) be a positive solution to (P). Then, if $u_0 = \sup_{\Omega} u > 0$, we have $-\operatorname{div}(|\nabla v|^{p(x)-2}\nabla v) \leqslant u_0^{c(x)} v^{e(x)}$ and Lemma 3 gives

$$v \geqslant u_0^{-\frac{c(x)}{e^- - p^+ + 1}} V_{e,0} \geqslant u_0^{-\frac{c(x)}{e^- - p^+ + 1}} d(x)^{-\beta_0(x)} B_{e,0},$$

where $\beta_0(x) = \frac{p(x)}{e(x)-q(x)+1}$, for $x \in \Omega$. We set $a_0 = u_0^{-\frac{c(x)}{e^- - p^+ + 1}} B_{e,0}$.

Using the first equation in (P), we obtain

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \geqslant (a_0 d(x)^{-\beta_0})^{b(x)} u^{a(x)}, \quad x \in \Omega.$$

Thus, thanks to Lemma 3, we have

$$u \leqslant a_0^{-\frac{b(x)}{a^+-p^-+1}} U_{a,\beta_0 b}^{a(x)} \leqslant a_0^{-\frac{b(x)}{a^+-p^-+1}} d(x)^{-\alpha_0(x)} A_{a,\beta_0 b},$$

where $\alpha_0(x) = \frac{p(x)-\beta_0(x)b(x)}{a(x)-p(x)+1}$, for $x \in \Omega$.

We can iterate this argument to obtain that

$$v \geqslant a_n d(x)^{-\beta_0(x)}, \tag{3.6}$$

$$u \leqslant a_{n+1} d(x)^{-\alpha_n(x)}, \tag{3.7}$$

where for $x \in \Omega$, we have

$$\alpha_n(x) = \frac{p(x) - \beta_n(x)b(x)}{a(x) - p(x) + 1},$$
$$\beta_n(x) = \frac{p(x) - \alpha_{n-1}(x)c(x)}{e(x) - p(x) + 1},$$

and

$$a_{n+1}(x) = a_n^{\frac{c(x)b(x)}{(a^+-p^-+1)(e^--p^++1)}} B_{e,\alpha_{n-1}c}^{-\frac{b(x)}{a^+-p^-+1}} A_{a,\beta_n b}. \tag{3.8}$$

It is easily seen that

$$\alpha_n(x) = \frac{p(x)(e(x)-p(x)+1-b(x))}{(a(x)-p(x)+1)(e(x)-p(x)+1)} + \frac{b(x)c(x)}{(a(x)-p(x)+1)(e(x)-p(x)+1)} \beta_{n-1}(x),$$

and that $\alpha_n < \alpha_{n-1}$, for $n > 0$. (α_n) is a decreasing sequence of positive numbers. Thus, (α_n) has a limit, which is easily seen to be $\alpha(x)$. This entails that $\beta_n \to \beta$.

By the hypotheses in Theorem 1, we deduce that $\alpha, \beta > 0$.

Moreover, the fact that $\alpha, \beta > 0$ implies that $\alpha_{n-1}(x)c(x) \leqslant p(x)$ and $\beta_n(x)b(x) \leqslant p(x)$, for $x \in \Omega$ and $n > 0$.

Thanks to Lemma 4, the quantities $A_{a,\beta_n b}$ and $B_{e,\alpha_{n-1}c}$ are bounded.

Hence, thanks to (3.8), there exists $C > 0$ such that $a_{n+1} \leqslant C a_n^{-\delta}$. Since the fact that $c(x) < a(x) - p(x) + 1$ and by (3.4), we have

$$\delta(x) = \frac{b(x)c(x)}{(a^+ - p^- + 1)(e^- - p^+ + 1)} < 1, \quad \text{for } x \in \Omega.$$

This gives, $a_{n+1} \leqslant C^{1+\delta+\ldots+\delta^n} a_0^{-\delta^{n+1}}$ and by passing to the limit, we obtain that

$$\lim_{n\to\infty} a_{n+1} \leqslant C^{\frac{1}{1-\delta}}.$$

Moreover, passing to the limit in (3.6) and (3.7), we find that, there exist positive constants C_1, C_2 such that

$$u \geqslant C_1 d(x)^{-\alpha} \quad \text{and} \quad v \leqslant C_2 d(x)^{-\beta}.$$

A symmetric argument with $u_0 = \inf_{\Omega} u$, proves the reversed inequalities since the fact that $b(x) < e(x) - q(x) + 1$ and by (3.5). Thus, we obtain, for $x \in \Omega$

$$C_1 d(x)^{-\alpha} \leqslant u(x) \leqslant C_2 d(x)^{-\alpha},$$

and

$$C_1 d(x)^{-\beta} \leqslant v(x) \leqslant C_2 d(x)^{-\beta}. \quad \square$$

3.4. Nonexistence

This section is devoted to prove the nonexistence of positive solutions (u,v) to (P) both for the boundary conditions (I), when the conditions in Theorem $1(ii)$ don't hold. We begin assuming that (u,v) is a positive solution to (P), satisfying $u = v = +\infty$ on $\partial\Omega$, with $c(x) \geqslant a(x) - p(x) + 1$ and $b(x) < e(x) - q(x) + 1$ and we will reach a contradiction.

