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GENERALIZATIONS OF SHERMAN'S INEQUALITY BY HERMITE'S INTERPOLATING POLYNOMIAL

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Abstract. Generalizations of Sherman's inequality for convex functions of higher order are obtained by applying Hermite's interpolating polynomials. The results for particular cases, namely, Lagrange, $(m,n-m)$ and two-point Taylor interpolating polynomials are also cosidered. The Grüss and Ostrowski type inequalities related to these generalizations are given.

1. Introduction

We start with the concept of majorization which is exactly a partial ordering of vectors and determines the degree of similarity between the vector elements.

For fixed $m \geqslant 2$, let $\mathbf{x} = (x_1,\dots,x_m)$ and $\mathbf{y} = (y_1,\dots,y_m)$ denote two m-tuples. Let $x_{[1]} \geqslant x_{[2]} \geqslant \dots \geqslant x_{[m]}$ and $y_{[1]} \geqslant y_{[2]} \geqslant \dots \geqslant y_{[m]}$ be their ordered components. We say that $\mathbf{x}$ *majorizes* $\mathbf{y}$ or $\mathbf{y}$ is *majorized* by $\mathbf{x}$ and write $\mathbf{y} \prec \mathbf{x}$ if

$$\sum_{i=1}^{k} y_{[i]} \leqslant \sum_{i=1}^{k} x_{[i]}, \quad k = 1,\dots,m-1, \quad \text{and} \quad \sum_{i=1}^{m} y_i = \sum_{i=1}^{m} x_i. \tag{1}$$

A notation from real vectors may be extended to real matrices. Let $\mathscr{M}_{ml}(\mathbb{R})$ denotes the space of $m \times l$ real matrices. A matrix $\mathbf{A} = (a_{ij}) \in \mathscr{M}_{ml}(\mathbb{R})$ is called *row stochastic* if all of its entries are greater than or equal to zero and the sum of the entries in each row is equal to 1. A square matrix $\mathbf{A} = (a_{ij}) \in \mathscr{M}_{ll}(\mathbb{R})$ is called *double stochastic* if all of its entries are greater than or equal to zero and the sum of the entries in each column and each row is equal to 1.

The majorization theorem, due to Hardy et al (1929 [6]), gives connections with matrix theory (see also [8, p. 333]).

THEOREM 1. *Let* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$. *Then the following statements are equivalent:*

(i) $\mathbf{y} \prec \mathbf{x}$;

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(ii) *There is a doubly stochastic matrix* $\mathbf{A}$ *such that* $\mathbf{y} = \mathbf{x}\mathbf{A}$*;*

(iii) *The inequality* $\sum_{i=1}^{m} \phi(y_i) \leqslant \sum_{i=1}^{m} \phi(x_i)$ *holds for each convex function* $\phi : \mathbb{R} \to \mathbb{R}$.

S. Sherman [10] obtained the following general result.

THEOREM 2. *Let* $[\alpha, \beta] \subset \mathbb{R}$ *and for fixed* $l, m \in \mathbb{N}$, $l, m \geqslant 2$, *let* $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^m$, $\mathbf{u} \in [0, \infty)^l$, $\mathbf{v} \in [0, \infty)^m$ *and*

$$\mathbf{y} = \mathbf{x}\mathbf{A}^T \text{ and } \mathbf{u} = \mathbf{v}\mathbf{A} \tag{2}$$

for some row stochastic matrix $\mathbf{A} \in \mathscr{M}_{ml}(\mathbb{R})$. *Then for every convex function* $\phi : [\alpha, \beta] \to \mathbb{R}$ *we have*

$$\sum_{q=1}^{m} v_q \phi(y_q) \leqslant \sum_{p=1}^{l} u_p \phi(x_p). \tag{3}$$

Sherman obtained this useful generalization replacing the classical concept of majorization $\mathbf{y} \prec \mathbf{x}$ by the notion of weighted majorization (2) for two pairs $(\mathbf{x}, \mathbf{u})$ and $(\mathbf{y}, \mathbf{v})$, where $\mathbf{x} = (x_1, ..., x_l)$ and $\mathbf{y} = (y_1, ..., y_m)$ are real vectors and $\mathbf{u} = (u_1, ..., u_l)$ and $\mathbf{v} = (v_1, ..., v_m)$ are corresponding nonnegative weights. Here $\mathbf{A}^T$ denotes the transpose of a matrix $\mathbf{A}$. In particular, if $m = l$ and $u_p = v_q$ for all $p, q = 1, ..., m$, the condition $\mathbf{u} = \mathbf{v}\mathbf{A}$ assures the stochasticity on columns, so in that case we deal with doubly stochastic matrices. Then, as a special case of Sherman's inequality, we get the weighted version of majorization's inequality:

