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RECURSIVELY DEFINED REFINEMENTS OF THE INTEGRAL FORM OF JENSEN'S INEQUALITY

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Abstract. In this paper we establish infinite chains of integral inequalities related to the classical Jensen's inequality by using special refinements of the discrete Jensen's inequality. As applications, we introduce and study new integral means (generalized quasi-arithmetic means), and give refinements of the left hand side of Hermite-Hadamard inequality.

1. Introduction

The integral form and the discrete version of Jensen's inequality provide the starting point for much of the discussion in this paper. They can be stated as follows:

THEOREM A. (classical Jensen's inequality, see [7]) *Let g be an integrable function on a probability space $(X,\mathscr{A},\mu)$ taking values in an interval $I\subset\mathbb{R}$. Then $\int_X g d\mu$ lies in I. If f is a convex function on I such that $f\circ g$ is integrable, then*

$$f\left(\int_X g d\mu\right)\leqslant\int_X f\circ g d\mu.$$

THEOREM B. (discrete Jensen's inequality, see [7]) *Let C be a convex subset of a real vector space V, and let $f:C\to\mathbb{R}$ be a convex function. If $p_1,\ldots,p_n$ are nonnegative numbers with $\sum_{i=1}^{n}p_i=1$, and $v_1,\ldots,v_n\in C$, then*

$$f\left(\sum_{i=1}^{n}p_iv_i\right)\leqslant\sum_{i=1}^{n}p_if\left(v_i\right).$$

Jensen obtained his famous inequality in [16]. There is an extensive theory for the study of refinements of the discrete Jensen's inequality, see [15], but there are only

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few papers dealing with refinements of the classical Jensen's inequality, see Rooin [18], Horváth [9] and Horváth and Pečarić [14]. In this paper we establish infinite chains of integral inequalities related to the classical Jensen's inequality. The key of our treatment is special refinements of the discrete Jensen's inequality which have been developed in Horváth [13]. As an immediate application, new infinite refinements of the classical Jensen's inequality are derived. We essentially follow the approach of Brnetić, Pearce and Pečarić [3], but our treatment is applicable in a more general environment. In Section 3 we consider our results in some interesting special cases. In Section 4 some new integral means (generalized quasi-arithmetic means) are introduced, and their properties are studied. Section 5 is devoted to refinements of the left hand side of Hermite-Hadamard inequality.

2. Preliminaries and the main inequalities

$\mathbb{N}$ and $\mathbb{N}_+$ denote the set of nonnegative and positive integers, respectively.

Before proceeding to the results we present some hypotheses, and an inequality from [13] which will be needed.

(H_1) Let $n \in \mathbb{N}_+$ be fixed, and denote

$$S_k := \left\{ (i_1,\ldots,i_n) \in \mathbb{N}_+^n \mid \sum_{j=1}^{n} i_j = n+k-1 \right\}, \quad k \in \mathbb{N}_+. \tag{1}$$

(H_2) Let

$$(a_j(m))_{m\in\mathbb{N}_+}, \quad 1 \leqslant j \leqslant n$$

be strictly increasing sequences such that

$$\alpha := a_1(1) = \ldots = a_n(1) > 0. \tag{2}$$

(H_3) Let $p_1,\ldots,p_n$ be nonnegative numbers with $\sum\limits_{j=1}^{n} p_j = 1$.

Under the hypotheses (H_1) and (H_2), define the finite sequences

$$(u_k(i_1,\ldots,i_n))_{(i_1,\ldots,i_n)\in S_k}, \quad k \in \mathbb{N}_+$$

recursively by

$$u_1(1,\ldots,1) := \frac{1}{\alpha}, \tag{3}$$

and for every $(i_1,\ldots,i_n) \in S_{k+1}$ (see (1))

$$u_{k+1}(i_1,\ldots,i_n) := \sum_{\{l\in\{1,\ldots,n\}\mid i_l\neq 1\}} \frac{1}{1+\frac{a_l(i_l-1)}{a_l(i_l)-a_l(i_l-1)}+\sum\limits_{\substack{j=1\\ j\neq l}}^{n}\frac{a_j(i_j)}{a_j(i_j+1)-a_j(i_j)}}$$
$$\times \frac{a_l(i_l-1)}{a_l(i_l)-a_l(i_l-1)} u_k(i_1,\ldots,i_{l-1},i_l-1,i_{l+1},\ldots,i_n).$$

Now we state one of the main results in [13].

THEOREM 1. *Assume (H_1–H_3). Let C be a convex subset of a real vector space X, and $\{x_1,\ldots,x_n\}$ be a finite subset of C. If $f : C \to \mathbb{R}$ is a convex function, then*

$$f\left(\sum_{j=1}^{n} p_j x_j\right) = T_1 \leqslant \ldots \leqslant T_k \leqslant T_{k+1} \leqslant \ldots \leqslant \sum_{j=1}^{n} p_j f(x_j),$$

where for each $k \in \mathbb{N}_+$

$$\begin{aligned} T_k &= T_{k,n}(x_1,\ldots,x_n;p_1,\ldots,p_n;a_1,\ldots,a_n) \\ &:= \sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)\,p_j\right) f\left(\frac{\sum_{j=1}^{n} a_j(i_j)\,p_j x_j}{\sum_{j=1}^{n} a_j(i_j)\,p_j}\right). \end{aligned}$$

We follow this section by introducing some notations. Let $(X_i,\mathscr{B}_i,\mu_i)$ $(i \in T_l := \{1,\ldots,l\})$ be probability spaces for some $l \in \mathbb{N}_+$, $l \geqslant 2$. The σ-algebra in $X^{T_l} := X_1 \times \ldots \times X_l$ generated by the projection mappings

$$pr_i : X_1 \times \ldots \times X_l \to X_i, \quad pr_i(x_1,\ldots,x_l) = x_i \quad (i = 1,\ldots,l)$$

is denoted by $\mathscr{B}^{T_l}$. μ^{T_l} means the product measure on $\mathscr{B}^{T_l}$: this is the only measure on $\mathscr{B}^{T_l}$ (the measures are σ-finite) which satisfies

$$\mu^{T_l}(B_1 \times \ldots \times B_l) = \mu_1(B_1)\ldots\mu_l(B_l), \quad B_i \in \mathscr{B}_i, \quad (i = 1,\ldots,l).$$

The l-fold product of the probability space $(X,\mathscr{B},\mu)$ is denoted by $\left(X^l,\mathscr{B}^l,\mu^l\right)$.