Notice that since $(a(x) - p(x) + 1)(e(x) - p(x) + 1) > b(x)c(x)$, for $x \in \Omega$, both conditions $c(x) \geqslant a(x) - p(x) + 1$ and $b(x) \geqslant e(x) - p(x) + 1$ cannot hold simultaneously. The second case $c(x) < a(x) - p(x) + 1$ and $b(x) \geqslant e(x) - p(x) + 1$, for $x \in \Omega$ is treated in the same way.

Let $u_0 = \inf_{\Omega} u > 0$, then $-\operatorname{div}(|\nabla v|^{p(x)-2}\nabla v) \geqslant u_0^{c(x)} v^{e(x)}$.

Using Lemma 3 and by definition of $B_{e,0}$, we obtain

$$v \leqslant u_0^{-\frac{c(x)}{e^+ - p^- + 1}} d(x)^{-\alpha_0(x)} B_{e,0},$$

with $\alpha_0(x) = \frac{p(x)}{e(x) - q(x) + 1}$, for $x \in \Omega$. Set $a_0 = u_0^{-\frac{c(x)}{e^+ - p^- + 1}} B_{e,0}$.

Using this, in the first equation in (P), we have

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \leqslant (a_0 d(x)^{-\alpha_0(x)})^{b(x)} u^{a(x)}.$$

Lemma 3 gives,

$$u \geqslant a_0^{-\frac{b(x)}{a^- - p^+ + 1}} d(x)^{-\beta_0(x)} A_{a,\alpha_0 b},$$

where $\beta_0(x) = \frac{p(x) - \alpha_0(x)b(x)}{a(x) - p(x) + 1}$, for $x \in \Omega$.

Proceeding inductively, we obtain

$$\begin{cases} v \leqslant a_n d(x)^{-\alpha_n(x)}, \\ u \geqslant a_{n+1} d(x)^{-\beta_n(x)}. \end{cases}$$

Where for $x \in \Omega$, we have

$$\begin{cases} \alpha_n(x) = \frac{p(x) - \beta_{n-1} c(x)}{e(x) - p(x) + 1}, \\ \beta_n(x) = \frac{p(x) - \alpha_n b(x)}{a(x) - p(x) + 1}, \\ a_{n+1}(x) = a_n^{\frac{c(x)b(x)}{(a^- - p^+ + 1)(e^+ - p^- + 1)}} B_{e,\beta_{n-1} c}^{-\frac{b(x)}{a^- - p^+ + 1}} A_{a,\alpha_n b}. \end{cases}$$

Let us see that the above quantities converge as $n \to +\infty$. Indeed, we have

$$\alpha_n(x) = \frac{p(x)(a(x)-p(x)+1-c(x))}{(a(x)-p(x)+1)(e(x)-p(x)+1)} + \frac{b(x)c(x)}{(a(x)-p(x)+1)(e(x)-p(x)+1)}\alpha_{n-1}(x),$$

and that $\alpha_n \leqslant \alpha_{n-1}$. Thus, we deduce that the sequence (α_n) is decreasing. hence, (α_n) has a limit, which is

$$\alpha(x) = \frac{p(x)(a(x)-p(x)+1-c(x))}{(a(x)-p(x)+1)(e(x)-p(x)+1)-b(x)c(x)} < 0.$$

So, $\beta_n \to \frac{p(x)(e(x)-p(x)+1-b(x))}{(a(x)-p(x)+1)(e(x)-p(x)+1)-b(x)c(x)}$ as $n \to \infty$.

Let n, the minimum positive integer such that $\beta_n(x)c(x) \geqslant q(x)$, for $x \in \Omega$. Hence, v is bounded. Thus no solution can exist.

4. The critical case

Now, we turn to the critical case $(a(x)-p(x)+1)(e(x)-q(x)+1) = b(x)c(x)$, for $x \in \Omega$. In this case, the solutions are not unique.