$$\sum_{p=1}^{m} u_p \phi(y_p) \leqslant \sum_{p=1}^{m} u_p \phi(x_p).$$

Denoting $U_m = \sum_{p=1}^{m} u_p$ and putting $y_1 = y_2 = ... = y_m = \frac{1}{U_m} \sum_{p=1}^{m} u_p x_p$, we obtain Jensen's inequality in the form

$$\phi\left(\frac{1}{U_m} \sum_{p=1}^{m} u_p x_p\right) \leqslant \frac{1}{U_m} \sum_{p=1}^{m} u_p \phi(x_p).$$

In this paper, we recall generalizations of Sherman's result for convex functions of the higher order. Moreover, we obtain extension to real, not necessary nonnegative weights $\mathbf{u}$, $\mathbf{v}$ and matrix $\mathbf{A}$. For some related results see also [1], [2], [7].

In sequel, we always assume that $[\alpha, \beta] \subset \mathbb{R}$ without having to be emphasized.

The notion of n-convexity was defined in terms of divided differences by Popoviciu [9]. A function $\phi : [\alpha, \beta] \to \mathbb{R}$ is n-convex, $n \geqslant 0$, if its nth order divided differences $[x_0, ..., x_n; \phi]$ are nonnegative for all choices of $(n+1)$ distinct points $x_i \in [\alpha, \beta]$, $i = 0, ..., n$. Thus, a 0-convex function is nonnegative, a 1-convex function is nondecreasing and a 2-convex function is convex in the usual sense. If $\phi^{(n)}$ exists then ϕ is n-convex iff $\phi^{(n)} \geqslant 0$ (see [8]).

2. Preliminaries

Let $\alpha \leqslant a_1 < a_2 < \ldots < a_r \leqslant \beta$, $(r \geqslant 2)$ be the given points. For $\phi \in C^n([\alpha,\beta])$ $(n \geqslant r)$ a unique polynomial $\rho_H(s)$ of degree $(n-1)$ exists, such that *Hermite conditions* hold:

$$\rho_H^{(i)}(a_j) = \phi^{(i)}(a_j), \quad 0 \leqslant i \leqslant k_j, \quad 1 \leqslant j \leqslant r, \tag{H}$$

where $\sum_{j=1}^{r} k_j + r = n$.

In particular, for $r = n$, $k_j = 0$ for all j, we have *Lagrange conditions*:

$$\rho_L(a_j) = \phi(a_j), \quad 1 \leqslant j \leqslant n.$$

For $r = 2$, $1 \leqslant m \leqslant n-1$, $k_1 = m-1$, $k_2 = n-m-1$, we have *Type* $(m,n-m)$ *conditions*:

$$\rho_{(m,n)}^{(i)}(\alpha) = \phi^{(i)}(\alpha), \quad 0 \leqslant i \leqslant m-1,$$

$$\rho_{(m,n)}^{(i)}(\beta) = \phi^{(i)}(\beta), \quad 0 \leqslant i \leqslant n-m-1.$$

For $n = 2m$, $r = 2$ and $k_1 = k_2 = m-1$, we have *Two-point Taylor conditions*:

$$\rho_{2T}^{(i)}(\alpha) = \phi^{(i)}(\alpha), \quad \rho_{2T}^{(i)}(\beta) = \phi^{(i)}(\beta), \quad 0 \leqslant i \leqslant m-1.$$

The following theorem and remark can be found in [3].