The following abbreviations will be used: $d\mu^{T_l}(\mathbf{x}) := d\mu^{T_l}(x_1,\ldots,x_l)$ and $d\mu^l(\mathbf{x}) := d\mu^l(x_1,\ldots,x_l)$.

λ^l is always means the Lebesgue measure on the Borel sets of $\mathbb{R}^l$.

Our first purpose is to obtain an extended and refined version of the classical Jensen's inequality.

THEOREM 2. *Assume (H_1–H_3). Suppose the following hypotheses are also hold*

(H_4) Let $(X_i,\mathscr{B}_i,\mu_i)$ $(i = 1,\ldots,n)$ be probability spaces.

(H_5) For each $i = 1,\ldots,n$, let g_i be a μ_i-integrable function on X_i taking values in an interval $I \subset \mathbb{R}$.

(H_6) Let f be a convex function on I such that $f \circ g_i$ is μ_i-integrable on X_i $(i = 1,\ldots,n)$.

Then

(a)

$$f\left(\sum_{i=1}^{n} p_i \int_{X_i} g_i\,d\mu_i\right) \leqslant \mathscr{T}_1 \leqslant \ldots \leqslant \mathscr{T}_k \leqslant \mathscr{T}_{k+1} \leqslant \ldots \leqslant \sum_{i=1}^{n} p_i \int_{X_i} f \circ g_i\,d\mu_i, \tag{4}$$

where

$$\begin{aligned}\mathscr{T}_k &= \mathscr{T}_{k,n}(f;g_i;\mu_i;p_i;a_i)\\ &:= \sum_{(i_1,\dots,i_n)\in S_k} u_k(i_1,\dots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)\,p_j\right)\\ &\quad\times\int_{X^{T_n}} f\left(\frac{\sum_{j=1}^{n} a_j(i_j)\,p_j g_j(x_j)}{\sum_{j=1}^{n} a_j(i_j)\,p_j}\right) d\mu^{T_n}(\mathbf{x}), \quad k\in\mathbb{N}_+. \end{aligned} \tag{5}$$

(b) For each $k\in\mathbb{N}_+$ and all $t\in[0,1]$

$$f\left(\sum_{i=1}^{n} p_i\int_{X_i} g_i\,d\mu_i\right) \leqslant H_k(0) \leqslant H_k(t) \leqslant \mathscr{T}_k, \tag{6}$$

where

$$\begin{aligned}H_k(t) &= H_{k,n}(t;f;g_i;\mu_i;p_i;a_i)\\ &:= \sum_{(i_1,\dots,i_n)\in S_k} u_k(i_1,\dots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)\,p_j\right)\\ &\quad\times\int_{X^{T_n}} f\left(t\frac{\sum_{j=1}^{n} a_j(i_j)\,p_j g_j(x_j)}{\sum_{j=1}^{n} a_j(i_j)\,p_j} + (1-t)\frac{\sum_{j=1}^{n} a_j(i_j)\,p_j\int_{X_j} g_j\,d\mu_j}{\sum_{j=1}^{n} a_j(i_j)\,p_j}\right) d\mu^{T_n}(\mathbf{x}). \end{aligned} \tag{7}$$

REMARK 1. By (H_3–H_6), the hypotheses of Lemma 2.1 in [8] are all satisfied, and so it yields that all the integrals in (5) and (7) exist and finite.

Assume (H_1–H_6).
(a) If $n=1$, then $S_k=\{k\}$ and $u_k(k)=\frac{1}{a_1(k)}$ for all $k\in\mathbb{N}_+$, and therefore

$$\mathscr{T}_k = \mathscr{T}_1 = \int_{X_1} f\circ g_1\,d\mu_1, \quad k\in\mathbb{N}_+,$$

and

$$H_k(t) = H_1(t) = \int_{X_1} f\left(t g_1(x_1) + (1-t)\int_{X_1} g_1\,d\mu_1\right) d\mu_1(x_1), \quad t\in[0,1], \quad k\in\mathbb{N}_+.$$

We can see that for $n = 1$ (4) is trivial (the classical Jensen's inequality), while (6) gives

$$f\left(\int_{X_1} g_1 d\mu_1\right) = H_1(0) \leqslant H_1(t) \leqslant H_1(1) = \int_{X_1} f \circ g_1 d\mu_1, \quad t \in [0,1].$$

(b) Suppose $n \geqslant 2$. Then $S_1 = \{(1,\ldots,1)\}$, and hence

$$\mathscr{T}_1 = \int_{X^{T_n}} f\left(\sum_{j=1}^{n} p_j g_j(x_j)\right) d\mu^{T_n}(\mathbf{x}),$$

and

$$H_1(t) = \int_{X^{T_n}} f\left(t\sum_{j=1}^{n} p_j g_j(x_j) + (1-t)\sum_{j=1}^{n} p_j \int_{X_j} g_j d\mu_j\right) d\mu^{T_n}(\mathbf{x}), \quad t \in [0,1].$$

Now we summarize the essential properties of the function H_k defined in (7).

THEOREM 3. *Assume (H_1–H_6). Then for each $k \in \mathbb{N}_+$*
(a) H_k is convex and increasing.
(b)

$$H_k(0) \geqslant f\left(\sum_{i=1}^{n} p_i \int_{X_i} g_i d\mu_i\right), \quad H_k(1) = \mathscr{T}_k.$$

(c) H_k is continuous on $[0,1[$.
(d) If f is continuous, then H_k is continuous on $[0,1]$.

It is easy to construct examples which show that H_k is not continuous at 1 in general.

The following refinements of the classical Jensen's inequality are immediate consequences of Theorem 2.