4.1. Existence of solutions

THEOREM 3. *Let us assume that* $(a(x)-p(x)+1)(e(x)-q(x)+1) = b(x)c(x)$, *and* $p(x) \leqslant q(x)$, *for* $x \in \Omega$.

(i) Problem (P) *admits a positive solution* $(u,v) \in W^{1,p(.)}(\Omega) \times W^{1,q(.)}(\Omega)$ *with the boundary condition* (F).

(ii) Problem (P) *admits a positive solution* $(u,v) \in W^{1,p(.)}(\Omega) \times W^{1,q(.)}(\Omega)$ *with the boundary condition* (I), *if and only if* $p(x) = q(x)$, $c(x) = a(x)-p(x)+1$ *and* $b(x) = e(x)-q(x)+1$, *for* $x \in \Omega$.

Moreover, for the problem

$$(4.1)\ \begin{cases} -\Delta_{p(x)}u = u^a v^b, & x \in \Omega, \\ -\Delta_{q(x)}v = u^c v^e, & x \in \Omega\,, \end{cases}$$

with $p(x) < a+1$ *and* $q(x) < e+1$, *for* $x \in \Omega$. *If* (u,v) *is a solution of* (4.1), *then* $(t^{\frac{b}{b+c}}u, t^{-\frac{c}{b+c}}v)$ *is also a solution, for every* $t > 0$. *Thus, there are infinitely many positive solutions.*

(iii) Problem (P) *admits a positive solution* $(u,v) \in W^{1,p(.)}(\Omega) \times W^{1,q(.)}(\Omega)$ *with the boundary condition* (SF), *if and only if* $b(x) < e(x)-q(x)+1$, *for* $x \in \Omega$.

Moreover, if (u,v) *is a solution of* (4.1), *then* $(t^{-\delta}u, tv)$ *is also a solution with the following boundary conditions* (4.2), *for every* $t > 0$ *and thus at least a solution with different boundary data* λ, *can be obtained from one of them, where* $p(x) = \frac{b-\delta a-\delta}{\delta}$, *for* $x \in \Omega$.

$$(4.2)\ \begin{cases} u = t^{-\delta}\lambda, \\ v = +\infty. \end{cases}$$

EXAMPLE 2. We give an example illustrating the main assumptions of Theorem 3. For this, we define $\Omega \subset \mathbb{R}^2$ by

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2_+ : x_1^2 + x_2^2 < 1\}.$$

There exists $\varepsilon \geqslant 1$ such that, for $x \in \Omega$,

$$\begin{cases} p(x) = x_1 + x_2 + 1,\ q(x) = x_1 + x_2 + \varepsilon, \\ a(x) = 2x_1 + 3x_2,\ e(x) = x_1 + 2x_2 + \varepsilon, \\ b(x) = x_1 + 2x_2,\ c(x) = x_2 + 1. \end{cases}$$

By the hypothesis of the constant ε, it is easy to see that the functions p, q, a, b, c and e satisfy the assumptions of Theorem 3.

Proof. (i) The finite case, boundary condition (F) is easy to handle. We will use the method of subsolutions and supersolutions. For this, we take $(\underline{u}, \underline{v}) = (0, m)$ and $(\bar{u}, \bar{v}) = (M, 0)$ as a subsolution and a supersolution respectively, for the small positive constant m and for the large positive constant M.

Theorem 4 guarantees the existence of a positive solution.

(ii) First, we prove that $c(x) = a(x) - p(x) + 1$ and $b(x) = e(x) - q(x) + 1$ are necessary conditions, for the existence of positive solutions. we will proceed by absurdity. Assuming that $c(x) > a(x) - p(x) + 1$ and thus $b(x) < e(x) - q(x) + 1$.

Notice that since $(a(x) - p(x) + 1)(e(x) - q(x) + 1) = b(x)c(x)$, both conditions $c(x) > a(x) - p(x) + 1$ and $b(x) > e(x) - q(x) + 1$ cannot hold simultaneously.

The remaining case $c(x) < a(x) - p(x) + 1$ and $b(x) > e(x) - q(x) + 1$ is treated in the same way.

Let (u, v) be a positive solution. Let $u_0 = \inf_{\Omega} u > 0$, we have $-\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) \geqslant u_0^{c(x)} v^{e(x)}$, Lemma 3 gives

$$v \leqslant u_0^{-\frac{c(x)}{e^+ - q^- + 1}} V_{e,0}.$$

By the definition of $B_{e,0}$, we have $v \leqslant u_0^{-\frac{c(x)}{e^+ - q^- + 1}} d(x)^{-\alpha_0(x)} B_{e,0}$.