THEOREM 3. *Let $\alpha \leqslant a_1 < a_2 < \ldots < a_r \leqslant \beta$, $(r \geqslant 2)$, be the given points and $\phi \in C^n([\alpha,\beta])$, $(n \geqslant r)$. Let $\rho_H(s)$ be the Hermite inrepolating polynomial. Then*

$$\begin{aligned}\phi(t) &= \rho_H(t) + R_{H,n}(\phi,t) \\ &= \sum_{j=1}^{r}\sum_{i=0}^{k_j} H_{ij}(t)\phi^{(i)}(a_j) + \int_\alpha^\beta G_{H,n}(t,s)\phi^{(n)}(s)ds,\end{aligned} \tag{4}$$

where H_{ij} are fundamental polynomials of the Hermite basis defined by

$$H_{ij}(t) = \frac{1}{i!}\frac{\omega(t)}{(t-a_j)^{k_j+1-i}} \sum_{k=0}^{k_j-i} \frac{1}{k!}\frac{d^k}{dt^k}\left(\frac{(t-a_j)^{k_j+1}}{\omega(t)}\right)\Bigg|_{t=a_j} (t-a_j)^k, \tag{5}$$

where

$$\omega(t) = \prod_{j=1}^{r}(t-a_j)^{k_j+1},$$

and $G_{H,n}(t,s)$ is defined by

$$G_{H,n}(t,s) = \begin{cases} \sum\limits_{j=1}^{l}\sum\limits_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t); \ s \leqslant t, \\ -\sum\limits_{j=l+1}^{r}\sum\limits_{i=0}^{k_j} \frac{(a_j-s)^{n-i-1}}{(n-i-1)!} H_{ij}(t); \ s \geqslant t, \end{cases} \tag{6}$$

for all $a_l \leqslant s \leqslant a_{l+1}$; $l = 0,\ldots,r$ with $a_0 = \alpha$ and $a_{r+1} = \beta$.

REMARK 1. For Lagrange conditions, from Theorem 3 we have

$$\phi(t) = \rho_L(t) + R_L(\phi, t)$$

where $\rho_L(t)$ is the Lagrange interpolating polynomial i.e.

$$\rho_L(t) = \sum_{j=1}^{n} \prod_{\substack{k=1\\k\neq j}}^{n} \left(\frac{t-a_k}{a_j-a_k}\right) \phi(a_j)$$

and the remainder $R_L(\phi, t)$ is given by

$$R_L(\phi, t) = \int_\alpha^\beta G_L(t,s)\phi^{(n)}(s)ds$$

with

$$G_L(t,s) = \frac{1}{(n-1)!} \begin{cases} \sum\limits_{j=1}^{l} (a_j - s)^{n-1} \prod\limits_{\substack{k=1\\k\neq j}}^{n} \left(\frac{t-a_k}{a_j-a_k}\right), & s \leqslant t \\ -\sum\limits_{j=l+1}^{n} (a_j - s)^{n-1} \prod\limits_{\substack{k=1\\k\neq j}}^{n} \left(\frac{t-a_k}{a_j-a_k}\right), & s \geqslant t \end{cases} \tag{7}$$

$a_l \leqslant s \leqslant a_{l+1}$, $l = 1, 2, ..., n-1$ with $a_1 = \alpha$ and $a_n = \beta$.

For type $(m, n-m)$ conditions, from Theorem 3 we have

$$\phi(t) = \rho_{(m,n)}(t) + R_{(m,n)}(\phi, t)$$

where $\rho_{(m,n)}(t)$ is $(m, n-m)$ interpolating polynomial, i.e.

$$\rho_{(m,n)}(t) = \sum_{i=0}^{m-1} \tau_i(t)\phi^{(i)}(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(t)\phi^{(i)}(\beta),$$

with

$$\tau_i(t) = \frac{1}{i!}(t-\alpha)^i \left(\frac{t-\beta}{\alpha-\beta}\right)^{n-m} \sum_{k=0}^{m-1-i} \binom{n-m+k-1}{k} \left(\frac{t-\alpha}{\beta-\alpha}\right)^k \tag{8}$$

and

$$\eta_i(t) = \frac{1}{i!}(t-\beta)^i \left(\frac{t-\alpha}{\beta-\alpha}\right)^{m} \sum_{k=0}^{n-m-1-i} \binom{m+k-1}{k} \left(\frac{t-\beta}{\alpha-\beta}\right)^k. \tag{9}$$