THEOREM 4. *Assume (H_1–H_3). The hypotheses (H_4–H_6) are replaced by*
($\hat{H}_4$) Let $(X,\mathscr{B},\mu)$ be a probability space.
($\hat{H}_5$) Let g be a μ-integrable function on X taking values in an interval $I \subset \mathbb{R}$.
($\hat{H}_6$) Let f be a convex function on I such that $f \circ g$ is μ-integrable on X.
Then
(a)

$$f\left(\int_X g d\mu\right) \leqslant \mathscr{T}_1 \leqslant \ldots \leqslant \mathscr{T}_k \leqslant \mathscr{T}_{k+1} \leqslant \ldots \leqslant \int_X f \circ g d\mu,$$

where

$$\begin{aligned}\mathscr{T}_k &= \mathscr{T}_{k,n}(f;g;\mu;p_i;a_i)\\ &= \sum_{(i_1,\dots,i_n)\in S_k} u_k(i_1,\dots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)\,p_j\right)\\ &\quad\times\int_{X^n} f\left(\frac{\sum_{j=1}^{n} a_j(i_j)\,p_j g(x_j)}{\sum_{j=1}^{n} a_j(i_j)\,p_j}\right) d\mu^n(\mathbf{x}), \quad k\in\mathbb{N}_+.\end{aligned}$$

(b) For each $k\in\mathbb{N}_+$ *and all* $t\in[0,1]$

$$f\left(\int_X g\,d\mu\right) = H_k(0) \leqslant H_k(t) \leqslant \mathscr{T}_k,$$

where

$$\begin{aligned}H_k(t) &= H_{k,n}(t;f;g;\mu;p_i;a_i)\\ &= \sum_{(i_1,\dots,i_n)\in S_k} u_k(i_1,\dots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)\,p_j\right)\\ &\quad\times\int_{X^n} f\left(t\,\frac{\sum_{j=1}^{n} a_j(i_j)\,p_j g(x_j)}{\sum_{j=1}^{n} a_j(i_j)\,p_j} + (1-t)\int_X g\,d\mu\right) d\mu^n(\mathbf{x}).\end{aligned}$$

3. Results when recursion is explicitly represented

We first recall the following example from [13].

EXAMPLE 1. Assume (H_1) and (H_3). *Let* $\alpha>0$, $a\geqslant 0$ and $b_j\in\mathbb{R}$ $(1\leqslant j\leqslant n)$ such that the numbers $a+b_j$ are all positive. Define the sequences $(a_j(m))_{m\in\mathbb{N}_+}$ by

$$a_j(m) := \alpha\prod_{i=1}^{m-1}\left(1+\frac{1}{ai+b_j}\right), \quad m\in\mathbb{N}_+, \quad 1\leqslant j\leqslant n. \tag{8}$$

Then these sequences are strictly increasing and

$$\alpha = a_1(1) = \ldots = a_n(1) > 0,$$

thus they satisfy (H_2).

In this case it can be proved that for every $k \in \mathbb{N}_+$

$$u_k(i_1,\ldots,i_n) = \frac{1}{\alpha}\prod_{j=1}^{k-1}\frac{1}{1+a(n+j-1)+\sum_{l=1}^{n}b_l}\prod_{j=1}^{n}\left(\prod_{m=1}^{i_j-1}(am+b_j)\right)$$
$$\times\frac{(k-1)!}{(i_1-1)!\ldots(i_n-1)!},\quad (i_1,\ldots,i_n)\in S_k. \tag{9}$$

As illustrations, we just consider Theorem 4 in two special cases of the previous example.

The first part of the next result can be considered as the integral version of Theorem 1 (a) in [12].

COROLLARY 1. *Assume (H_1), (H_3), and ($\hat{H}_4$–$\hat{H}_6$). By choosing $\alpha = a = 1$ and $b_j = 0$ $(1 \leqslant j \leqslant n)$ in (8), we have*

(a)

$$f\left(\int_X g d\mu\right) \leqslant \mathscr{T}_1 \leqslant \ldots \leqslant \mathscr{T}_k \leqslant \mathscr{T}_{k+1} \leqslant \ldots \leqslant \int_X f\circ g d\mu,$$

where

$$\mathscr{T}_k = \mathscr{T}_{k,n}(f;g;\mu;p_i)$$
$$= \frac{1}{\binom{n+k-1}{k-1}}\sum_{(i_1,\ldots,i_n)\in S_k}\left(\sum_{j=1}^{n}i_jp_j\right)\int_{X^n} f\left(\frac{\sum_{j=1}^{n}i_jp_jg(x_j)}{\sum_{j=1}^{n}i_jp_j}\right)d\mu^n(\mathbf{x}),\quad k\in\mathbb{N}_+.$$

(b) For each $k \in \mathbb{N}_+$ and all $t \in [0,1]$

$$f\left(\int_X g d\mu\right) = H_k(0) \leqslant H_k(t) \leqslant \mathscr{T}_k,$$

where

$$H_k(t) = H_{k,n}(t;f;g;\mu;p_i)$$
$$= \frac{1}{\binom{n+k-1}{k-1}}\sum_{(i_1,\ldots,i_n)\in S_k}\left(\sum_{j=1}^{n}i_jp_j\right)\int_{X^n} f\left(t\frac{\sum_{j=1}^{n}i_jp_jg(x_j)}{\sum_{j=1}^{n}i_jp_j}+(1-t)\int_X g d\mu\right)d\mu^n(\mathbf{x}).$$

(c) For each $k \in \mathbb{N}_+$

$$\mathscr{T}_{k,n}\left(f;g;\mu;\frac{1}{n}\right) \leqslant \mathscr{T}_{k,n}(f;g;\mu;p_i),$$

where

$$\mathcal{T}_{k,n}\left(f;g;\mu;\frac{1}{n}\right)=\frac{1}{\binom{n+k-2}{k-1}}\sum_{(i_1,\ldots,i_n)\in S_k}\int_{X^n} f\left(\frac{1}{n+k-1}\sum_{j=1}^{n} i_j g(x_j)\right)d\mu^n(\mathbf{x})$$