For $x \in \Omega$, we note $\alpha_0(x) = \frac{q(x)}{e(x) - q(x) + 1}$, for $x \in \Omega$. We set $a_0 = u_0^{-\frac{c(x)}{e^+ - q^- + 1}} B_{e,0}$, using this in the first equation in (P), we have

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \leqslant (a_0 d(x)^{-\alpha_0(x)})^{b(x)} u^{a(x)}, \quad x \in \Omega.$$

Using Lemma 3 again and by the definition of $A_{a,\alpha_0 b}$, we obtain

$$u \geqslant a_0^{-\frac{b(x)}{a^- - p^+ + 1}} d(x)^{-\beta_0(x)} A_{a,\alpha_0 b},$$

where $\beta_0(x) = \frac{p(x) - \alpha_0(x)b(x)}{a(x) - p(x) + 1}$, for $x \in \Omega$.

Proceeding inductively, we obtain

$$v(x) \leqslant a_n d(x)^{-\alpha_n(x)}. \tag{4.3}$$

$$u(x) \geqslant a_{n+1} d(x)^{-\beta_n(x)}. \tag{4.4}$$

Where for $x \in \Omega$, we have

$$\begin{cases} \alpha_n(x) = \frac{q(x)-\beta_{n-1}(x)c(x)}{e(x)-q(x)+1}, \\ \beta_n(x) = \frac{p(x)-\alpha_n(x)b(x)}{a(x)-p(x)+1}, \\ a_{n+1}(x) = a_n^{\frac{c(x)b(x)}{(a^- - p^+ +1)(e^+ - q^- +1)}} B_{e,\beta_{n-1}c}^{-\frac{b(x)}{a^- - p^+ +1}} A_{a,\alpha_n b}. \end{cases}$$

Now, let us see that all these quantities converge as $n \to +\infty$. It is a simple calculation to check that, for $x \in \Omega$

$$\alpha_n(x) = \frac{q(x)(a(x)-p(x)+1) - c(x)p(x)}{(a(x)-p(x)+1)(e(x)-q(x)+1)} + \alpha_{n-1}(x).$$

Hence $\alpha_n < \alpha_{n-1}$, for all $n > 0$. Thus, $\alpha_n \to -\infty$ and $\beta_n \to +\infty$ as $n \to +\infty$.

We have by (4.4) that

$$-\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) = u^{c(x)} v^{e(x)} \geqslant a_{n+1}^{c(x)} d(x)^{-c(x)\beta_n(x)} v^{e(x)}.$$

Thus, by Lemma 3, we obtain

$$v \leqslant a_{n+1}^{-\frac{c(x)}{e^+ - q^- +1}} d(x)^{-\alpha_{n+1}(x)} B_{e,\beta_n c}.$$

Moreover, let n the minimum positive integer such that $\beta_n(x)c(x) \geqslant q(x)$, for $x \in \Omega$. We conclude that v is bounded. Hence, there is no solutions. Thus, $c(x) = a(x) - p(x) + 1$ and $b(x) = e(x) - q(x) + 1$.

Next, we consider the boundary condition (I), since $p(x) = q(x)$ for $x \in \Omega$, it is easily to see that (U,U) is a solution to (P). Using Lemma 2, with $a(x)+e(x)-q(x)+1 > p(x)-1$, we find that U satisfies

$$\begin{cases} -\Delta_{p(x)} U = U^{a(x)+e(x)-q(x)+1}, & x \in \Omega, \\ U = +\infty, & x \in \partial\Omega. \end{cases}$$

Thus, $u = v = U_{a+e-q+1,0}$.

Moreover, it is easily to see that $(u,v) = (t^{\frac{b}{b+c}} U_{a+e-q+1,0}, t^{-\frac{c}{b+c}} U_{a+e-q+1,0})$ is also a positive solution to (4.1), for $t > 0$. Indeed,

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) &= t^{\frac{b(p(x)-1)}{b+c}} U_{a+e-q+1,0}^{a+e-q(x)+1} \\ &= t^{\frac{bc-b(a-p(x)+1)}{b+c}} u^a v^b. \end{aligned}$$

But $p(x) = a - c + 1$, for $x \in \Omega$.

Thus, $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = u^a v^b, -\operatorname{div}(|\nabla v|^{q(x)-2}\nabla v) = u^c v^e$.

(iii) We will easily show that the condition $b(x) < e(x) - q(x) + 1$ is necessary.