and the remainder $R_{(m,n)}(\phi, t)$ is given by

$$R_{(m,n)}(\phi, t) = \int_\alpha^\beta G_{(m,n)}(t,s)\phi^{(n)}(s)ds$$

with

$$G_{(m,n)}(t,s) = \begin{cases} \sum\limits_{j=0}^{m-1} \left[\sum\limits_{p=0}^{m-1-j} \binom{n-m+p-1}{p} \left(\frac{t-\alpha}{\beta-\alpha}\right)^p\right] \frac{(t-\alpha)^j(\alpha-s)^{n-j-1}}{j!(n-j-1)!} \left(\frac{\beta-t}{\beta-\alpha}\right)^{n-m}, & s \leqslant t \\ -\sum\limits_{i=0}^{n-m-1} \left[\sum\limits_{q=0}^{n-m-i-1} \binom{m+q-1}{q} \left(\frac{\beta-t}{\beta-\alpha}\right)^q \frac{(t-\beta)^i(\beta-s)^{n-i-1}}{i!(n-i-1)!}\right] \left(\frac{t-\alpha}{\beta-\alpha}\right)^m, & t \leqslant s. \end{cases} \tag{10}$$

For Type Two-point Taylor conditions, from Theorem 3 we have

$$\phi(t) = \rho_{2T}(t) + R_{2T}(\phi, t)$$

where $\rho_{2T}(t)$ is the two-point Taylor interpolating polynomial i.e,

$$\rho_{2T}(t) = \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \binom{m+k-1}{k} \left[\phi^{(i)}(\alpha) \frac{(t-\alpha)^i}{i!} \left(\frac{t-\beta}{\alpha-\beta}\right)^m \left(\frac{t-\alpha}{\beta-\alpha}\right)^k + \phi^{(i)}(\beta) \frac{(t-\beta)^i}{i!} \left(\frac{t-\alpha}{\beta-\alpha}\right)^m \left(\frac{t-\beta}{\alpha-\beta}\right)^k \right] \tag{11}$$

and the remainder $R_{2T}(\phi, t)$ is given by

$$R_{2T}(\phi, t) = \int_\alpha^\beta G_{2T}(t,s) \phi^{(n)}(s) ds$$

with

$$G_{2T}(t,s) = \begin{cases} \frac{(-1)^m}{(2m-1)!} p^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (t-s)^{m-1-j} q^j(t,s), & s \leqslant t; \\ \frac{(-1)^m}{(2m-1)!} q^m(t,s) \sum_{j=0}^{m-1} \binom{m-1+j}{j} (s-t)^{m-1-j} p^j(t,s), & s \geqslant t; \end{cases} \tag{12}$$

where $p(t,s) = \frac{(s-\alpha)(\beta-t)}{\beta-\alpha}$, $q(t,s) = p(s,t), \forall t, s \in [\alpha, \beta]$.

3. Generalizations of Sherman's inequality

Applying Hermite's interpolating polynomial we obtain a generalization of Sherman's theorem which holds for real, not necessary nonnegative weights $\mathbf{u}$, $\mathbf{v}$ and a matrix $\mathbf{A}$ and without assumption (2).

THEOREM 4. *Let $\alpha \leqslant a_1 < a_2 < \ldots < a_r \leqslant \beta$ $(r \geqslant 2)$ be the given points, $k_j \geqslant 0$, $j = 1, \ldots, r$, with $\sum_{j=1}^{r} k_j + r = n$. Let $\phi \in C^n([\alpha, \beta])$ be n-convex and $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^m$, $\mathbf{u} \in \mathbb{R}^l$ and $\mathbf{v} \in \mathbb{R}^m$. If*

$$\sum_{p=1}^{l} u_p G_{H,n}(x_p, s) - \sum_{q=1}^{m} v_q G_{H,n}(y_q, s) \geqslant 0, \quad s \in [\alpha, \beta], \tag{13}$$

then

$$\sum_{p=1}^{l} u_p \phi(x_p) - \sum_{q=1}^{m} v_q \phi(y_q) \geqslant \sum_{p=1}^{l} u_p \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_j) H_{ij}(x_p) - \sum_{q=1}^{m} v_q \sum_{j=1}^{r} \sum_{i=0}^{k_j} \phi^{(i)}(a_j) H_{ij}(y_q), \tag{14}$$

where $G_{H,n}$ and H_{ij} are defined as in (6) and (5), respectively.