(d) If $p_i>0\ (1\leqslant i\leqslant n)$, *then for each* $k\in\mathbb{N}_+$ *and for each* $l\in\mathbb{N}_+$

$$f\left(\int_X g\,d\mu\right)\leqslant\mathcal{T}_k\leqslant\mathcal{A}_l\leqslant\int_X f\circ g\,d\mu,$$

where

$$\begin{aligned}\mathcal{A}_l &= \mathcal{A}_{l,n}(f;g;\mu;p_i)\\ &:=\frac{1}{\binom{n+l-1}{l-1}}\sum_{\substack{i_1+\ldots+i_n=l\\ i_j\in\mathbb{N};\quad 1\leqslant j\leqslant n}}\left(\sum_{j=1}^{l} i_j p_j\right)\int_{X^n} f\left(\frac{\sum_{j=1}^{l} i_j p_j g(x_j)}{\sum_{j=1}^{l} i_j p_j}\right)d\mu^n(\mathbf{x}).\end{aligned}$$

(e) Suppose $p_i>0\ (1\leqslant i\leqslant n)$. *Then*

$$\lim_{k\to\infty}\mathcal{T}_k=\lim_{l\to\infty}\mathcal{A}_l=n!\int_{X^n}\left(\int_{E_n} h(t_1,\ldots,t_{n-1},x_1,\ldots,x_n)\,d\lambda^{n-1}(\mathbf{t})\right)d\mu^n(\mathbf{x}),\quad (10)$$

where

$$E_n:=\left\{(t_1,\ldots,t_{n-1})\in\mathbb{R}^{n-1}\mid\sum_{j=1}^{n-1}t_j\leqslant 1,\quad t_j\geqslant 0,\ j=1,\ldots,n-1\right\},$$

the function h *defined on* $E_n\times X^n$ *by*

$$h(t_1,\ldots,t_{n-1},x_1,\ldots,x_n):=\left(\sum_{j=1}^{n}t_j p_j\right)f\left(\frac{1}{\sum_{j=1}^{n}t_j p_j}\left(\sum_{j=1}^{n}t_j p_j g(x_j)\right)\right)$$

with the notation $t_n:=1-\sum_{j=1}^{n-1}t_j$.

Proof. (a) and (b) come from Theorem 4.

(c) Theorem 1 (b) in [12] can be applied.

(d) According to Proposition 2 in [14], the sequence $(\mathcal{A}_l)_{l\in\mathbb{N}_+}$ is decreasing and

$$f\left(\int_X g\,d\mu\right)\leqslant\mathcal{A}_l\leqslant\int_X f\circ g\,d\mu,\quad l\in\mathbb{N}_+.$$

Define for all $(x_1,\dots,x_n)\in X^n$ the expressions

$$G_{k,n}(x_1,\dots,x_n) := \frac{1}{\binom{n+k-1}{k-1}} \sum_{(i_1,\dots,i_n)\in S_k} \left(\sum_{j=1}^{n} i_j p_j\right) f\left(\frac{1}{\sum\limits_{j=1}^{n} i_j p_j}\sum_{j=1}^{n} i_j p_j g(x_j)\right), \quad k\in\mathbb{N}_+,$$

and

$$B_{l,n}(x_1,\dots,x_n) := \frac{1}{\binom{n+l-1}{l-1}} \sum_{\substack{i_1+\ldots+i_n=l \\ i_j\in\mathbb{N};\quad 1\leqslant j\leqslant n}} \left(\sum_{j=1}^{n} i_j p_j\right) f\left(\frac{1}{\sum\limits_{j=1}^{n} i_j p_j}\sum_{j=1}^{n} i_j p_j g(x_j)\right), \quad l\in\mathbb{N}_+.$$

Theorem 1 (a) in [12] shows that the sequence

$$\left(G_{k,n}(x_1,\dots,x_n)\right)_{k\in\mathbb{N}_+} \tag{11}$$

is increasing for all $(x_1,\dots,x_n)\in X^n$. By Example 3 in [10], the sequence

$$\left(B_{l,n}(x_1,\dots,x_n)\right)_{l\in\mathbb{N}_+} \tag{12}$$

is decreasing for all $(x_1,\dots,x_n)\in X^n$. It follows from Theorem 3 in [12] that

$$\begin{aligned}\lim_{k\to\infty} G_{k,n}(x_1,\dots,x_n) &= \lim_{l\to\infty} B_{l,n}(x_1,\dots,x_n)\\ &= \int\limits_{E_n} h(t_1,\dots,t_{n-1},x_1,\dots,x_n)\,d\lambda^{n-1}(\mathbf{t}), \quad (x_1,\dots,x_n)\in X^n. \end{aligned}\tag{13}$$

Putting all this together gives that for each $k\in\mathbb{N}_+$ and for each $l\in\mathbb{N}_+$

$$G_{k,n}(x_1,\dots,x_n)\leqslant B_{l,n}(x_1,\dots,x_n), \quad (x_1,\dots,x_n)\in X^n.$$

By integrating both sides over X^n, we have $\mathscr{T}_k\leqslant\mathscr{A}_l$ $(k\in\mathbb{N}_+,\ l\in\mathbb{N}_+)$.

(e) Assuming that the function h is $\lambda^{n-1}\times\mu^n$-integrable over $E_n\times X^n$ for the present, the monotonicity properties of the sequences (11) and (12), the limit formula (13), and the Fubini's theorem imply (10).

The measurability of h is obvious. To justify the supposed integrability condition, choose a fixed interior point a of I. Since f is convex

$$f(t)\geqslant f(a)+f'_+(a)(t-a), \quad t\in I,$$

where $f'_+(a)$ means the right-hand derivative of f at a. By using this and the discrete Jensen's inequality, we have

$$\begin{aligned}&\left(\sum_{j=1}^{n} t_j p_j\right) f(a)+f'_+(a)\left(\sum_{j=1}^{n} t_j p_j g(x_j)-a\left(\sum_{j=1}^{n} t_j p_j\right)\right)\\ &\leqslant h(t_1,\dots,t_{n-1},x_1,\dots,x_n)\leqslant \sum_{j=1}^{n} t_j p_j f(g(x_j))\end{aligned}\tag{14}$$

for all $(t_1,\ldots,t_{n-1},x_1,\ldots,x_n) \in E_n \times X^n$. It is enough to prove that the functions on the left hand side and the right hand side of the previous inequalities are $\lambda^{n-1} \times \mu^n$-integrable over $E_n \times X^n$. We consider only the function on the right hand side of (14), the other case can be handled similarly. In proving this, we may suppose that f is nonnegative on I, and therefore by the Fubini's theorem, and then by Lemma 2.1 (a) in [8]

$$\begin{aligned}
&\int\limits_{E_n \times X^n} \left(\sum_{j=1}^{n} t_j p_j f(g(x_j)) \right) d\lambda^{n-1} \times \mu^n (t_1,\ldots,t_{n-1},x_1,\ldots,x_n) \\
&= \int\limits_{E_n} \left(\int\limits_{X^n} \sum_{j=1}^{n} t_j p_j f(g(x_j))\, d\mu^n(\mathbf{x}) \right) d\lambda^{n-1}(\mathbf{t}) \\
&= \left(\int\limits_{X} f \circ g\, d\mu \right) \left(\int\limits_{E_n} \left(\sum_{j=1}^{n} t_j p_j \right) d\lambda^{n-1}(\mathbf{t}) \right) < \infty.
\end{aligned}$$