Moreover, if (u,v) is a solution to the problem (4.1)

$$\begin{cases} -\Delta_{p(x)}(t^{-\delta}u) = t^{-\delta(p(x)-1)}u^a v^b = t^{\delta(a-p(x)+1)-b}(t^{-\delta}u)^a(tv)^b, \\ -\Delta_{q(x)}(tv) = t^{q(x)-1}u^a v^b = t^{c\delta-(e-q(x)+1)}(t^{-\delta}u)^c(tv)^e. \end{cases}$$

Due to the fact that, for $x\in\Omega,\ p(x)=a-\frac{b}{\delta}+1$ and thus $q(x)=e-c\delta+1,$ thanks to the subcriticality condition. We obtain

$$\begin{cases} -\Delta_{p(x)}(t^{-\delta}u) = (t^{-\delta}u)^a(tv)^b, \\ -\Delta_{q(x)}(tv) = (t^{-\delta}u)^c(tv)^e. \end{cases}$$

Thus, $(t^{-\delta}u,tv)$ is also a solution for (P), with the boundary condition (SF).

We can easily obtain the solution $(t^{-\delta}u,tv)$, with the following conditions

$$\begin{cases} t^{-\delta}u = t^{-\delta}\lambda, \\ tv = +\infty. \end{cases}$$

For every $t>0,\ t^{-\delta}\lambda$ can obtain any positive number. Thus, a solution with different boundary λ can be obtained, from one of them.

Fix $\varepsilon>0$, such as $c<a+\varepsilon-(p(x)-1)$, for $x\in\Omega$. We consider the following problem with the boundary condition (SF)

$$(4.5)\ \begin{cases} -\Delta_{p(x)}u = u^{a+\varepsilon}v^b, & x\in\Omega, \\ -\Delta_{q(x)}v = u^c v^e, & x\in\Omega, \\ u=\lambda,\ v=+\infty, & x\in\partial\Omega. \end{cases}$$

Since $(a+\varepsilon-(p(x)-1))(e-q(x)+1)>bc$, for $x\in\Omega$. The system (4.5) has a solution (u_λ,v_λ), by Theorem $1(iii)$.

We can choose λ large enough, so that $\inf\limits_{\Omega} u_\lambda \geqslant 1$.

On the one hand, one has

$$-\Delta_{p(x)}u_\lambda = u_\lambda^{a+\varepsilon}v_\lambda^b \geqslant u_\lambda^a v_\lambda^b,$$

and

$$-\Delta_{q(x)}v_\lambda = u_\lambda^c v_\lambda^e \geqslant u_\lambda^c v_\lambda^e.$$

Hence (u_λ,v_λ) is a supersolution to (4.5).

On another hand, it is not hard to show that $(\underline{u},\underline{v}) = (\lambda, \lambda^{-\frac{c}{e-q^-+1}}U_{e,0})$ is a subsolution. Indeed, we have

$$\begin{aligned} -\Delta_{q(x)}\underline{v} &= \lambda^{-\frac{c(q(x)-1)}{e-q^-+1}}U_{e,0}^e = \lambda^{\frac{c(e-q(x)+1)-c(e-q^-+1)}{e-q^-+1}}\lambda^c\underline{v}^e \\ &\leqslant \lambda^{\frac{c(e-q(x)+1-e+q(x)-1)}{e-q^-+1}}\underline{u}^c\underline{v}^e \leqslant \underline{u}^c\underline{v}^e, \end{aligned}$$

and

$$-\operatorname{div}(|\nabla\lambda|^{p(x)-2}\nabla\lambda) = 0 \leqslant \lambda^a\underline{v}^b.$$

Hence, by Theorem 6 in Appendix, (4.5) has a solution (u,v) with $u=\lambda,\ v=+\infty$ on $\partial\Omega$, for every $\lambda>0$. $\square$

4.2. Nonexistence

We will now take in the consideration the nonexistence of positive solutions to (P), with the boundary conditions (I). Assuming that $c(x) < a(x) - p(x) + 1$ and thus $b(x) > e(x) - q(x) + 1$, since $(a(x) - p(x) + 1)(e(x) - q(x) + 1) = b(x)c(x)$ and let (u, v) be a positive solution. By means of iterative procedure in section 3 (Theorem $2(ii)$), we obtain

$$u \leqslant a_n d(x)^{-\alpha_n(x)} A_{a,\beta_n b} \quad \text{and} \quad v \geqslant a_{n+1} d(x)^{-\beta_n(x)},$$

where, for $x \in \Omega$

$$\alpha_n(x) = \frac{p(x)(e(x) - q(x) + 1) - q(x)b(x)}{(a(x) - p(x) + 1)(e(x) - q(x) + 1)} + \alpha_{n-1}(x).$$

Hence, $\alpha_n \to -\infty$ and $\beta_n \to +\infty$ as $n \to +\infty$.