Proof. Since $\phi \in C^n([\alpha,\beta])$, applying Theorem 3 on $\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q)$, we get the identity

$$\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i)}(a_j)\left[\sum_{p=1}^{l} u_p H_{ij}(x_p) - \sum_{q=1}^{m} v_q H_{ij}(y_q)\right] + \int_{\alpha}^{\beta}\left[\sum_{p=1}^{l} u_p G_{H,n}(x_p,s) - \sum_{q=1}^{m} v_q G_{H,n}(y_q,s)\right]\phi^{(n)}(s)ds. \tag{15}$$

Since ϕ is n-convex on $[\alpha,\beta]$, then we have $\phi^{(n)} \geqslant 0$ on $[\alpha,\beta]$. Moreover, the inequality (14) holds. □

Under Sherman's assumptions the following generalizations hold.

THEOREM 5. *Let all the assumptions of Theorem 4 be satisfied. Additionally, let vectors* $\mathbf{u}$, $\mathbf{v}$ *be nonnegative and let (2) holds for some row stochastic matrix* $\mathbf{A} \in \mathscr{M}_{ml}(\mathbb{R})$. *If (14) holds and the function*

$$\bar{F}(\cdot) = \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i)}(a_j)H_{ij}(\cdot) \tag{16}$$

is convex on $[\alpha,\beta]$ *then the inequality (3) holds.*

Proof. If (14) holds, the right hand side of (14) can be written in the form

$$\sum_{p=1}^{l} u_p\bar{F}(x_p) - \sum_{q=1}^{m} v_q\bar{F}(y_q),$$

where $\bar{F}$ is defined by (16). If $\bar{F}$ is convex, then by Sherman's theorem we have

$$\sum_{p=1}^{l} u_p\bar{F}(x_p) - \sum_{q=1}^{m} v_q\bar{F}(y_q) \geqslant 0,$$

i.e. the right-hand side of (14) is nonnegative, so (3) immediately follows. □

By using Lagrange conditions we get the following generalization of Sherman's theorem.

COROLLARY 1. *Let* $\alpha \leqslant a_1 < a_2 < \ldots < a_n \leqslant \beta$ $(n \geqslant 2)$ *be the given points and* $\phi \in C^n([\alpha,\beta])$ *be* n*-convex. Let* $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^w$, $\mathbf{u} \in [0,\infty)^l$ *and* $\mathbf{v} \in [0,\infty)^w$ *be such that (2) holds for some row stochastic matrix* $\mathbf{A} \in \mathscr{M}_{wl}(\mathbb{R})$.

(i) If

$$\sum_{p=1}^{l} u_p G_L(x_p,s) - \sum_{q=1}^{w} v_q G_L(y_q,s) \geqslant 0, \quad s \in [\alpha,\beta],$$

then

$$\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{w} v_q\phi(y_q) \geqslant \sum_{p=1}^{l} u_p \sum_{j=1}^{n} \phi(a_j) \prod_{\substack{u=1\\u\neq j}}^{n} \left(\frac{x_p - a_u}{a_j - a_u}\right) - \sum_{q=1}^{w} v_q \sum_{j=1}^{n} \phi(a_j) \prod_{\substack{u=1\\u\neq j}}^{n} \left(\frac{y_q - a_u}{a_j - a_u}\right), \tag{17}$$

where G_L is defined as in (7).

(ii) If (17) holds and the function

$$\tilde{F}(\cdot) = \sum_{j=1}^{n} \phi(a_j) \prod_{\substack{u=1\\u\neq j}}^{n} \left(\frac{\cdot - a_u}{a_j - a_u}\right)$$

is convex on $[\alpha,\beta]$ then

$$\sum_{q=1}^{w} v_q\phi(y_q) \leqslant \sum_{p=1}^{l} u_p\phi(x_p).$$

By using type $(m, n-m)$ conditions we can give the following result.