The proof is complete. □

REMARK 2. We stress that the sequences $(\mathscr{T}_k)_{k\in\mathbb{N}_+}$ and $(\mathscr{A}_l)_{l\in\mathbb{N}_+}$, compared in part (d) of the previous result, are generated from such refinements of the discrete Jensen's inequality which have been obtained by essentially different methods.

Now, the integral variant of Theorem 1 (a) is obtained.

COROLLARY 2. *Assume (H_1), (H_3), and $(\hat{H}_4$–$\hat{H}_6)$. By choosing $\alpha = 1$, $a = 0$ and $b_j = \frac{1}{\lambda_j - 1}$ $(1 \leqslant j \leqslant n)$, where $\lambda_j > 1$ $(1 \leqslant j \leqslant n)$ in (8), we have with the notation* $d(\lambda) := \sum_{j=1}^{n} \frac{1}{\lambda_j - 1}$

(a)

$$f\left(\int\limits_X g\, d\mu \right) \leqslant \mathscr{T}_1 \leqslant \ldots \leqslant \mathscr{T}_k \leqslant \mathscr{T}_{k+1} \leqslant \ldots \leqslant \int\limits_X f \circ g\, d\mu,$$

where for every $k \in \mathbb{N}_+$

$$\begin{aligned}
\mathscr{T}_k &= \mathscr{T}_{k,n}(f;g;\mu;p_i) \\
&= \frac{1}{(d(\lambda)+1)^{k-1}} \sum_{(i_1,\ldots,i_n)\in S_k} \frac{(k-1)!}{(i_1-1)!\ldots(i_n-1)!} \\
&\quad \times \prod_{j=1}^{n} \frac{1}{(\lambda_j - 1)^{i_j - 1}} \left(\sum_{j=1}^{n} \lambda_j^{i_j-1} p_j \right) \int\limits_{X^n} f \left(\frac{\sum_{j=1}^{n} \lambda_j^{i_j-1} p_j g(x_j)}{\sum_{j=1}^{n} \lambda_j^{i_j-1} p_j} \right) d\mu^n(\mathbf{x}).
\end{aligned}$$

(b) For each $k \in \mathbb{N}_+$ *and all* $t \in [0,1]$

$$f\left(\int_X g d\mu\right) = H_k(0) \leqslant H_k(t) \leqslant \mathscr{T}_k,$$

where

$$\begin{aligned} H_k(t) &= H_{k,n}(t; f; g; \mu; p_i) \\ &= \frac{1}{(d(\lambda)+1)^{k-1}} \sum_{(i_1,\dots,i_n)\in S_k} \frac{(k-1)!}{(i_1-1)!\dots(i_n-1)!} \prod_{j=1}^{n} \frac{1}{(\lambda_j-1)^{i_j-1}} \\ &\quad \times \left(\sum_{j=1}^{n} \lambda_j^{i_j-1} p_j\right) \int_{X^n} f\left(\frac{\sum_{j=1}^{n} \lambda_j^{i_j-1} p_j g(x_j)}{\sum_{j=1}^{n} \lambda_j^{i_j-1} p_j} + (1-t)\int_X g d\mu\right) d\mu^n(\mathbf{x}). \end{aligned}$$

(c)

$$\lim_{k\to\infty} \mathscr{T}_k = \int_X f \circ g d\mu.$$

Proof. We have only to apply Theorem 4 to get (a) and (b).

(c) Let for all $(x_1,\dots,x_n) \in X^n$

$$\begin{aligned} D_{k,n}(\lambda; x_1,\dots,x_n) = \frac{1}{(d(\lambda)+1)^{k-1}} \sum_{(i_1,\dots,i_n)\in S_k} \frac{(k-1)!}{(i_1-1)!\dots(i_n-1)!} \\ \times \prod_{j=1}^{n} \frac{1}{(\lambda_j-1)^{i_j-1}} \left(\sum_{j=1}^{n} \lambda_j^{i_j-1} p_j\right) f\left(\frac{\sum_{j=1}^{n} \lambda_j^{i_j-1} p_j g(x_j)}{\sum_{j=1}^{n} \lambda_j^{i_j-1} p_j}\right). \end{aligned}$$

By Theorem 1 (a) in [11], the sequence

$$\left(D_{k,n}(\lambda; x_1,\dots,x_n)\right)_{k\in\mathbb{N}_+}$$

is increasing, and by (b) of the same theorem

$$\lim_{k\to\infty} D_{k,n}(\lambda; x_1,\dots,x_n) = \sum_{j=1}^{n} p_j f(g(x_j)), \quad (x_1,\dots,x_n) \in X^n.$$

It follows from these facts that

$$\begin{aligned} \lim_{k\to\infty} \mathscr{T}_k &= \lim_{k\to\infty} \int_{X^n} D_{k,n}(\lambda; x_1,\dots,x_n) d\mu^n(\mathbf{x}) \\ &= \sum_{j=1}^{n} p_j \int_{X^n} f(g(x_j)) \mu^n(\mathbf{x}) = \int_X f \circ g d\mu. \end{aligned}$$

The proof is now complete. □

4. Means generated by the expressions in the new refinements

In this section we introduce some new integral means (generalized quasi-arithmetic means) and study their properties.