We deduce that

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \geqslant a_{n+1}^{b(x)} d(x)^{-\beta_n(x)b(x)} u^{a(x)}, \quad x \in \Omega,$$

and it follows as in Lemma 3 that, $u \leqslant a_{n+1}^{-\frac{b(x)}{a^+ - p^- + 1}} d(x)^{-\alpha_{n+1}(x)} A_{a,\beta_n c}$.

If n is the minimum positive integer such that $\beta_n(x)c(x) \geqslant p(x)$, for $x \in \Omega$. Thus, u is bounded.

Thus, there is no solutions.

5. Appendix

In this part, we recall two results related to the method of subsolutions and supersolutions [10], for the following system

$$(P) \begin{cases} -\Delta_{p(x)} u = u^{a(x)} v^{b(x)}, & x \in \Omega, \\ -\Delta_{q(x)} v = u^{c(x)} v^{e(x)}, & x \in \Omega, \end{cases}$$

where $a(x) > p(x) - 1$, $e(x) > q(x) - 1$ and $b(x), c(x) > 0$, for $x \in \Omega$.

We start by giving the following definition.

DEFINITION 1. (see [10]) Let $\underline{u}, \bar{u} \in W^{1,p(.)}(\Omega)$ and $\underline{v}, \bar{v} \in W^{1,q(.)}(\Omega)$ four positive functions, such as $\underline{u} \leqslant \bar{u}$ and $\underline{v} \leqslant \bar{v}$ a.e. in Ω. The couple $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ are said to be supersolutions and subsolutions pairs respectively to (P), if the following inequalities are satisfied in the distribution case:

$$\begin{aligned}
-\Delta_{p(x)} \bar{u} &\geqslant a(x)\bar{u}^{\alpha_1} v^{\beta_1}, \ x \in \Omega, && \text{for any } v \in [\underline{v}, \bar{v}], \\
-\Delta_{q(x)} \bar{v} &\geqslant b(x)u^{\alpha_2} \bar{v}^{\beta_2}, \ x \in \Omega, && \text{for any } u \in [\underline{u}, \bar{u}], \\
-\Delta_{p(x)} \underline{u} &\leqslant a(x)\underline{u}^{\alpha_1} v^{\beta_1}, \ x \in \Omega, && \text{for any } v \in [\underline{v}, \bar{v}], \\
-\Delta_{q(x)} \underline{v} &\leqslant b(x)u^{\alpha_2} \underline{v}^{\beta_2}, \ x \in \Omega, && \text{for any } u \in [\underline{u}, \bar{u}].
\end{aligned}$$

We begin by considering the system (P), with finite boundary conditions $u=\lambda$, $v=\mu$ with $\lambda,\mu>0$.

THEOREM 4. *Let us assume that $(\underline{u},\underline{v})$ is a subsolution and $(\bar{u},\bar{v})$ is a supersolution to (P) with $\underline{u}\leqslant\lambda\leqslant\bar{u}$, $\underline{v}\leqslant\mu\leqslant\bar{v}$ on $\partial\Omega$.*

Then, the problem (P) admits at least a weak solution $(u,v)\in W^{1,p(.)}(\Omega)\times W^{1,q(.)}(\Omega)$ with $\underline{u}\leqslant u\leqslant\bar{u}$, $\underline{v}\leqslant v\leqslant\bar{v}$ in Ω and $u=\lambda$, $v=\mu$ on $\partial\Omega$.

Proof. Denote by u_1 the unique positive solution to the problem (5.1) (see [1] and the Remark 1)

$$(5.1)\ \begin{cases} -\Delta_{p(x)}u=\underline{v}^{b(x)}u^{a(x)}, & x\in\Omega,\\ u=\lambda, \quad x\in\partial\Omega. \end{cases}$$

Due to the fact that $\underline{u},\bar{u}$ are subsolution and supersolution of (P), respectively. Thus, $\underline{u},\bar{u}$ are also subsolution and supersolution of (5.1). By uniqueness of solution, we have $\underline{u}\leqslant u_1\leqslant\bar{u}$ in Ω.

Now, we consider v_1 to be the unique solution to the problem

$$(5.2)\ \begin{cases} -\Delta_{q(x)}v=u_1^{c(x)}v^{e(x)}, & x\in\Omega,\\ v=\mu, \quad x\in\partial\Omega. \end{cases}$$

It follows similarly that $\underline{v}\leqslant v_1\leqslant\bar{v}$. Thus, in this way, we define u_n to be the unique solution to (5.1), with $\underline{v}$ replaced by v_{n-1}, such that $u_n=\lambda$ on $\partial\Omega$ and we define the unique solution v_n to (5.2), with u_1 replaced by u_n, such that $v_n=\mu$ on $\partial\Omega$.