COROLLARY 2. *Let $n \geqslant 2$, $1 \leqslant m \leqslant n-1$ and $\phi \in C^n([\alpha,\beta])$ be n-convex. Let $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^w$, $\mathbf{u} \in [0,\infty)^l$ and $\mathbf{v} \in [0,\infty)^w$ be such that (2) holds for some row stochastic matrix $\mathbf{A} \in \mathscr{M}_{wl}(\mathbb{R})$.*

(i) If

$$\sum_{p=1}^{l} u_p G_{(m,n)}(x_p,s) - \sum_{q=1}^{w} v_q G_{(m,n)}(y_q,s) \geqslant 0, \quad s \in [\alpha,\beta],$$

then

$$\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{w} v_q\phi(y_q) \geqslant \sum_{p=1}^{l} u_p \left(\sum_{i=0}^{m-1} \tau_i(x_p)\phi^{(i)}(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(x_p)\phi^{(i)}(\beta)\right) - \sum_{q=1}^{w} v_q \left(\sum_{i=0}^{m-1} \tau_i(y_q)\phi^{i}(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(y_q)\phi^{(i)}(\beta)\right), \tag{18}$$

where τ_i, η_i and $G_{(m,n)}$ are defined as in (8), (9) and (10), respectively.

(ii) If (18) holds and the function

$$\hat{F}(\cdot) = \sum_{i=0}^{m-1} \tau_i(\cdot)\phi^{(i)}(\alpha) + \sum_{i=0}^{n-m-1} \eta_i(\cdot)\phi^{(i)}(\beta)$$

is convex on $[\alpha,\beta]$ *then*

$$\sum_{q=1}^{u} v_q\phi(y_q) \leqslant \sum_{p=1}^{w} u_p\phi(x_p).$$

By using Two-point Taylor conditions we can give the following result.

COROLLARY 3. *Let* $m \geqslant 1$ *and* $\phi \in C^{2m}([\alpha,\beta])$ *be* $2m$*-convex. Let* $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^w$, $\mathbf{u} \in [0,\infty)^l$ *and* $\mathbf{v} \in [0,\infty)^w$ *be such that (2) holds for some row stochastic matrix* $\mathbf{A} \in \mathscr{M}_{wl}(\mathbb{R})$.

(i) If

$$\sum_{p=1}^{l} u_p G_{2T}(x_p,s) - \sum_{q=1}^{w} v_q G_{2T}(y_q,s) \geqslant 0, \quad s \in [\alpha,\beta],$$

then

$$\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{w} v_q\phi(y_q) \geqslant \sum_{p=1}^{l} u_p\rho_{2T}(x_p) - \sum_{q=1}^{w} v_q\rho_{2T}(y_q), \tag{19}$$

where ρ_{2T} *and* G_{2T} *are defined as in (11) and (12), respectively.*

(ii) Moreover, if the function ρ_{2T} *is convex on* $[\alpha,\beta]$, *then*

$$\sum_{q=1}^{w} v_q\phi(y_q) \leqslant \sum_{p=1}^{l} u_p\phi(x_p).$$

REMARK 2. Motivated by the inequality (14), under the assumptions of Theorem 4, we define the linear functional $A: C^n([\alpha,\beta]) \to \mathbb{R}$ by

$$A(\phi) = \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) \tag{20}$$
$$- \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i)}(a_j)\left[\sum_{p=1}^{l} u_p H_{ij}(x_p) - \sum_{q=1}^{m} v_q H_{ij}(y_q)\right].$$

Then for every n-convex functions $\phi \in C^n([\alpha,\beta])$ we have $A(\phi) \geqslant 0$. Using the linearity and positivity of this functional we may derive corresponding mean-value theorems applying the same method as given in [2]. Moreover, we could produce new classes of exponentially convex functions and as outcome we get new means of the Cauchy type. Here we also refer to [7] with related results.

4. Grüss and Ostrowski type inequalities

P. L. Chebyshev [5] obtained the following inequality

$$|T(f,g)| \leqslant \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty$$

where $f,g:[\alpha,\beta]\to\mathbb{R}$ are absolutely continuous functions whose derivatives f' and g' are bounded and $T(f,g)$ is so-called Chebyshev functional defined as

$$T(f,g):=\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}f(t)g(t)dt-\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}f(t)dt\cdot\frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}g(t)dt. \tag{21}$$

Here $\|\cdot\|_\infty$ denotes the norm in $L_\infty[\alpha,\beta]$, the space of essentially bounded functions on $[\alpha,\beta]$, defined by $\|f\|_\infty = ess\sup_{t\in[\alpha,\beta]}|f(t)|$. We also use notation $\|\cdot\|_p$, $p\geqslant 1$, for L_p norm.