DEFINITION 1. Assume (H_1–H_3) and

(H_4) Let $(X_i, \mathscr{B}_i, \mu_i)$ $(i = 1, \ldots, n)$ be probability spaces.

Assume further

(H_7) For each $i = 1, \ldots, n$, let g_i be a measurable function on X_i taking values in an interval $I \subset \mathbb{R}$.

(H_8) Let $\varphi,\ \psi : I \to \mathbb{R}$ be continuous and strictly monotone functions.

(a) For each $k \in \mathbb{N}_+$, we define integral means with respect to (5) by

$$\begin{aligned} & M_{\psi,\varphi}(k) \\ = {} & M_{\psi,\varphi}(g_i;\mu_i;p_i;a_i;k) \\ := {} & \psi^{-1}\Bigg(\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n) \\ & \times \left(\sum_{j=1}^{n} a_j(i_j)p_j\right)\int_{X^{T_n}} (\psi\circ\varphi^{-1})\left(\frac{\sum_{j=1}^{n} a_j(i_j)p_j\varphi(g_j(x_j))}{\sum_{j=1}^{n} a_j(i_j)p_j}\right) d\mu^{T_n}(\mathbf{x})\Bigg), \end{aligned} \tag{15}$$

if the integrals exist and finite.

(b) For each $k \in \mathbb{N}_+$ and for all $t \in [0,1]$, integral means can be defined with respect to (7) by

$$\begin{aligned} & M_{\psi,\varphi}(t;k) \\ = {} & M_{\psi,\varphi}(t;g_i;\mu_i;p_i;a_i;k) \\ := {} & \psi^{-1}\Bigg(\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)p_j\right)\int_{X^{T_n}} (\psi\circ\varphi^{-1}) \\ & \times \left(t\frac{\sum_{j=1}^{n} a_j(i_j)p_j\varphi(g_j(x_j))}{\sum_{j=1}^{n} a_j(i_j)p_j} + (1-t)\frac{\sum_{j=1}^{n} a_j(i_j)p_j\int_{X_j}\varphi\circ g_j d\mu_j}{\sum_{j=1}^{n} a_j(i_j)p_j}\right) d\mu^{T_n}(\mathbf{x})\Bigg), \end{aligned} \tag{16}$$

if the integrals exist and finite.

It has been shown in [13] that for any $j = 1, \ldots, n$

$$\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)a_j(i_j) = 1, \tag{17}$$

and therefore by (H_3)

$$\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)\,p_j\right) = 1, \quad k\in\mathbb{N}_+. \tag{18}$$

This implies that

$$M_{\psi,\varphi}(k)\in I \text{ and } M_{\psi,\varphi}(t;k)\in I, \quad k\in\mathbb{N}_+, \quad t\in[0,1],$$

that is they really define means.

By Remark 1, if $\varphi\circ g_i$ and $\psi\circ g_i$ are μ_i-integrable on X_i $(i=1,\ldots,n)$, and $\psi\circ\varphi^{-1}$ is either convex or concave, then the integrals in (15) and (16) exist and finite.

The following integral mean is also needed: if (H_3–H_4) and (H_7) are satisfied, and $\chi : I\to\mathbb{R}$ is a continuous and strictly monotone function, then define

$$M_\chi = M_\chi(g_i;\mu_i;p_i) := \chi^{-1}\left(\sum_{i=1}^{n} p_i\int_{X_i}\chi\circ g_i\,d\mu_i\right), \tag{19}$$

if the integrals exist and finite.

Let q, $g:[a,b]\to\mathbb{R}$ be positive and Lebesgue-integrable functions, and let $\chi :]0,\infty[\to\mathbb{R}$ be a continuous and strictly monotone function. The so called generalized weighted quasi-arithmetic mean of g with respect to the weight function q

$$M_\chi = M_\chi(g;q) := \chi^{-1}\left(\frac{\int_a^b q(x)\,\chi(g(x))\,dx}{\int_a^b q(x)\,dx}\right) \tag{20}$$

is a special case of (19), and it contains different remarkable means (for example, weighted arithmetic, harmonic and geometric means). The properties of means (20) are studied intensively, we just mention two papers dealing with integral means: Haluška and Hutnik [6] and Sun, Long and Chu [19].

We continue this section with a discussion on the monotonicity of the introduced means.

THEOREM 5. *Assume (H_1–H_4), (H_7–H_8), and assume that $\varphi\circ g_i$ and $\psi\circ g_i$ are μ_i-integrable on X_i $(i=1,\ldots,n)$. Then*

(a)

$$M_\varphi \leqslant M_{\psi,\varphi}(1)\leqslant\ldots\leqslant M_{\psi,\varphi}(k)\leqslant\ldots\leqslant M_\psi, \quad k\in\mathbb{N}_+, \tag{21}$$

and

$$M_\varphi \leqslant M_{\psi,\varphi}(0;k)\leqslant M_{\psi,\varphi}(t;k)\leqslant M_{\psi,\varphi}(k), \quad k\in\mathbb{N}_+, \quad t\in[0,1],$$

if either $\psi\circ\varphi^{-1}$ is convex and ψ is increasing or $\psi\circ\varphi^{-1}$ is concave and ψ is decreasing.

(b)

$$M_{\varphi} \geqslant M_{\psi,\varphi}(1) \geqslant \ldots \geqslant M_{\psi,\varphi}(k) \geqslant \ldots \geqslant M_{\psi}, \quad k \in \mathbb{N}_{+}, \tag{22}$$

and

$$M_{\varphi} \geqslant M_{\psi,\varphi}(0;k) \geqslant M_{\psi,\varphi}(t;k) \geqslant M_{\psi,\varphi}(k), \quad k \in \mathbb{N}_{+}, \quad t \in [0,1],$$

if either $\psi \circ \varphi^{-1}$ is convex and ψ is decreasing or $\psi \circ \varphi^{-1}$ is concave and ψ is increasing.