We obtain two increasing sequences $(u_n)_n$ and $(v_n)_n$ such that $\underline{u}\leqslant u_n\leqslant\bar{u}$ and $\underline{v}\leqslant v_n\leqslant\bar{v}$.

Indeed, we have for $\varphi\in W^{1,p(.)}(\Omega)$

$$\int_\Omega(|\nabla u_1|^{p(x)-2}\nabla u_1\nabla\varphi-\underline{v}^{b(x)}u_1^{a(x)}\varphi)\,dx\leqslant\int_\Omega(|\nabla u_2|^{p(x)-2}\nabla u_2\nabla\varphi-\underline{v}^{b(x)}u_2^{a(x)}\varphi)\,dx,$$

and $u_1=u_2=\lambda$ on $\partial\Omega$. By the principe of the comparison [13], $u_1\leqslant u_2$ in Ω. Thus, $\underline{u}\leqslant u_1\leqslant u_2\leqslant\ldots\leqslant u_{n-1}\leqslant u_n\leqslant\ldots\leqslant\bar{u}$ and with the same argument, we obtain that $\underline{v}\leqslant v_1\leqslant v_2\leqslant\ldots\leqslant v_{n-1}\leqslant v_n\leqslant\ldots\leqslant\bar{v}$.

According to the above result, we have $u_n\to u$, $v_n\to v$ as $n\to+\infty$, for a.e $x\in\Omega$.

Moreover, by the regularity result in [9] and [25], we have $(u_n)_n,(v_n)_n\subset C^{1,\alpha}(\Omega)$. Thus, by Ascoli-Arzela Theorem, we obtain that

$$u_n\to u \text{ in } C^1(\bar{\Omega}) \quad\text{and}\quad v_n\to v \text{ in } C^1(\bar{\Omega}) \text{ as } n\to+\infty.$$

Thus, by the dominate convergence Theorem, we obtain that (u,v) is a weak solution for (P). Moreover, $u=\lambda$, $v=\mu$ on $\partial\Omega$ and $\underline{u}\leqslant u\leqslant\bar{u}$, $\underline{v}\leqslant v\leqslant\bar{v}$ in Ω. □

Now, we prove a version of the method, which is directly applicable to the problem with infinite boundary conditions.

THEOREM 5. *Let us assume that $(\underline{u},\underline{v})$ and $(\bar{u},\bar{v})$ are subsolutions and supersolutions to (P) respectively with $\underline{u}=\bar{u}=\underline{v}=\bar{v}=+\infty$ on $\partial\Omega$ and $\underline{u}\leqslant\bar{u}$, $\underline{v}\leqslant\bar{v}$ in Ω.*

Then, the problem (P) admits at least a weak solution $(u,v)\in W^{1,p(.)}(\Omega)\times W^{1,q(.)}(\Omega)$ with $\underline{u}\leqslant u\leqslant\bar{u}$, $\underline{v}\leqslant v\leqslant\bar{v}$ in Ω and $u=v=+\infty$ on $\partial\Omega$.

Proof. Let $n>0$, we consider the problem

$$(5.3)\quad \begin{cases} -\Delta_{p(x)}u=u^{a(x)}v^{b(x)}, & x\in\Omega_n,\\ -\Delta_{q(x)}v=u^{c(x)}v^{e(x)}, & x\in\Omega_n,\\ u=\lambda_n,\ v=\mu_n, & x\in\partial\Omega_n,\end{cases}$$

where $\Omega_n=\{x\in\Omega : d(x)>n\}$ and $\lambda_n,\mu_n>0$, for all $n>0$, with $\underline{u}\leqslant\lambda_n\leqslant\bar{u}$ and $\underline{v}\leqslant\mu_n\leqslant\bar{v}$ on $\partial\Omega_n$.

By Theorem 4, there exists a solution $(u_n,v_n)\in W^{1,p(.)}(\Omega_n)\times W^{1,q(.)}(\Omega_n)$ of (5.3), such that $\underline{u}\leqslant u_n\leqslant\bar{u}$, $\underline{v}\leqslant v_n\leqslant\bar{v}$ in Ω_n and $u_n=\lambda_n$, $v_n=\mu_n$ on $\partial\Omega_n$.

Thus, we construct two decreasing sequences $(u_n)_n\subset W^{1,p(.)}(\Omega_n)$ and $(v_n)_n\subset W^{1,q(.)}(\Omega_n)$. According to the above result, we have for a.e. $x\in\Omega$, $u_n\to u$ and $v_n\to v$ as $n\to 0$.