P. Cerone and S. S. Dragomir [4], considering the Chebyshev functional (21), obtained the following two related results.

THEOREM 6. *Let $f:[\alpha,\beta]\to\mathbb{R}$ be Lebesgue integrable and $g:[\alpha,\beta]\to\mathbb{R}$ be absolutely continuous with $(\cdot-\alpha)(\beta-\cdot)(g')^2\in L_1[\alpha,\beta]$. Then*

$$|T(f,g)|\leqslant\frac{1}{\sqrt{2}}[T(f,f)]^{\frac{1}{2}}\frac{1}{\sqrt{\beta-\alpha}}\left(\int_{\alpha}^{\beta}(x-\alpha)(\beta-x)[g'(x)]^2dx\right)^{\frac{1}{2}}. \tag{22}$$

The constant $\frac{1}{\sqrt{2}}$ in (22) is the best possible.

THEOREM 7. *Let $g:[\alpha,\beta]\to\mathbb{R}$ be monotonic nondecreasing and $f:[\alpha,\beta]\to\mathbb{R}$ be absolutely continuous with $f'\in L_\infty[\alpha,\beta]$. Then*

$$|T(f,g)|\leqslant\frac{1}{2(\beta-\alpha)}\|f'\|_\infty\int_{\alpha}^{\beta}(x-\alpha)(\beta-x)dg(x). \tag{23}$$

The constant $\frac{1}{2}$ in (23) is the best possible.

In following results we consider the function $\mathscr{B}:[\alpha,\beta]\to\mathbb{R}$, defined under assumptions of Theorem 4, by

$$\mathscr{B}(s)=\sum_{p=1}^{l}u_pG_{H,n}(x_p,s)-\sum_{q=1}^{m}v_qG_{H,n}(y_q,s), \tag{24}$$

where $\mathbf{x}\in[\alpha,\beta]^l$, $\mathbf{y}\in[\alpha,\beta]^m$, $\mathbf{u}\in\mathbb{R}^l$, $\mathbf{v}\in\mathbb{R}^m$ and $G_{H,n}$ is defined as in (6).

THEOREM 8. *Let $\alpha\leqslant a_1<a_2<\ldots<a_r\leqslant\beta$ ($r\geqslant 2$) be the given points, $k_j\geqslant 0$, $j=1,\ldots,r$, with $\sum_{j=1}^{r}k_j+r=n$. Let $\phi:[\alpha,\beta]\to\mathbb{R}$ be such that $\phi^{(n)}$ is an absolutely continuous on $[\alpha,\beta]$ with $(\cdot-\alpha)(\beta-\cdot)(\phi^{(n+1)})^2\in L_1[\alpha,\beta]$. Let $\mathbf{x}\in[\alpha,\beta]^l$, $\mathbf{y}\in[\alpha,\beta]^m$, $\mathbf{u}\in\mathbb{R}^l$, $\mathbf{v}\in\mathbb{R}^m$ and H_{ij} and $\mathscr{B}$ be defined as in (5) and (24), respectively.*

Then the remainder $R(\phi;\alpha,\beta)$ *defined by*

$$\begin{aligned} R(\phi;\alpha,\beta) = & \sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) \\ & - \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i)}(a_j)\left[\sum_{p=1}^{l} u_p H_{ij}(x_p) - \sum_{q=1}^{m} v_q H_{ij}(y_q)\right] \\ & - \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_{\alpha}^{\beta}\mathscr{B}(s)ds \end{aligned} \tag{25}$$

satisfies the estimation

$$|R(\phi;\alpha,\beta)| \leqslant \frac{\sqrt{\beta-\alpha}}{\sqrt{2}}[T(\mathscr{B},\mathscr{B})]^{\frac{1}{2}}\left(\int_{\alpha}^{\beta}(s-\alpha)(\beta-s)[\phi^{(n+1)}(s)]^2 ds\right)^{\frac{1}{2}}. \tag{26}$$

Proof. Comparing (15) and (25) we have

$$\begin{aligned} R(\phi;\alpha,\beta) &= \int_{\alpha}^{\beta}\mathscr{B}(s)\phi^{(n)}(s)ds - \frac{\phi^{(n-1)}(\beta)-\phi^{(n-1)}(\alpha)}{\beta-\alpha}\int_{\alpha}^{\beta}\mathscr{B}(s)ds \\ &= \int_{\alpha}^{\beta}\mathscr{B}(s)\phi^{(n)}(s)ds - \frac{1}{\beta-\alpha}\int_{\alpha}^{\beta}\phi^{(n)}ds\int_{\alpha}^{\beta}\mathscr{B}(s)ds = (\beta-\alpha)T(\mathscr{B},\phi^{(n)}). \end{aligned}$$

Applying Theorem 6 on the functions $\mathscr{B}$ and $\phi^{(n)}$ we obtain (26). □

Using Theorem 7 we obtain the Grüss type inequality.