Proof. (a) and (b) can be obtained by applications of Theorem 2 to the functions $\psi \circ \varphi^{-1}$ and $\varphi \circ g_i$ $(i=1,\ldots,n)$ ($\varphi(I)$ is an interval), if $\psi \circ \varphi^{-1}$ is convex, and to the functions $-\psi \circ \varphi^{-1}$ and $\varphi \circ g_i$ $(i=1,\ldots,n)$, if $\psi \circ \varphi^{-1}$ is concave, and then upon taking ψ^{-1}. □

Recently, in [17] by Khuram Ali Khan and Pečarić the inequalities (21) and (22) have been proved for the mean $M_{\psi,\varphi}(1)$. It can be seen that our approach allows us to essentially generalize and extend some of the results from [17].

5. Connections to Hermite-Hadamard inequality

Different refinements of the left hand side of Hermite-Hadamard inequality can be got from Theorem 4.

THEOREM 6. *Assume (H_1–H_3), and let f be a convex function on $[a,b]$. Then*
(a)

$$f\left(\frac{a+b}{2}\right) \leqslant \hat{\mathscr{T}}_1 \leqslant \ldots \leqslant \hat{\mathscr{T}}_k \leqslant \hat{\mathscr{T}}_{k+1} \leqslant \ldots \leqslant \frac{1}{b-a}\int_a^b f,$$

where

$$\begin{aligned}\hat{\mathscr{T}}_k &= \hat{\mathscr{T}}_{k,n}(f;p_i;a_i)\\ &= \frac{1}{(b-a)^n}\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)\,p_j\right)\\ &\quad\times \int_{[a,b]^n} f\left(\frac{\sum_{j=1}^{n} a_j(i_j)\,p_j x_j}{\sum_{j=1}^{n} a_j(i_j)\,p_j}\right) d\lambda^n(\mathbf{x}), \quad k \in \mathbb{N}_{+}.\end{aligned}$$

(b) For each $k \in \mathbb{N}_{+}$ and all $t \in [0,1]$

$$f\left(\frac{a+b}{2}\right) = \hat{H}_k(0) \leqslant \hat{H}_k(t) \leqslant \hat{H}_k(1) = \hat{\mathscr{T}}_k,$$

where

$$\begin{aligned}\hat{H}_k(t) &= \hat{H}_{k,n}(t;f;p_i;a_i)\\ &= \frac{1}{(b-a)^n}\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)\,p_j\right)\\ &\quad\times \int_{[a,b]^n} f\left(t\frac{\sum_{j=1}^{n} a_j(i_j)\,p_j x_j}{\sum_{j=1}^{n} a_j(i_j)\,p_j} + (1-t)\frac{a+b}{2}\right) d\lambda^n(\mathbf{x}).\end{aligned}$$

(c) $\hat{H}_k$ is convex and increasing for each $k\in\mathbb{N}_+$. If f is continuous, then $\hat{H}_k$ is also continuous.

Proof. We can apply Theorem 4 and Theorem 3, when the probability space is $\left([a,b],\mathscr{B},\frac{1}{b-a}\lambda\right)$ ($\mathscr{B}$ now means the σ-algebra of Borel sets of $[a,b]$), $I:=[a,b]$, g is the identity function on $[a,b]$, and f is a convex function on $[a,b]$. $\square$

The investigation of functions like $\hat{H}_k$, seems to be due to Dragomir, who has introduced and studied among others the function $\hat{H}_{1,1}$ in [4]. Many papers deal with similar functions, for example see Abdallah El Farissi [1], Dragomir and Agarwal [5], Yang and Wang [20] and Yang and Tseng [21]. Our result gives a new approach in treating the problem.

6. Proofs of the main inequalities

We need the following well known result:

LEMMA 1. (see [2], 16.1 Lemma) *Let $(\Omega,\mathscr{A},\mu)$ be a measure space. Let E be a metric space, and $f:E\times\Omega\to\mathbb{R}$ a function with the properties*
(i) $\omega\to f(x,\omega)$ is μ-integrable for each $x\in E$,
(ii) $x\to f(x,\omega)$ is continuous at $x_0\in E$ for every $\omega\in\Omega$,
(iii) there is a nonnegative μ-integrable function h on Ω such that

$$|f(x,\omega)|\leqslant h(\omega),\quad (x,\omega)\in E\times\Omega.$$

Then the function φ defined on E by

$$\varphi(x)=\int_{\Omega} f(x,\omega)\,d\mu(\omega)$$

is continuous at x_0.

Proof of Theorem 3. Fix $k\in\mathbb{N}_+$.

(a) Since convexity is invariant under affine maps, the integral is monotonic, and the sum of convex functions is also convex, H_k is convex on $[0,1]$.

By applying the classical Jensen's inequality, we get for all $t \in [0,1]$ that

$$H_k(t) \geqslant \sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)p_j\right)$$
$$\times f\left(\int_{X^{T_n}}\left(t\frac{\sum_{j=1}^{n} a_j(i_j)p_jg_j(x_j)}{\sum_{j=1}^{n} a_j(i_j)p_j}+(1-t)\frac{\sum_{j=1}^{n} a_j(i_j)p_j\int_{X_j} g_jd\mu_j}{\sum_{j=1}^{n} a_j(i_j)p_j}\right)d\mu^{T_n}(\mathbf{x})\right)$$
$$= \sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)p_j\right)f\left(\frac{\sum_{j=1}^{n} a_j(i_j)p_j\int_{X_j} g_jd\mu_j}{\sum_{j=1}^{n} a_j(i_j)p_j}\right)$$
$$= H_k(0). \tag{23}$$

Suppose $0 \leqslant t_1 < t_2 \leqslant 1$. The convexity of H_k, and $H_k(t) \geqslant H_k(0)$ $(t \in [0,1])$ mean that

$$\frac{H_k(t_2)-H_k(t_1)}{t_2-t_1} \geqslant \frac{H_k(t_2)-H_k(0)}{t_2} \geqslant 0,$$

and thus

$$H_k(t_2) \geqslant H_k(t_1).$$

(b) When (18) is combined with (23) and with the discrete Jensen's inequality, it follows that

$$H_k(0) \geqslant f\left(\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\left(\sum_{j=1}^{n} a_j(i_j)p_j\int_{X_j} g_jd\mu_j\right)\right)$$
$$= f\left(\sum_{j=1}^{n} p_j\int_{X_j} g_jd\mu_j\left(\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)a_j(i_j)\right)\right) = f\left(\sum_{j=1}^{n} p_j\int_{X_j} g_jd\mu_j\right).$$

$H_k(1) = \mathscr{T}_k$ is obvious.