However, using the regularity result in [9] and [25], we get that (u_n) is bounded in $C^{1,\alpha}(\bar{\Omega}_n)$.

Thus, by the Ascoli-Arzela Theorem, (u_n) is relatively compact in $C^1(\bar{\Omega}_n)$. Thus, there exists a subsequence (u_{n_k}) such as $u_{n_k}\to u$ as $k\to 0$ in $C^1(\bar{\Omega}_n)$.

The same argument gives that $v_{n_k}\to v$ in $C^1(\bar{\Omega}_n)$ as $k\to 0$.

Thus, (u,v) is a weak solution to (P) verifying in addition that, $\underline{u}\leqslant u\leqslant\bar{u}$ and $\underline{v}\leqslant v\leqslant\bar{v}$ in Ω. In particular, $u=v=+\infty$ on $\partial\Omega$. □

THEOREM 6. *Let us assume that $(\underline{u},\underline{v})$ and $(\bar{u},\bar{v})$ are subsolutions and supersolutions to (P) respectively with $\underline{u}\leqslant\lambda\leqslant\bar{u}$, $\underline{v}=\bar{v}=+\infty$ on $\partial\Omega$ and $\underline{u}\leqslant\bar{u}$, $\underline{v}\leqslant\bar{v}$ in Ω.*

Then, the problem (P) admits at least a weak solution $(u,v)\in W^{1,p(.)}(\Omega)\times W^{1,q(.)}(\Omega)$ with $\underline{u}\leqslant u\leqslant\bar{u}$, $\underline{v}\leqslant v\leqslant\bar{v}$ in Ω and $u=\lambda$, $v=+\infty$ on $\partial\Omega$.

Proof. The problem

$$(5.4)\quad \begin{cases} -\Delta_{p(x)}u=\underline{v}^{b(x)}u^{a(x)}, & x\in\Omega,\\ u=\lambda, & x\in\partial\Omega,\end{cases}$$

has a unique positive solution denoting u_1, which exists thanks to [10] (see also the Remark 1).

Moreover, since $\underline{u}$ and $\bar{u}$ are subsolution and supersolution for (P) respectively, we have $-\Delta_{p(x)}\underline{u}\leqslant\underline{v}^{b(x)}\underline{u}^{a(x)}$ and $-\Delta_{p(x)}\bar{u}\geqslant\underline{v}^{b(x)}\bar{u}^{a(x)}$. Thus, $\underline{u}$ and $\bar{u}$ are subsolution and supersolution for (5.4) such that $\underline{u}\leqslant u_1\leqslant\bar{u}$.

We now define v_1 as the unique solution to

$$(5.5)\ \begin{cases} -\Delta_{q(x)}v = u_1^{c(x)}v^{e(x)}, & x \in \Omega, \\ v = +\infty, & x \in \partial\Omega. \end{cases}$$

It follows similarly that $\underline{v} \leqslant v_1 \leqslant \bar{v}$.

We continue this procedure and define u_2 as the unique solution to

$$(5.4)'\ \begin{cases} -\Delta_{p(x)}u = v_1^{b(x)}u^{a(x)}, & x \in \Omega, \\ u = \lambda, & x \in \partial\Omega. \end{cases}$$

Then, it follows as previously that $\underline{u} \leqslant u_2 \leqslant \bar{u}$ in Ω. In addition, by the principle of comparison, we obtain that $u_1 \leqslant u_2$. We can recursively define u_n as the unique solution to $(5.4)'$, replacing v_1 by v_{n-1} with $u_n = \lambda$ on $\partial\Omega$ and we define the unique solution v_n to (5.5), replacing u_1 by u_{n-1} with $v_n = +\infty$ on $\partial\Omega$.

In this way, we obtain two increasing sequences $(u_n)_n$ and $(v_n)_n$ such that $\underline{u} \leqslant u_n \leqslant \bar{u}$ and $\underline{v} \leqslant v_n \leqslant \bar{v}$ in Ω, for all $n > 0$.

With the same process in the Theorem 4, we conclude that there exists a subsequence (labeled again by (u_n) and (v_n)) such that $u_n \to u$ in $C^1(\bar{\Omega})$ and $v_n \to v$ in $C^1(\bar{\Omega})$, where (u,v) is a solution to (P) and $\underline{u} \leqslant u \leqslant \bar{u}$, $\underline{v} \leqslant v \leqslant \bar{v}$ in Ω.

As a consequence, $u = \lambda$ and $v = +\infty$ on $\partial\Omega$. □

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