THEOREM 9. *Let* $\alpha \leqslant a_1 < a_2 < \ldots < a_r \leqslant \beta$ *(* $r \geqslant 2$ *) be the given points,* $k_j \geqslant 0$, $j = 1,\ldots,r$, *with* $\sum_{j=1}^{r} k_j + r = n$. *Let* $\phi \in C^n([\alpha,\beta])$ *be such that* $\phi^{(n+1)} \geqslant 0$ *on* $[\alpha,\beta]$ *and* $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^m$, $\mathbf{u} \in \mathbb{R}^l$, $\mathbf{v} \in \mathbb{R}^m$ *and* H_{ij} *and* $\mathscr{B}$ *be defined as in (5) and (24), respectively. Then the remainder* $R(\phi;\alpha,\beta)$ *defined by (25) satisfies the estimation*

$$|R(\phi;\alpha,\beta)| \leqslant \|\mathscr{B}'\|_{\infty}\left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{27}$$

Proof. Since $R(\phi;\alpha,\beta) = (\beta-\alpha)T(\mathscr{B},\phi^{(n)})$, applying Theorem 7 on the functions $\mathscr{B}$ and $\phi^{(n)}$ we obtain (27). □

We present the Ostrowski type inequality related to generalizations of Sherman's inequality.

THEOREM 10. *Let* $\alpha \leqslant a_1 < a_2 < \ldots < a_r \leqslant \beta$ $(r \geqslant 2)$ *be the given points,* $k_j \geqslant 0$, $j = 1,...,r$, *with* $\sum_{j=1}^{r} k_j + r = n$. *Let* $\phi \in C^n([\alpha,\beta])$ *and* $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^m$, $\mathbf{u} \in \mathbb{R}^l$ *and* $\mathbf{v} \in \mathbb{R}^m$. *Let* $1 \leqslant p,q \leqslant \infty$, $1/p+1/q = 1$ *and* $\left|\phi^{(n)}\right|^p \in L_p[\alpha,\beta]$. *Then*

$$\left|\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) - \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i)}(a_j)\left[\sum_{p=1}^{l} u_pH_{ij}(x_p) - \sum_{q=1}^{m} v_qH_{ij}(y_q)\right]\right| \leqslant \left\|\phi^{(n)}\right\|_p \|\mathscr{B}\|_q, \tag{28}$$

where H_{ij} *and* $\mathscr{B}$ *are defined as in (5) and (24), respectively.*

The constant $\|\mathscr{B}\|_q$ *is sharp for* $1 < p \leqslant \infty$ *and the best possible for* $p = 1$.

Proof. Under ussumption of theorem the identity (15) holds. Applying the well-known Hölder inequality to (15), we have

$$\begin{aligned}
&\left|\sum_{p=1}^{l} u_p\phi(x_p) - \sum_{q=1}^{m} v_q\phi(y_q) - \sum_{j=1}^{r}\sum_{i=0}^{k_j} \phi^{(i)}(a_j)\left[\sum_{p=1}^{l} u_pH_{ij}(x_p) - \sum_{q=1}^{m} v_qH_{ij}(y_q)\right]\right| \\
&= \left|\int_\alpha^\beta \left[\sum_{p=1}^{l} u_pG_{H,n}(x_p,s) - \sum_{q=1}^{m} v_qG_{H,n}(y_q,s)\right]\phi^{(n)}(s)ds\right| \\
&= \left|\int_\alpha^\beta \mathscr{B}(s)\phi^{(n)}(s)ds\right| \leqslant \left\|\phi^{(n)}\right\|_p \left(\int_\alpha^\beta |\mathscr{B}(s)|^q ds\right)^{\frac{1}{q}}
\end{aligned}$$

The proof of the sharpness is analog to one in proof of Theorem 11 in [2]. □

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