(c) It follows from (a).

(d) It remains only to show that H_k is continuous at 1. We check the conditions of Lemma 1.

(i) See Remark 1.

(ii) Since f is continuous, the function

$$t \to f\left(t\frac{\sum_{j=1}^{n} a_j(i_j)p_j g_j(x_j)}{\sum_{j=1}^{n} a_j(i_j)p_j} + (1-t)\frac{\sum_{j=1}^{n} a_j(i_j)p_j \int_{X_j} g_j d\mu_j}{\sum_{j=1}^{n} a_j(i_j)p_j}\right)$$

is continuous at 1 for every $\mathbf{x} \in X^{T_n}$.

(iii) By applying the discrete Jensen's inequality, we have

$$f\left(t\frac{\sum_{j=1}^{n} a_j(i_j)p_j g_j(x_j)}{\sum_{j=1}^{n} a_j(i_j)p_j} + (1-t)\frac{\sum_{j=1}^{n} a_j(i_j)p_j \int_{X_j} g_j d\mu_j}{\sum_{j=1}^{n} a_j(i_j)p_j}\right)$$

$$\leqslant tf\left(\frac{\sum_{j=1}^{n} a_j(i_j)p_j g_j(x_j)}{\sum_{j=1}^{n} a_j(i_j)p_j}\right) + (1-t)f\left(\frac{\sum_{j=1}^{n} a_j(i_j)p_j \int_{X_j} g_j d\mu_j}{\sum_{j=1}^{n} a_j(i_j)p_j}\right)$$

$$\leqslant \max\left(f\left(\frac{\sum_{j=1}^{n} a_j(i_j)p_j g_j(x_j)}{\sum_{j=1}^{n} a_j(i_j)p_j}\right), f\left(\frac{\sum_{j=1}^{n} a_j(i_j)p_j \int_{X_j} g_j d\mu_j}{\sum_{j=1}^{n} a_j(i_j)p_j}\right)\right)$$

for all $t \in [0,1]$ and $\mathbf{x} \in X^{T_n}$.

Choose a fixed interior point a of I. Since f is convex

$$f(t) \geqslant f(a) + f'_+(a)(z-a), \quad z \in I,$$

where $f'_+(a)$ means the right-hand derivative of f at a. It follows from this that

$$f\left(t\frac{\sum_{j=1}^{n} a_j(i_j)p_j g_j(x_j)}{\sum_{j=1}^{n} a_j(i_j)p_j} + (1-t)\frac{\sum_{j=1}^{n} a_j(i_j)p_j \int_{X_j} g_j d\mu_j}{\sum_{j=1}^{n} a_j(i_j)p_j}\right)$$

$$\geqslant f(a) + f'_+(a)\left(t\frac{\sum_{j=1}^{n} a_j(i_j)p_j g_j(x_j)}{\sum_{j=1}^{n} a_j(i_j)p_j} + (1-t)\frac{\sum_{j=1}^{n} a_j(i_j)p_j \int_{X_j} g_j d\mu_j}{\sum_{j=1}^{n} a_j(i_j)p_j} - a\right)$$

$$\geqslant \min\left(f(a)+f'_{+}(a)\left(\frac{\sum_{j=1}^{n} a_j(i_j)\,p_j g_j(x_j)}{\sum_{j=1}^{n} a_j(i_j)\,p_j}-a\right)\right.$$

$$\left.\times\, f(a)+f'_{+}(a)\left(\frac{\sum_{j=1}^{n} a_j(i_j)\,p_j \int\limits_{X_j} g_j d\mu_j}{\sum_{j=1}^{n} a_j(i_j)\,p_j}-a\right)\right)$$

for all $t\in[0,1]$ and $\mathbf{x}\in X^{T_n}$.

The result now follows from Lemma 1.

The proof is complete. □

Proof of Theorem 2. (a) Since $S_1=\{(1,\ldots,1)\}$, (3), (2) and (H_3) give that

$$\mathscr{T}_1=\int\limits_{X^{T_n}} f\left(\sum_{j=1}^{n} p_j g_j(x_j)\right) d\mu^{T_n}(\mathbf{x}).$$

From the classical Jensen's inequality we therefore have

$$\mathscr{T}_1\geqslant f\left(\int\limits_{X^{T_n}}\sum_{j=1}^{n} p_j g_j(x_j)\,d\mu^{T_n}(\mathbf{x})\right)=f\left(\sum_{i=1}^{n} p_i\int\limits_{X_i} g_i d\mu_i\right).$$

According to Theorem 1

$$\begin{aligned}&T_{k,n}(g(x_1),\ldots,g(x_n);p_1,\ldots,p_n;a_1,\ldots,a_n)\\ &\leqslant T_{k+1,n}(g(x_1),\ldots,g(x_n);p_1,\ldots,p_n;a_1,\ldots,a_n),\quad k\in\mathbb{N}_+\end{aligned}$$

for all fixed $(x_1,\ldots,x_n)\in X^{T_l}$, and hence

$$\mathscr{T}_k\leqslant\mathscr{T}_{k+1},\quad k\in\mathbb{N}_+.$$

Finally, it follows from the discrete Jensen's inequality that

$$\begin{aligned}\mathscr{T}_k&\leqslant\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\int\limits_{X^{T_n}}\sum_{j=1}^{n} a_j(i_j)\,p_j f(g_j(x_j))\,d\mu^{T_n}(\mathbf{x})\\ &=\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\sum_{j=1}^{n} a_j(i_j)\,p_j\int\limits_{X_j} f\circ g_j d\mu_j\\ &=\sum_{j=1}^{n}\left(\sum_{(i_1,\ldots,i_n)\in S_k} u_k(i_1,\ldots,i_n)\,a_j(i_j)\right) p_j\int\limits_{X_j} f\circ g_j d\mu_j.\end{aligned}\tag{24}$$

This and (17) imply that

$$\mathscr{T}_k \leqslant \sum_{j=1}^{n} p_j \int_{X_j} f \circ g_j d\mu_j, \quad k \in \mathbb{N}_+.$$

(b) Apply Theorem 3 (b).

The proof is complete. □

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